

琉球大学学術リポジトリ

Approximation Processes of Bernstein-type Operators

メタデータ	言語: 出版者: Department of Mathematical Sciences, Faculty of Science, University of the Ryukyus 公開日: 2007-03-03 キーワード (Ja): キーワード (En): 作成者: Nishishiraho, Toshihiko メールアドレス: 所属:
URL	http://hdl.handle.net/20.500.12000/77

APPROXIMATION PROCESSES OF BERNSTEIN-TYPE OPERATORS

TOSHIHIKO NISHISHIRAH

ABSTRACT. We give a generalization of the Bernstein polynomials on the closed unit interval of the real line, and consider the uniform convergence and the degree of approximation by the generalized Bernstein-type operators.

1. Introduction

Let \mathbb{N} denote the set of all natural numbers. Let f be a real-valued continuous function on the closed unit interval $\mathbb{I} = [0, 1]$ of the real line \mathbb{R} and let $n \in \mathbb{N}$. Then n th Bernstein polynomial of f is defined by

$$(1) \quad B_n(f)(x) = \sum_{j=0}^n f\left(\frac{j}{n}\right) \binom{n}{j} x^j (1-x)^{n-j} \quad (x \in \mathbb{I}).$$

It is well-known that the sequence $\{B_n(f)\}_{n \in \mathbb{N}}$ converges uniformly to f on \mathbb{I} (cf. [7]).

Nowadays there are various generalizations of (1) and one of them is the following ([3], cf. [2], [12]):

$$(2) \quad C_n(f, s_n, x) = \frac{1}{s_n} \sum_{j=0}^n \sum_{k=0}^{s_n-1} f\left(\frac{j+k}{n+s_n-1}\right) \binom{n}{j} x^j (1-x)^{n-j},$$

where $\{s_n\}_{n \in \mathbb{N}}$ is a sequence of natural numbers. If $s_n = 1$ for all $n \in \mathbb{N}$, then $C_n(f, s_n, x) = B_n(f)(x)$.

Received November 30, 2006.

In this paper, we further generalize (2) to the multidimensional case and consider its uniform convergence with rates in terms of the modulus of continuity of functions to be approximated.

2. A theorem of Korovkin-Volkov type

Let $1 \leq p \leq \infty$ be fixed and let X be a locally closed subset of the r -dimensional Euclidean space \mathbb{R}^r with the metric

$$d_p(x, y) = \begin{cases} \left(\sum_{i=1}^r |x_i - y_i|^p \right)^{1/p} & (1 \leq p < \infty) \\ \max\{|x_i - y_i| : 1 \leq i \leq r\} & (p = \infty), \end{cases}$$

where

$$x = (x_1, x_2, \dots, x_r), \quad y = (y_1, y_2, \dots, y_r) \in \mathbb{R}^r.$$

Let $V(X)$ denote the linear space of all real-valued functions on X . Let $B(X)$ denote the Banach space of all real-valued bounded functions on X with the supremum norm $\|\cdot\|_X$. Also, we denote by $C(X)$ the linear subspace consisting of all functions $f \in V(X)$ which are continuous on X and set $BC(X) = B(X) \cap C(X)$. 1_X stands for the unit function defined by $1_X(x) = 1$ for all $x \in X$. For $i = 1, 2, \dots, r$, e_i denotes the i th coordinate function on \mathbb{R}^r defined by

$$e_i(x) = x_i$$

for all $x = (x_1, x_2, \dots, x_r) \in \mathbb{R}^r$. Then we have

$$(3) \quad (d_p(x, y))^q \leq C(p, q, r) \sum_{i=1}^r |e_i(x) - e_i(y)|^q \quad (x, y \in \mathbb{R}^r, q > 0),$$

where

$$C(p, q, r) = \begin{cases} r^{q/p} & (1 \leq p < \infty, p \neq q) \\ 1 & (1 \leq p < \infty, p = q) \\ 1 & (p = \infty). \end{cases}$$

Let Y be a compact subset of X . A subset G of $C(X)$ is called a Korovkin test system for $BC(X)$ if for any sequence $\{L_n\}_{n \in \mathbb{N}}$ of positive linear operators of $C(X)$ into $V(Y)$, the limit relation

$$\lim_{n \rightarrow \infty} \|L_n(g) - g\|_Y = 0 \quad \text{for all } g \in G$$

implies that

$$\lim_{n \rightarrow \infty} \|L_n(f) - f\|_Y = 0 \quad \text{for all } f \in BC(X).$$

For the background of the Korovkin-type approximation theory, we refer to the book of Altomare and Campiti [1], in which an excellent source and a vast literature of this theory as well as the Bernstein-type operators can be found (cf. [4], [5], [6]).

Theorem 1. *Let s be any fixed even positive integer. Let*

$$G_s := \{1_X\} \cup \{e_i^j : 1 \leq i \leq r, 1 \leq j \leq s\}$$

and

$$K_s := G_{s-1} \cup \{e_1^s + e_2^s + \cdots + e_r^s\}.$$

Then K_s (and hence G_s) is a Korovkin test system for $BC(X)$. In particular, K_2 (and hence G_2) is a Korovkin test system for $BC(X)$.

Proof. In view of (3), this follows from [9, Theorem 4].

3. An estimate of the rate of convergence

Let $f \in B(X)$ and $\delta \geq 0$. Then we define

$$\omega_p(f, \delta) = \sup\{|f(x) - f(y)| : x, y \in X, d_p(x, y) \leq \delta\},$$

which is called the modulus of continuity of f . Obviously, $\omega_p(f, \cdot)$ is a monotone increasing function on $[0, \infty)$ and

$$\omega_p(f, 0) = 0, \quad \omega_p(f, \delta) \leq 2\|f\|_X \quad (\delta \geq 0).$$

Also, f is uniformly continuous on X if and only if

$$\lim_{\delta \rightarrow +0} \omega_p(f, \delta) = 0.$$

Now we here suppose that X is convex.

Theorem 2. *Let $\{L_n\}_{n \in \mathbb{N}}$ be a sequence of positive linear operators of $C(X)$ into $V(Y)$ and let $q \geq 1$. Then for all $f \in BC(X)$, $x \in Y$, $n \in \mathbb{N}$ and all $\delta > 0$,*

$$(4) \quad |L_n(f)(x) - f(x)| \leq |f(x)| |L_n(1_X)(x) - 1| \\ + (L_n(1_X)(x) + \zeta_n(q, \delta, x)) \omega_p(f, \delta),$$

where

$$\zeta_n(q, \delta, x)$$

$$= \min\{C(p, q, r)\delta^{-q}\mu_n(q, x), (L_n(1_X)(x))^{1-1/q}C(p, q, r)^{1/q}\delta^{-1}\mu_n(q, x)^{1/q}\}$$

and

$$\mu_n(q, x) = \sum_{i=1}^r L_n(|e_i - e_i(x)1_X|^q)(x).$$

In particular, if $L_n(1_X) = 1_X$ for all $n \in \mathbb{N}$, then (4) reduces to

$$|L_n(f)(x) - f(x)| \leq (1 + \zeta_n(q, \delta, x))\omega_p(f, \delta)$$

and

$$\zeta_n(q, \delta, x) = \min\{C(p, q, r)\delta^{-q}\mu_n(q, x), C(p, q, r)^{1/q}\delta^{-1}\mu_n(q, x)^{1/q}\}.$$

Proof. We have

$$\omega_p(f, \xi\delta) \leq (1 + \xi)\omega_p(f, \delta)$$

for all $\xi, \delta \geq 0$ (cf. [8, Lemma 3 (ii)], [10, Lemma 1 (b), [11, Lemma 2.4 (b)]]. Therefore, in view of (3) the desired inequality (4) follows from [9, Corollary 3].

4. Bernstein-type operators

Let $n \in \mathbb{N}$. Then we define

$$p_{n,j}(\alpha, \beta) = \binom{n}{j} \beta^j (\alpha - \beta)^{n-j} \quad (\alpha, \beta \in \mathbb{R}, j = 0, 1, 2, \dots, n).$$

Lemma 1. *The following equalities hold:*

$$(5) \quad \sum_{j=0}^n \sum_{k=0}^{m-1} p_{n,j}(\alpha, \beta) = m\alpha^n.$$

$$(6) \quad \sum_{j=0}^n \sum_{k=0}^{m-1} (j+k)p_{n,j}(\alpha, \beta) = m\alpha^{n-1} \left(n\beta + \frac{\alpha(m-1)}{2} \right).$$

$$(7) \quad \begin{aligned} & \sum_{j=0}^n \sum_{k=0}^{m-1} (j+k)^2 p_{n,j}(\alpha, \beta) \\ &= m \left(n(n-1)\alpha^{n-2}\beta^2 + mn\alpha^{n-1}\beta + \frac{(m-1)(2m-1)\alpha^n}{6} \right). \end{aligned}$$

Proof. This follows from elementary properties of binomial coefficients.

Lemma 2. Let α, β and γ be real numbers. Then the following equality holds:

$$(8) \quad \begin{aligned} & \sum_{j=0}^n \sum_{k=0}^{m-1} (\gamma(j+k) - \beta)^2 p_{n,j}(\alpha, \beta) \\ & = m\beta^2 ((\alpha - n\gamma)^2 - n\gamma^2) \alpha^{n-2} + m\beta\gamma (n\gamma - (m-1)(\alpha - n\gamma)) \alpha^{n-1} \\ & \quad + \frac{1}{3} m(m-1) \left(m - \frac{1}{2} \right) \gamma^2 \alpha^n. \end{aligned}$$

Proof. By (5), (6) and (7) in Lemma 1 we have

$$\begin{aligned} & \sum_{j=0}^n \sum_{k=0}^{m-1} (\gamma(j+k) - \beta)^2 p_{n,j}(\alpha, \beta) = \gamma^2 \sum_{j=0}^n \sum_{k=0}^{m-1} (j+k)^2 p_{n,j}(\alpha, \beta) \\ & \quad - 2\beta\gamma \sum_{j=0}^n \sum_{k=0}^{m-1} (j+k) p_{n,j}(\alpha, \beta) + \beta^2 \sum_{j=0}^n \sum_{k=0}^{m-1} p_{n,j} p_{n,j}(\alpha, \beta) \\ & = m(n(n-1)\gamma^2 - 2n\alpha\gamma + \alpha^2) \alpha^{n-2} \beta^2 + m\gamma(mn\gamma - \alpha(m-1)) \alpha^{n-1} \beta \\ & \quad + \frac{m(m-1)(2m-1)\alpha^n \gamma^2}{6} = m((n\gamma - \alpha)^2 - n\gamma^2) \alpha^{n-2} \beta^2 \\ & \quad + m\gamma(n\gamma - (m-1)(\alpha - n\gamma)) \alpha^{n-1} \beta + m\gamma^2 \left(\frac{(m-1)^2}{3} + \frac{m-1}{6} \right) \alpha^n, \end{aligned}$$

which implies the desired equality (8).

From now on let X be a locally closed subset of the region of the first hyperquadrant

$$\mathbb{R}_+^r = \{(x_1, x_2, \dots, x_r) \in \mathbb{R}^r : x_i \geq 0, 1 \leq i \leq r\}$$

and let $b = (b_1, b_2, \dots, b_r) \in X$, where $b_i > 0$ for $i = 1, 2, \dots, r$. Let Y be a closed subset of $X \cap H_b$, where

$$H_b := \{(x_1, x_2, \dots, x_r) \in \mathbb{R}^r : x_i \leq b_i, i = 1, 2, \dots, r\}.$$

Let $\{\nu_{n,i}\}_{n \in \mathbb{N}}, i = 1, 2, \dots, r$, be strictly monotone increasing sequences of positive integers and let $\{m_{n,i}\}_{n \in \mathbb{N}}, i = 1, 2, \dots, r$, be sequences of positive integers. Let $\{\gamma_{n,i}\}_{n \in \mathbb{N}}, i = 1, 2, \dots, r$, be sequences of positive real-valued functions defined on Y which satisfy

$$\begin{aligned} & \gamma_n(j_1, j_2, \dots, j_r; k_1, k_2, \dots, k_r; x) \\ & := (\gamma_{n,1}(x)(j_1 + k_1), \gamma_{n,2}(x)(j_2 + k_2), \dots, \gamma_{n,r}(x)(j_r + k_r)) \in X \end{aligned}$$

for all $x \in Y$ and all $n \in \mathbb{N}$, where $j_i = 0, 1, 2, \dots, \nu_{n,i}$ and $k_i = 0, 1, 2, \dots, m_{n,i} - 1$ ($i = 1, 2, \dots, r$). Let $n \in \mathbb{N}, f \in C(X)$ and $x = (x_1, x_2, \dots, x_r) \in Y$. Then we define

$$(9) \quad B_{\nu_{n,1}, \dots, \nu_{n,r}}(f; m_{n,1}, \dots, m_{n,r}; \gamma_{n,1}, \dots, \gamma_{n,r}; b)(x)$$

$$= \prod_{i=1}^r \frac{1}{m_{n,i} b_i^{\nu_{n,i}}} \sum_{j_1=0}^{\nu_{n,1}} \sum_{k_1=0}^{m_{n,1}-1} \sum_{j_2=0}^{\nu_{n,2}} \sum_{k_2=0}^{m_{n,2}-1} \cdots \sum_{j_r=0}^{\nu_{n,r}} \sum_{k_r=0}^{m_{n,r}-1}$$

$$f(\gamma_n(j_1, j_2, \dots, j_r; k_1, k_2, \dots, k_r; x)) \prod_{i=1}^r p_{\nu_{n,i}, j_i}(b_i, x_i),$$

which forms a positive linear operator of $C(X)$ into $V(Y)$.

Remark 1. Let $r = 1, b_1 = 1$ and $X = Y = \mathbb{I}$. If $\nu_{n,1} = n, m_{n,1} = s_n$ and $\gamma_{n,1}(x) = 1/(n + s_n - 1)$ for all $n \in \mathbb{N}$ and all $x \in Y$, then (9) reduces to (2). Also, let $X = Y = \mathbb{I}^r$ be the unit r -cube and $b = (1, 1, \dots, 1)$. If $m_{n,i} = 1$ and $\gamma_{n,i}(x) = 1/\nu_{n,i}$ for all $n \in \mathbb{N}, x \in Y$ and for $i = 1, 2, \dots, r$, then (9) reduces to the form

$$B_{\nu_{n,1}, \nu_{n,2}, \dots, \nu_{n,r}}(f)(x) := \sum_{j_1=0}^{\nu_{n,1}} \sum_{j_2=0}^{\nu_{n,2}} \cdots \sum_{j_r=0}^{\nu_{n,r}} f\left(\frac{j_1}{\nu_{n,1}}, \frac{j_2}{\nu_{n,2}}, \dots, \frac{j_r}{\nu_{n,r}}\right)$$

$$\times \prod_{i=1}^r \binom{\nu_{n,i}}{j_i} x_i^{j_i} (1 - x_i)^{\nu_{n,i} - j_i}$$

(cf. [7]).

Theorem 3. *If*

$$(10) \quad \lim_{n \rightarrow \infty} \|\gamma_{n,i}\|_Y = 0 \quad (i = 1, 2, \dots, r),$$

$$(11) \quad \lim_{n \rightarrow \infty} \|m_{n,i} \gamma_{n,i}\|_Y = 0 \quad (i = 1, 2, \dots, r),$$

and

$$(12) \quad \lim_{n \rightarrow \infty} \|\nu_{n,i} \gamma_{n,i} - b_i 1_X\|_Y = 0 \quad (i = 1, 2, \dots, r),$$

then for every $f \in BC(X)$,

$$(13) \quad \lim_{n \rightarrow \infty} \|B_{\nu_{n,1}, \dots, \nu_{n,r}}(f; m_{n,1}, \dots, m_{n,r}; \gamma_{n,1}, \dots, \gamma_{n,r}; b) - f\|_Y = 0.$$

Proof. We set

$$T_n(f)(x) := B_{\nu_{n,1}, \dots, \nu_{n,r}}(f; m_{n,1}, \dots, m_{n,r}; \gamma_{n,1}, \dots, \gamma_{n,r}; b)(x)$$

$$(n \in \mathbb{N}, f \in C(X), x \in Y).$$

Then by (5), (6) and (7) in Lemma 1, we have that for all $n \in \mathbb{N}$, $x = (x_1, x_2, \dots, x_r) \in Y$ and for $i = 1, 2, \dots, r$,

$$T_n(1_X)(x) = 1,$$

$$T_n(e_i)(x) - x_i = \frac{1}{b_i}(\nu_{n,i}\gamma_{n,i}(x) - b_i)x_i + \frac{(m_{n,i} - 1)\gamma_{n,i}(x)}{2},$$

and

$$T_n(e_i^2)(x) - x_i^2 = \left(\frac{x_i}{b_i}\right)^2 \left((\nu_{n,i}\gamma_{n,i}(x) - b_i)^2 + 2b_i(\nu_{n,i}\gamma_{n,i}(x) - b_i) - \nu_{n,i}\gamma_{n,i}^2(x) \right)$$

$$+ \frac{m_{n,i}\nu_{n,i}\gamma_{n,i}^2(x)}{b_i}x_i + \frac{1}{3}((m_{n,i} - 1)\gamma_{n,i}(x))^2 + \frac{1}{6}(m_{n,i} - 1)\gamma_{n,i}^2(x).$$

Therefore, it follows from (10), (11) and (12) that

$$\lim_{n \rightarrow \infty} \|T_n(1_X) - 1_X\|_Y = 0,$$

$$\lim_{n \rightarrow \infty} \|T_n(e_i) - e_i\|_Y = 0 \quad (i = 1, 2, \dots, r)$$

and

$$\lim_{n \rightarrow \infty} \|T_n(e_i^2) - e_i^2\|_Y = 0 \quad (i = 1, 2, \dots, r),$$

and so Theorem 1 establishes the desired result (13).

Theorem 4. Let $\{\epsilon_n\}_{n \in \mathbb{N}}$ be a sequence of positive real numbers. Then for all $f \in BC(X)$, $x = (x_1, x_2, \dots, x_r) \in Y$ and all $n \in \mathbb{N}$,

$$(14) \quad |B_{\nu_{n,1}, \dots, \nu_{n,r}}(f; m_{n,1}, \dots, m_{n,r}; \gamma_{n,1}, \dots, \gamma_{n,r}; b)(x)|$$

$$\leq (1 + \mu_n(x))\omega_p(f, \epsilon_n),$$

where

$$\mu_n(x) = \min \left\{ C(p, r)\epsilon_n^{-2} \sum_{i=1}^r \mu_{n,i}(x), \sqrt{C(p, r)\epsilon_n^{-1}} \sqrt{\sum_{i=1}^r \mu_{n,i}(x)} \right\},$$

$$C(p, r) = \begin{cases} r^{2/p} & (1 \leq p < \infty, p \neq 2) \\ 1 & (p = 2, \infty) \end{cases}$$

and

$$\mu_{n,i}(x) = ((b_i - \nu_{n,i}\gamma_{n,i}(x))^2 - \nu_{n,i}\gamma_{n,i}^2(x)) \left(\frac{x_i}{b_i}\right)^2$$

$$\begin{aligned}
& + \gamma_{n,i}(x) \left(\nu_{n,i} \gamma_{n,i}(x) - (m_{n,i} - 1)(b_i - \nu_{n,i} \gamma_{n,i}(x)) \right) \frac{x_i}{b_i} \\
& + \frac{1}{3} (m_{n,i} - 1) \left(m_{n,i} - \frac{1}{2} \right) \gamma_{n,i}^2(x).
\end{aligned}$$

Proof. Let T_n be as in the proof of Theorem 3. Then by (8) in Lemma 2, for $i = 1, 2, \dots, r$, we have

$$\begin{aligned}
& T_n(|e_i - x_i 1_X|^2)(x) \\
& \frac{1}{m_{n,i} b_i^{\nu_{n,i}}} \sum_{j_i=0}^{\nu_{n,i}} \sum_{k_i=0}^{m_{n,i}-1} (\gamma_{n,i}(x)(j_i + k_i) - x_i)^2 p_{\nu_{n,i}, j_i}^\circ(b_i, x_i) = \mu_{n,i}(x).
\end{aligned}$$

Therefore, applying Theorem 2 to $L_n = T_n, q = 2$ and $\delta = \epsilon_n$, we obtain the desired inequality (14).

Theorem 5. *Let $\{\epsilon_n\}_{n \in \mathbb{N}}$ be a sequence of positive real numbers. Suppose that $\nu_{n,i} \gamma_{n,i}(x) \leq b_i$ for all $n \in \mathbb{N}, x \in Y$ and for $i = 1, 2, \dots, r$. Then for all $BC(X), x = (x_1, x_2, \dots, x_r) \in Y$ and all $n \in \mathbb{N}$,*

$$\begin{aligned}
(15) \quad & |B_{\nu_{n,1}, \dots, \nu_{n,r}}(f; m_{n,1}, \dots, m_{n,r}; \gamma_{n,1}, \dots, \gamma_{n,r}; b)(x)| \\
& \leq (1 + \tau_n(x)) \omega_p(f, \epsilon_n),
\end{aligned}$$

where

$$\tau_n(x) = \min \left\{ C(p, r) \epsilon_n^{-2} \sum_{i=1}^r \tau_{n,i}(x), \sqrt{C(p, r)} \epsilon_n^{-1} \sqrt{\sum_{i=1}^r \tau_{n,i}(x)} \right\}$$

and

$$\begin{aligned}
\tau_{n,i}(x) &= \left(\frac{x_i}{b_i} \right)^2 (b_i - \nu_{n,i} \gamma_{n,i}(x))^2 + \frac{x_i}{b_i} \nu_{n,i} \gamma_{n,i}^2(x) \\
&+ \frac{1}{3} (m_{n,i} - 1) \left(m_{n,i} - \frac{1}{2} \right) \gamma_{n,i}^2(x).
\end{aligned}$$

Proof. We have $\mu_{n,i}(x) \leq \tau_{n,i}(x)$ for all $n \in \mathbb{N}, x \in Y$ and for $i = 1, 2, \dots, r$ and so $\mu_n(x) \leq \tau_n(x)$. Therefore, the desired inequality (15) immediately follows from (14).

Let $\{\varphi_{n,i}\}_{n \in \mathbb{N}}, i = 1, 2, \dots, r$, be sequences of nonnegative real-valued functions defined on Y . We define

$$\gamma_{n,i}(x) = \frac{1}{\nu_{n,i} + m_{n,i} + \varphi_{n,i}(x) - 1}$$

for all $n \in \mathbb{N}, x \in Y$ and for $i = 1, 2, \dots, r$. Suppose that

$$\gamma_n(j_1, j_2, \dots, j_r; k_1, k_2, \dots, k_r; x) \in X$$

for all $x \in Y$ and all $n \in \mathbb{N}$, where $j_i = 0, 1, 2, \dots, \nu_{n,i}$ and $k_i = 0, 1, 2, \dots, m_{n,i} - 1$ ($i = 1, 2, \dots, r$). Then Theorems 3, 4 and 5 can be applied.

In the rest of this section, we restrict ourselves to the special case where

$$b = (1, 1, \dots, 1), \quad \varphi_{n,i}(x) = 0 \quad (n \in \mathbb{N}, x \in Y, i = 1, 2, \dots, r).$$

Therefore, (9) reduces to the form

$$\begin{aligned} & B_{\nu_{n,1}, \nu_{n,2}, \dots, \nu_{n,r}}(f; m_{n,1}, m_{n,2}, \dots, m_{n,r})(x) \\ &:= \prod_{i=1}^r \frac{1}{m_{n,i}} \sum_{j_1=0}^{\nu_{n,1}} \sum_{k_1=0}^{m_{n,1}-1} \sum_{j_2=0}^{\nu_{n,2}} \sum_{k_2=0}^{m_{n,2}-1} \cdots \sum_{j_r=0}^{\nu_{n,r}} \sum_{k_r=0}^{m_{n,r}-1} \\ & f\left(\frac{j_1 + k_1}{\nu_{n,1} + m_{n,1} - 1}, \frac{j_2 + k_2}{\nu_{n,2} + m_{n,2} - 1}, \dots, \frac{j_r + k_r}{\nu_{n,r} + m_{n,r} - 1}\right) \\ & \times \prod_{i=1}^r \binom{\nu_{n,i}}{j_i} x_i^{j_i} (1 - x_i)^{\nu_{n,i} - j_i}. \end{aligned}$$

Theorem 6. Suppose that

$$\lim_{n \rightarrow \infty} \frac{m_{n,i}}{\nu_{n,i}} = 0 \quad (i = 1, 2, \dots, r).$$

Then for all $f \in BC(X)$,

$$(16) \quad \lim_{n \rightarrow \infty} \|B_{\nu_{n,1}, \nu_{n,2}, \dots, \nu_{n,r}}(f; m_{n,1}, m_{n,2}, \dots, m_{n,r}) - f\|_Y = 0.$$

Proof. We have

$$\lim_{n \rightarrow \infty} \|\gamma_{n,i}\|_Y = \lim_{n \rightarrow \infty} \frac{1}{\nu_{n,i} + m_{n,i} - 1} = 0 \quad (i = 1, 2, \dots, r),$$

$$\lim_{n \rightarrow \infty} \|m_{n,i} \gamma_{n,i}\|_Y = \lim_{n \rightarrow \infty} \frac{m_{n,i}}{\nu_{n,i} + m_{n,i} - 1} = 0 \quad (i = 1, 2, \dots, r)$$

and

$$\lim_{n \rightarrow \infty} \|\nu_{n,i} \gamma_{n,i} - 1_X\|_Y = \lim_{n \rightarrow \infty} \frac{m_{n,i} - 1}{\nu_{n,i} + m_{n,i} - 1} = 0 \quad (i = 1, 2, \dots, r).$$

Therefore, by Theorem 3, (13) implies the desired result (16).

Theorem 7. Let $\{\epsilon_n\}_{n \in \mathbb{N}}$ be a sequence of positive real numbers. Then for all $BC(X)$, $x = (x_1, x_2, \dots, x_r) \in Y$ and all $n \in \mathbb{N}$,

$$(17) \quad |B_{\nu_{n,1}, \nu_{n,2}, \dots, \nu_{n,r}}(f; m_{n,1}, m_{n,2}, \dots, m_{n,r})(x) - f(x)| \leq (1 + \eta_n(x))\omega_p(f, \epsilon_n),$$

where

$$\eta_n(x) = \min \left\{ C(p, r) \epsilon_n^{-2} \sum_{i=1}^r \eta_{n,i}(x), \sqrt{C(p, r)} \epsilon_n^{-1} \sqrt{\sum_{i=1}^r \eta_{n,i}(x)} \right\}$$

and

$$\begin{aligned} \eta_{n,i}(x) &= x_i^2 \left(\frac{m_{n,i} - 1}{\nu_{n,i} + m_{n,i} - 1} \right)^2 + \frac{\nu_{n,i} x_i}{(\nu_{n,i} + m_{n,i} - 1)^2} \\ &\quad + \frac{1}{3} (m_{n,i} - 1) \left(m_{n,i} - \frac{1}{2} \right) \frac{1}{(\nu_{n,i} + m_{n,i} - 1)^2}. \end{aligned}$$

Proof. This immediately follows from Theorem 5, since $b_i = 1$, $\gamma_{n,i}(x) = 1/(\nu_{n,i} + m_{n,i} - 1)$ for all $n \in \mathbb{N}$, $x \in Y$ and for $i = 1, 2, \dots, r$.

Theorem 8. For all $f \in BC(X)$ and all $n \in \mathbb{N}$,

$$(18) \quad \|B_{\nu_{n,1}, \nu_{n,2}, \dots, \nu_{n,r}}(f; m_{n,1}, m_{n,2}, \dots, m_{n,r}) - f\|_Y \leq (1 + \min \{M_{r,Y} C(p, r), \sqrt{M_{r,Y} C(p, r)}\}) \omega_p(f, \theta_n),$$

where

$$M_{r,Y} = \max \left\{ \|e_i^2 + \frac{1}{3} 1_X\|_Y : i = 1, 2, \dots, r \right\}$$

and

$$\theta_n = \sqrt{\sum_{i=1}^r \left(\left(\frac{m_{n,i} - 1}{\nu_{n,i} + m_{n,i} - 1} \right)^2 + \frac{1}{\nu_{n,i}} \right)}.$$

Proof. For all $n \in \mathbb{N}$, $x = (x_1, x_2, \dots, x_r) \in Y$ and for $i = 1, 2, \dots, r$, we have

$$\begin{aligned} \eta_{n,i}(x) &= \left(x_i^2 + \frac{1}{3} \right) \left(\frac{m_{n,i} - 1}{\nu_{n,i} + m_{n,i} - 1} \right)^2 \\ &\quad + \frac{\nu_{n,i} x_i}{(\nu_{n,i} + m_{n,i} - 1)^2} + \frac{1}{6} \frac{m_{n,i} - 1}{(\nu_{n,i} + m_{n,i} - 1)^2} \\ &< \left(x_i^2 + \frac{1}{3} \right) \left(\frac{m_{n,i} - 1}{\nu_{n,i} + m_{n,i} - 1} \right)^2 + \frac{\nu_{n,i} x_i}{\nu_{n,i}^2} + \frac{1}{12\nu_{n,i}} \\ &= \left(x_i^2 + \frac{1}{3} \right) \left(\frac{m_{n,i} - 1}{\nu_{n,i} + m_{n,i} - 1} \right)^2 + \left(x_i + \frac{1}{12} \right) \frac{1}{\nu_{n,i}} \end{aligned}$$

$$\leq M_{r,Y} \left(\left(\frac{m_{n,i} - 1}{\nu_{n,i} + m_{n,i} - 1} \right)^2 + \frac{1}{\nu_{n,i}} \right).$$

Therefore, we obtain

$$\eta_n(x) < \min\{M_{r,Y}C(p,r)\epsilon_n^{-2}\theta_n^2, \sqrt{M_{r,Y}C(p,r)}\epsilon_n^{-1}\theta_n\},$$

and so taking $\epsilon_n = \theta_n$, the inequality (17) reduces to

$$\begin{aligned} & |B_{\nu_{n,1},\nu_{n,2},\dots,\nu_{n,r}}(f; m_{n,1}, m_{n,2}, \dots, m_{n,r})(x) - f(x)| \\ & \leq \left(1 + \min\{M_{r,Y}C(p,r), \sqrt{M_{r,Y}C(p,r)}\} \right) \omega_p(f, \theta_n), \end{aligned}$$

which implies the desired inequality (18).

Remark 2. Since $M_{r,Y} \leq 4/3$ and

$$\begin{aligned} \theta_n & \leq \sum_{i=1}^r \left(\frac{m_{n,i} - 1}{\nu_{n,i} + m_{n,i} - 1} + \frac{1}{\sqrt{\nu_{n,i}}} \right) \\ & \leq \sum_{i=1}^r \left(\frac{m_{n,i} - 1}{\nu_{n,i}} + \frac{1}{\sqrt{\nu_{n,i}}} \right) =: \zeta_n, \end{aligned}$$

the inequality (18) yields

$$\begin{aligned} (19) \quad & \|B_{\nu_{n,1},\nu_{n,2},\dots,\nu_{n,r}}(f; m_{n,1}, m_{n,2}, \dots, m_{n,r}) - f\|_Y \\ & \leq \left(1 + 2\sqrt{\frac{C(p,r)}{3}} \right) \omega_p(f, \zeta_n) \quad (f \in BC(X), n \in \mathbb{N}). \end{aligned}$$

Therefore, the inequality (19) improves the estimate given in [3, Theorem 2] for $r = 1, b_1 = 1, X = Y = \mathbb{I}$ and $\nu_{n,1} = n$. Also, we can get the following estimate for all $f \in C(\mathbb{T}^r)$ and all $n \in \mathbb{N}$ (cf. [10], [11]):

$$(20) \quad \|B_{\nu_{n,1},\nu_{n,2},\dots,\nu_{n,r}}(f) - f\|_{\mathbb{T}^r} \leq \xi_n(p,r) \omega_p \left(f, \epsilon_n \sqrt{\sum_{i=1}^r \frac{1}{\nu_{n,i}}} \right),$$

where $\{\epsilon_n\}_{n \in \mathbb{N}}$ is a sequence of positive real numbers and

$$\xi_n(p,r) = 1 + \min \left\{ \frac{C(p,r)}{4\epsilon_n^2}, \frac{\sqrt{C(p,r)}}{2\epsilon_n} \right\}.$$

In particular, if $\nu_{n,i} = n$ for all $n \in \mathbb{N}, i = 1, 2, \dots, r$ and $B_n := B_{\nu_{n,1},\nu_{n,2},\dots,\nu_{n,r}}$, then (20) establishes the inequality

$$\|B_n(f) - f\|_{\mathbb{T}^r} \leq \left(1 + \min \left\{ \frac{rC(p,r)}{4}, \frac{\sqrt{rC(p,r)}}{2} \right\} \right) \omega_p \left(f, \frac{1}{\sqrt{n}} \right)$$

for all $f \in C(\mathbb{T}^r)$ and all $n \in \mathbb{N}$.

References

- [1] F. Altomare and M. Campiti, *Korovkin-Type Approximation Theory and its Applications*, Walter de Gruyter, Berlin/New York, 1994.
- [2] J.-D. Cao, *On Sikkema-Kantorovič polynomials of order k*, Approx. Theory & its Appl., **5** (1989), 99-109.
- [3] J.-D. Cao, *A generalization of the Bernstein polynomials*, J. Math. Anal. Appl., **209** (1997), 140-146.
- [4] K. Donner, *Extension of Positive Operators and Korovkin Theorems*, Lecture Notes in Math. **904**, Springer-Verlag, Berlin/Heidelberg/New York, 1982.
- [5] K. Keimel and W. Roth, *Ordered Cones and Approximation*, Lecture Notes in Math. **1517**, Springer-Verlag, Berlin/Heidelberg/New York, 1992.
- [6] P. P. Korovkin, *Linear Operators and Approximation Theory*, Hindustan Publ. Corp., Delhi, 1960.
- [7] G. G. Lorentz, *Bernstein Polynomials*, Univ. of Toronto Press, Toronto, 1953.
- [8] T. Nishishiraho, *Convergence of positive linear approximation processes*, Tôhoku Math. J., **35** (1983), 441-458.
- [9] T. Nishishiraho, *Refinements of Korovkin-type approximation processes*, Proc. the 4th Internat. Conf. on Functional Analysis and Approximation Theory, Acquafredda di Maratea, 2000, Suppl. Rend. Circ. Mat. Palermo, **68** (2002), 711-725.
- [10] T. Nishishiraho, *The degree of interpolation type approximation processes for vector-valued functions*, Ryukyu Math. J., **17** (2004), 21-37.
- [11] T. Nishishiraho, *Quantitative equi-uniform approximation processes of integral operators in Banach spaces*, Taiwanese J. Math., **10** (2006), 441-465.
- [12] P. C. Sikkema, *Über die Schurerschen Linearen Positiven Operatoren. I*, Indag. Math., **37** (1975), 230-242; *II*, ibid., 243-253.

Department of Mathematical Sciences
Faculty of Science
University of the Ryukyus
Nishihara-cho, Okinawa 903-0213
JAPAN