On Quasicharacters Associated to Distinguished Metaplectic Forms

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1. Introduction

Let F be a non-archimedean local field such that $Card.\mu_n(F) = n$ where

$$\mu_n(F) = \{x \in F : x^n = 1\}.$$

Let

$$(\ ,\ )_F : F^\times \times F^\times \to \mu_n(F)$$

be the n-th order Hilbert symbolof F. The metaplectic group $\tilde{G}_r^{(c)}$(see [2]) is a central extension of the general linear group $G_r = GL(r, F)$ over F by $\mu_n(F)$ $(c \in \mathbb{Z}/n\mathbb{Z})$:

$$I \longrightarrow \mu_n(F) \xrightarrow{\mathrm{i}} \tilde{G}_r^{(c)} \xrightarrow{\mathrm{p}} G_r \longrightarrow I.$$

Here we choose a section $\underline{s}$: $G_r \to \tilde{G}_r^{(c)}$ of p so that $\underline{s}$ is homomorphism on the upper-triangular unipotent subgroup N and the $\mu_n(F)$-valued 2-cocycle σ on G_r, which arise from $\underline{s}$, is given on the diagonal subgroup H of G_r by

$$\sigma(h, h') = \prod_{i<j}(h_i, h'_j)_F \cdot (det.h, det.h')_F^c$$

for $h = diag(h_i), h' = diag(h'_i) \in H$ with $h_i, h'_i \in F^\times$ $(1 \le i \le r)$.
We fixe an injective character $\epsilon : \mu_n(F) \to \mathbb{C}^\times$ once for all.
Let us consider an admissible irreducible ϵ-representation $\tilde{\pi}$ of $\tilde{G}_r$ on a complex vector space V, of which the restriction to $\mathrm{i}(\mu_n(F))$ coincides to $\epsilon \cdot I_V$ where I_V is the ientity operator of V.
If e is a non-degenerate character of the upper-triangular unipotent subgroup N of G_r, we define $Wh(\tilde{\pi})$ by

$$\{\lambda \in V' :< \lambda, \tilde{\pi}(n)f >= e(n) < \lambda, f > \quad \text{for } n \in N, f \in V\}$$

where V' is the algebraic dual space of V with a bilinear form $<\ ,\ >$. We call $\tilde{\pi}$ *distinguished* if $dim_{\mathbb{C}} Wh(\tilde{\pi}) = 1$.

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Let Z denote the center of G_r and set

$$\tilde{Z} = \mathbf{p}^{-1}(Z).$$

We find that $\tilde{Z}$ plays an interesting role in the following sense. First $\tilde{Z}$ is isomorphic to a central extension of $F^\times$ by $\mu_n(F)$. On the other hand, for any admissible irreducible represetation $\tilde{\pi}$ of $\tilde{G}_r$, $\tilde{Z}$ acts naturally on $Wh(\tilde{\pi})$ where $\mathbf{i}(\mu_n(F))$ acts by ϵ. We shall prove in the next section.

Proposition 1. *If $\tilde{G}_r$ has a distinguished ϵ-representation, then $\tilde{Z}$ is abelian.*

Proposition 2. *The group $\tilde{Z}$ is abelian if and only if*

$$r(r-1+2cr) \equiv 0(n).$$

Let us see the relation between $\tilde{Z}$ and the center $Z(\tilde{G}_r)$ of $\tilde{G}_r$. For any divisor m of n, set

$$\tilde{Z}^m = \mathbf{p}^{-1}\{z^m : z \in Z\}.$$

We recall that the center $Z(\tilde{G}_r)$ is given by $\tilde{Z}^{n'}$ where $n' = n/(n, r-1+2cr)$ (see [2]);

$$\tilde{Z}^n \subset Z(\tilde{G}_r) \subset \tilde{Z}.$$

Thus the center $Z(\tilde{G}_r)$ is equal to $\tilde{Z}$ if and only if $r-1+2cr \equiv 0(n)$. This is the case where $Z(\tilde{G}_r)$ is the biggest. If the center of $\tilde{G}_r$ is the smallest ,i.e. that is given by $\tilde{Z}^n$, and if $\tilde{Z}$ is abelian, then $r \equiv 0(n)$. Conversely if $r \equiv 0(n)$, then $\tilde{Z}$ is abelian and the center of $\tilde{G}_r$ is given by $\tilde{Z}^n$.

If $\tilde{\pi}$ is a distinguished ϵ-representation of $\tilde{G}_r$, we obtain a one-dimen sional representation $\tilde{\omega}(\tilde{\pi})$ of $\tilde{Z}$ by acting $\tilde{Z}$ on $Wh(\tilde{\pi})$. The restriction of $\tilde{\omega}(\tilde{\pi})$ to the center $Z(\tilde{G}_r)$ is nothing but the central quasicharacter of $\tilde{\pi}$. When $r \equiv 0(n)$, the quasicharacter $\tilde{\omega}(\tilde{\pi})$ contains information about $\tilde{\pi}$, more than its central quasicharacter, which is rather mysterious.

From $\tilde{\omega}(\tilde{\pi})$ we can easily move to a quasicharacter $\omega(\tilde{\pi})$ of $F^\times$. Then we obtain a mapping

$$\tilde{\pi} \to \omega(\tilde{\pi})$$

from the set of the isomorphism classes of distinguished ϵ-representations of $\tilde{G}_{n\ell}$ to the set of quasicharacters of $F^\times$.

When $r = n$ in which case is the most interesting, this mapping seems to be the inverse of the conjectural correspondence

$$\omega \to \tilde{\pi}(\omega)$$

from the set of quasicharacters of $F^\times$ to the set of the isomorphism classes of distinguished ϵ-representations of $\tilde{G}_n$. We can calculate that in the following cases.

(1) If ω is a quasicharacter of $F^\times$ such that $\omega(\mu_n(F)) = 1$, we can construct a distinguished ϵ-representation $\tilde{\pi}(\omega)$ as an image of an intertwining operator

of certain principal series representations of $\tilde{G}_n$ such that $\omega(\tilde{\pi}(\omega)) = \omega$ (see [2]).

(2) If n is prime to the residual charactreristic of F and if ω is a quasicharacter of $F^\times$ of which restriction to $\mu_n(F)$ is injective, then we can construct a supercuspidal distingushed ϵ-representation $\tilde{\pi}(\omega)$ of $\tilde{G}_n$ inducing from a modulo center compact subgroup such that $\omega(\tilde{\pi}(\omega)) = \omega$ (see [1]).

The advantage of our mapping is that the automorphic property is almost trivial. Let k be a global field containing a primitive root of unity and A the ring of its adeles. Let $\tilde{G}_r(A)$ be the global metaplectic group which is a central extension of $G_r(A) = GL(r, A)$ by $\mu_n(k)$ and contains a discrete subgroup $G_r^*(k)$ isomorphic to $G_r(k) = GL(r, k)$. Let $\epsilon : \mu_n(k) \to C^\times$ be an injective character. We consider irreducible admissible ϵ-representations π_A of $\tilde{G}_r(A)$ of which the restriction to the central subgroup isomorphic to $\mu_n(k)$ is given by ϵ. Then we have a "tensor decomposition"

$$\pi_A \cong \otimes_v \pi_v$$

in the sense of [2] where v runs over all places of k and π_v is an irreducible admissible ϵ-representation of $\tilde{G}_r(k_v)$. We call π_A distinguished if all π_v are distinguished, and automorphic if it has a realization on the space of automorphic functions on $\tilde{G}_r(A)$ with respect to $G_r^*(k)$. If π_A is distinguished, we define

$$\omega(\pi_A) = \prod_v \omega(\pi_v)$$

as a quasicharacter of $A^\times = \prod_v k_v^\times$.
In the last section we prove

Theorem. *If a distinguished ϵ-representation π_A of $\tilde{G}_{n\ell}(A)$ is automorphic, then $\omega(\pi_A)$ is also automorphic.*

2. The subgroup $\tilde{Z}$

Proof of Proposition 1. Let $\tilde{\pi}$ be a distingished representation of $\tilde{G}_r$. Then by definition $dim_{\mathbf{C}} Wh(\tilde{\pi}) = 1$. We obtain an irreducible 1-dimen sional $\tilde{Z}$-module $Wh(\tilde{\pi})$. Here the central subgroup $\mathbf{i}(\mu_n(F)$ of $\tilde{Z}$ acts on $Wh(\tilde{\pi})$ by ϵ. If $\tilde{Z}$ is not abelian, one has $dim_{\mathbf{C}} Wh(\tilde{\pi}) > 1$. Therefore $\tilde{Z}$ is abelian.

Proof of Proposition 2. For $\tilde{h}, \tilde{h'} \in \tilde{H} = \mathbf{p}^{-1}(H)$ with $\mathbf{p}(\tilde{h}) = diag(h_i)$ and $\mathbf{p}(\tilde{h'}) = diag(h'_i)$ we have

$$\tilde{h}\tilde{h'}\tilde{h}^{-1}\tilde{h'}^{-1} = \mathbf{i}(\prod_{i=1}^{r}(h_i, h'_i)_F^{-1} \cdot (det.p(\tilde{h}), det.p(\tilde{h'}))_F^{1+2c}).$$

If $\tilde{z}, \tilde{z'} \in \tilde{Z}$, then $\mathbf{p}(\tilde{z}) = z \cdot I$ and $\mathbf{p}(\tilde{z'}) = z' \cdot I$ with $z, z' \in F^\times$. Hence

$$\begin{aligned}\tilde{z}\tilde{z'}\tilde{z}^{-1}\tilde{z'}^{-1} &= \mathbf{i}((z, z')_F^{-r} \cdot (z^r, z'^r)_F^{1+2c}) \\ &= \mathbf{i}((z, z')_F^{-r+r^2(1+2c)}) \\ &= \mathbf{i}((z, z')_F^{r(r-1+2cr)}).\end{aligned}$$

This proves the proposition.

3. The quasi-character $\tilde{\omega}(\pi)$

Let (π, V) be a distinguished representation of $\tilde{G}_{n\ell}$. Then $\tilde{Z}$ is abelian and $Wh(\pi)$ is a 1-dimensional $\tilde{Z}$-module. Hence there exists a quasi-character $\tilde{\omega}(\pi)$ of the abelian group $\tilde{Z}$ such that

$$\lambda(\pi(x)f) = (\tilde{\omega}(\pi))(x)\lambda(f)$$

for all $\lambda \in Wh(\pi)$, $f \in V$ and $x \in \tilde{Z}$. For a quasi-character χ of $F^\times$, we write $\pi \otimes (\chi \circ det)$ for the representation of $\tilde{G}_{n\ell}$ defined by

$$g \to \pi(g)\chi(det.\mathbf{p}(g))$$

for $g \in \tilde{G}_{n\ell}$. Then $\pi \otimes (\chi \circ det)$ is also distinguished because

$$Wh(\pi) \cong Wh(\pi \otimes (\chi \circ det).$$

Propossition 3. *One has*

$$\tilde{\omega}(\pi \otimes (\chi \circ det)) = \tilde{\omega}(\pi)\chi^{n\ell}.$$

Here $\chi^{n\ell}$ is considered a quasicharacter of $\tilde{Z}$ by $\mathbf{p} : \tilde{Z} \to Z \cong F^\times$.

Proof. Clear.

4. From $\tilde{\omega}(\pi)$ to $\omega(\pi)$

If n is odd, then $\tilde{Z} = \mathbf{i}(\mu_n(F)) \cdot \underline{s}(Z)$ is canonically isomorphic to $\mu_n(F) \times F^\times$; hence $\tilde{\omega}(\pi)$ is expressed as

$$\epsilon \times \omega(\pi)$$

with a uniquely determined quasi-character $\omega(\pi)$ of $F^\times$. If n is even, then $\tilde{Z}$ is isomorphic to the fiber product

$$\mu_n(F) \times_{\mu_2(F)} \tilde{F}^\times$$

where $\tilde{F}^\times$ is the unique nontrivial extension

$$I \to \mu_2(F) \to \tilde{F}^\times \to F^\times \to I.$$

There exists a genuine character ψ of $\tilde{F}^\times$ derived from Weil representation (see [1]). Let $\tilde{\epsilon}$ be the character of $\tilde{Z}$ associated to ϵ and ψ. Then $\tilde{\omega}(\pi)$ is expressed as

$$\tilde{\epsilon} \times \omega(\pi)$$

with a uniquely determined quasi-character $\omega(\pi)$ of $F^\times$.

Thus,to any distinguished representation π of $\tilde{G}_n$,we can associate the quasi-character $\omega(\pi)$ of $F^\times$:

$$\pi \to \omega(\pi).$$

If χ is any quasicharacter of $F^\times$, then

$$\omega(\pi \otimes (\chi \circ det)) = \omega(\pi)\chi^n.$$

4. The theta representation

Each metaplectic group $\tilde{G}_r^{(c)}$ has an interesting ϵ-representation, which corresponds to the trivial representation of a reductive group over a local field. This representation , which we call the theta representation, is constructed as the image of an intertwining operator of reducible principal series representations (see [2]).

For the sake of simplicity, we assume that n is prime to the residual characteristic of F and that the center $Z(\tilde{G}_r^{(c)})$ is equal to $\tilde{Z}^n$. The integer ring R_F of F is maximal isotropic with respect to the n-th Hilbert symbol. The metaplectic cover $\mathbf{p} : \tilde{G}_r \to G_r = GL(r, F)$ splits over $K = GL(r, R_F)$; $\tilde{G}_r$ has a subgroup K^* which is isomorphic to K. If we denote by W the group of permutation matrices in K, we use the same symbol W for the corresponding subgroup of K^*. Let H_* be the subgroup of H defined by

$$\{h = diag(h_i) : h_i \in (F^\times)^n R_F^\times\}$$

where R_F is the integer rin of F. Then the subgroup

$$\tilde{H}_* = \mathbf{p}^{-1}(H_*) \subset \tilde{H}$$

is maximal abelian . For a quasi-character ω of $\tilde{H}_*$ such that $\omega_\theta \circ \mathbf{i} = \epsilon$ we define $V(\omega)$ to be the space of smooth functions f on $\tilde{G}_r^(c)$ satisfing

$$f(nhg) = \omega(h)\mu(\mathbf{p}(h))f(g) \quad \text{for} \quad h \in \tilde{H}_* n \in N$$

where μ is the modulus character i.e. $\mu(h) = \prod_{i<j} |h_i/h_j|_F^{1/2}$ for $h = diag(h_i) \in H$. Then we obtain an admissible representation $(\pi, V(\omega))$ by right translation. If $\eta \in \tilde{H}$,we define an element $\lambda_\eta \in Wh(V(\omega))$ by

$$< \lambda_\eta, f >= \mu(\mathbf{p}(\eta))^{-1} \int_N f(\eta w_r n)\bar{e}(n)dn \quad \text{for} \quad f \in V(\omega)$$

where w_r is the longest element in W. Then

$$\lambda_{h\eta} = \omega(h)\lambda_\eta \quad \text{for} \quad h \in \tilde{H}_*.$$

It is proved (see 2]) that the vecor space $Wh(V(\omega))$ over $\mathbb{C}$ has a basis

$$\{\lambda_\eta : \eta \in \tilde{H}_* \backslash \tilde{H}\}.$$

Let ω_θ(resp. ω'_θ) be a quasicharacter of $\tilde{H}_*$ defined by

$$\omega_\theta(\underline{s}(h)) = \prod_{i<j} |h_i/h_j|_F \quad \text{for} \quad h \in H_*$$

$$(\text{resp.} \quad \omega'_\theta(\underline{s}(h)) = \prod_{i<j} |h_i/h_j|_F^{-1} \quad \text{for} \quad h \in H_*)$$

Then we have an intertwining operater

$$I(w_r) : V(\omega_\theta) \to V(\omega'_\theta).$$

where

$$(I(w_r)f)(g) = \int_N f(w_r n)dn.$$

The theta representation $(\pi_\theta, V(\theta))$ is defined by

$$V(\theta) = Im(I(w_r) : V(\omega_\theta) \to V(\omega'_\theta)).$$

This operator induces a linear map

$$T_\theta : Wh(V(\omega'_\theta)) \to Wh(V(\omega_\theta))$$

whose matrix coefficients $\tau_r(\eta', \eta)$ for $\eta', \eta \in \tilde{H}_* \backslash \tilde{H}$ are defined by

$$T_\theta \lambda_{\eta'} = \sum_{\eta \in \tilde{H}_* \backslash \tilde{H}} \tau_r(\eta', \eta)\lambda_\eta.$$

The space $Wh(V(\theta))$ is identified with $Im(T_\theta)$; in particular

$$dim_{\mathbb{C}} Wh(V(\theta)) = rank.T_\theta.$$

Any element in $Wh(V(\theta))$ is written as

$$\sum_{\eta \in \tilde{H}_* \backslash \tilde{H}} \mathbf{c}(\eta)\lambda_\eta \qquad (\lambda_\eta \in Wh(V(\omega_\theta))$$

where $\mathbf{c}$ is a function $\tilde{H}$ satisfying

$$\mathbf{c}(h\eta) = \omega_\theta(h)^{-1}\mathbf{c}(\eta) \quad \text{for} \quad h \in \tilde{H}_*$$

and the following condition.
For $\mathbf{f} = (f_1, f_2, \cdots, f_r) \in \mathbb{Z}^r$ we set

$$\pi^{\mathbf{f}} = diag(\pi^{f_i}) \in H.$$

Let $\alpha = (i, i+1)$ with $1 \leq i < r$. We set $\mathbf{f}^\alpha = f_i - f_{i+1}$ and for $x \in F^\times$

$$h_\alpha(x) = \underline{s}(diag(1, \cdots, x, x^{-1}, \cdots, 1)) \in \tilde{H}$$

where x is in the i-th position. We define "Gauss sum" $g(\pi^\ell)$ for $1 \leq \ell < n$ by

$$g(\pi^\ell) = \int_{R_F^\times} \epsilon(\pi^\ell, x)_F e(\frac{x}{\pi})dx.$$

Then the condition on $\mathbf{c}$ is that

(1) if $\mathbf{f}^\alpha = n-1$, then $\mathbf{c}(\underline{s}(\pi^{\mathbf{f}})) = 0$,
(2) if $0 \leq \mathbf{f}^\alpha < n-1$, then

$$\mathbf{c}(h_\alpha(\pi^{-\mathbf{f}^\alpha - 1}) \cdot \underline{s}(\pi^{\mathbf{f}})) = g(\pi^{\mathbf{f}_\alpha})\mathbf{c}(\underline{s}(\pi^{\mathbf{f}})).$$

Proposition 4. *If $r = n$, then* $\mathbf{c}$ *is determined up to constant multiple by the above conditions, hence*
$$dim_{\mathbb{C}} Wh(V(\theta)) = 1.$$
In particular we have
$$\mathbf{c}(\underline{s}(\pi^{(1,1,\cdots,1)})) = \begin{cases} \mathbf{c}(\underline{s}(I)) & \text{if } n \text{ is odd,} \\ \epsilon(\pi,-1)_4 \frac{g(\pi^2)}{|g(\pi^2)|}\mathbf{c}(\underline{s}(I)) & \text{if } n \text{ is even.} \end{cases}$$

Proof. The first part is one of main results of [2] (see [2]). We prove the second part. In $\tilde{H}$ we have
$$\underline{s}(\pi^{(-1,-1,\cdots,-1)}) =$$
$$\underline{s}(\pi^{(0,\cdots,0,-n)})h_{(n-1,n)}(\pi^{-n+1})\cdots h_{(23)}(\pi^{-2})h_{(12)}(\pi^{-1}).$$
Therefore
$$\begin{aligned}\mathbf{c}(\underline{s}(\pi^{(-1,-1,\cdots,-1)})) &= \omega_\theta(\underline{s}(\pi^{(0,\cdots,0,-n)}))g(\pi^{n-1})\cdots g(\pi^2)g(\pi)\mathbf{c}(\underline{s}(I)) \\ &= q_F^{\frac{n-1}{2}} g(\pi^{n-1})\cdots g(\pi^2)g(\pi)\mathbf{c}(\underline{s}(I))\end{aligned}$$
Since $g(\pi^\ell)g(\pi^{n-\ell}) = \epsilon(\pi^\ell,-1)_F q_F^{-1}$, the assertion follows.

5. Automorphic distinguished representations

Proof of Theorem. Let V_A be a representation space of π_A consisting of automorphic functions on
$$G^*_{\ell n}\backslash\tilde{G}_{\ell n}(A).$$
Let e_A be a non-trivial character of the adelic unipotent group $N(A)$ which is trivial on the subgroup $N(k)$ of rational points. The metaplectic group $\tilde{G}_{\ell n}(A)$ has a subgroup which is canonically isomorphic to $N(A)$; we denote it by $N(A)$ too. If $\phi \in V_A$, then we set
$$\Lambda(\phi) = \int_{N(A)/N(k)} \phi(n)\bar{e}_A(n)dn.$$
Then Λ is a Whittaker functional on V_A with respect to e_A.
It is easy to see that
$$\Lambda(\pi_A(x)\phi) = \Lambda(\phi) \quad \text{for} \quad x \in \tilde{Z}_A \cap G^*_{\ell n}(k).$$
On the other hand, since
$$(\pi_A, V_A) \cong \otimes_v(\pi_v, V_v),$$
we have
$$\Lambda(\pi_A(y)\phi) = \tilde{\omega}(\pi_A)(y)\Lambda(\phi) \quad \text{for} \quad y \in \tilde{Z}_A.$$
Thus $\tilde{\omega}(\pi_A)$ is trivial on $\tilde{Z}_A \cap G^*_{\ell n}(k)$.
If n is odd, then the quasicharacter $\omega(\pi_A)$ on $A^\times$ is trivial on the subgroup $k^\times$. If n is even, then the same conclusion holds by the automorphic property of the Weil representation.

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