

琉球大学学術リポジトリ

On Quasicharacters Associated to Distinguished Metaplectic Forms

メタデータ	言語: 出版者: Department of Mathematical Sciences, Faculty of Science, University of the Ryukyus 公開日: 2007-03-03 キーワード (Ja): キーワード (En): 作成者: Suzuki, Toshiaki メールアドレス: 所属:
URL	http://hdl.handle.net/20.500.12000/84

ON QUASICHARACTERS ASSOCIATED TO
DISTINGUISHED METAPLECTIC FORMS

TOSHIAKI SUZUKI

1. Introduction

Let F be a non-archimedean local field such that $Card.\mu_n(F) = n$ where

$$\mu_n(F) = \{x \in F : x^n = 1\}.$$

Let

$$(\ , \)_F : F^\times \times F^\times \rightarrow \mu_n(F)$$

be the n -th order Hilbert symbol of F . The metaplectic group $\tilde{G}_r^{(c)}$ (see [2]) is a central extension of the general linear group $G_r = GL(r, F)$ over F by $\mu_n(F)$ ($c \in \mathbb{Z}/n\mathbb{Z}$):

$$I \rightarrow \mu_n(F) \xrightarrow{i} \tilde{G}_r^{(c)} \xrightarrow{p} G_r \rightarrow I.$$

Here we choose a section $\underline{g} : G_r \rightarrow \tilde{G}_r^{(c)}$ of p so that \underline{g} is homomorphism on the upper-triangular unipotent subgroup N and the $\mu_n(F)$ -valued 2-cocycle σ on G_r , which arise from \underline{g} , is given on the diagonal subgroup H of G_r by

$$\sigma(h, h') = \prod_{i < j} (h_i, h'_j)_F \cdot (\det.h, \det.h')_F^c$$

for $h = \text{diag}(h_i), h' = \text{diag}(h'_i) \in H$ with $h_i, h'_i \in F^\times$ ($1 \leq i \leq r$).

We fix an injective character $\epsilon : \mu_n(F) \rightarrow \mathbb{C}^\times$ once for all.

Let us consider an admissible irreducible ϵ -representation $\tilde{\pi}$ of \tilde{G}_r , on a complex vector space V , of which the restriction to $i(\mu_n(F))$ coincides to $\epsilon \cdot I_V$ where I_V is the identity operator of V .

If e is a non-degenerate character of the upper-triangular unipotent subgroup N of G_r , we define $Wh(\tilde{\pi})$ by

$$\{\lambda \in V' : \langle \lambda, \tilde{\pi}(n)f \rangle = e(n) \langle \lambda, f \rangle \text{ for } n \in N, f \in V\}$$

where V' is the algebraic dual space of V with a bilinear form $\langle \ , \ \rangle$. We call $\tilde{\pi}$ distinguished if $\dim_{\mathbb{C}} Wh(\tilde{\pi}) = 1$.

Received November 30 2006.

Let Z denote the center of G_r and set

$$\tilde{Z} = \mathfrak{p}^{-1}(Z).$$

We find that \tilde{Z} plays an interesting role in the following sense. First \tilde{Z} is isomorphic to a central extension of F^\times by $\mu_n(F)$. On the other hand, for any admissible irreducible representation $\tilde{\pi}$ of \tilde{G}_r , \tilde{Z} acts naturally on $Wh(\tilde{\pi})$ where $\mathfrak{i}(\mu_n(F))$ acts by ϵ . We shall prove in the next section.

Proposition 1. *If \tilde{G}_r has a distinguished ϵ -representation, then \tilde{Z} is abelian.*

Proposition 2. *The group \tilde{Z} is abelian if and only if*

$$r(r-1+2cr) \equiv 0(n).$$

Let us see the relation between \tilde{Z} and the center $Z(\tilde{G}_r)$ of \tilde{G}_r . For any divisor m of n , set

$$\tilde{Z}^m = \mathfrak{p}^{-1}\{z^m : z \in Z\}.$$

We recall that the center $Z(\tilde{G}_r)$ is given by $\tilde{Z}^{n'}$ where $n' = n/(n, r-1+2cr)$ (see [2]);

$$\tilde{Z}^n \subset Z(\tilde{G}_r) \subset \tilde{Z}.$$

Thus the center $Z(\tilde{G}_r)$ is equal to \tilde{Z} if and only if $r-1+2cr \equiv 0(n)$. This is the case where $Z(\tilde{G}_r)$ is the biggest. If the center of \tilde{G}_r is the smallest, i.e. that is given by \tilde{Z}^n , and if \tilde{Z} is abelian, then $r \equiv 0(n)$. Conversely if $r \equiv 0(n)$, then \tilde{Z} is abelian and the center of \tilde{G}_r is given by \tilde{Z}^n .

If $\tilde{\pi}$ is a distinguished ϵ -representation of \tilde{G}_r , we obtain a one-dimensional representation $\tilde{\omega}(\tilde{\pi})$ of \tilde{Z} by acting \tilde{Z} on $Wh(\tilde{\pi})$. The restriction of $\tilde{\omega}(\tilde{\pi})$ to the center $Z(\tilde{G}_r)$ is nothing but the central quasicharacter of $\tilde{\pi}$. When $r \equiv 0(n)$, the quasicharacter $\tilde{\omega}(\tilde{\pi})$ contains information about $\tilde{\pi}$, more than its central quasicharacter, which is rather mysterious.

From $\tilde{\omega}(\tilde{\pi})$ we can easily move to a quasicharacter $\omega(\tilde{\pi})$ of F^\times . Then we obtain a mapping

$$\tilde{\pi} \rightarrow \omega(\tilde{\pi})$$

from the set of the isomorphism classes of distinguished ϵ -representations of \tilde{G}_r to the set of quasicharacters of F^\times .

When $r = n$ in which case is the most interesting, this mapping seems to be the inverse of the conjectural correspondence

$$\omega \rightarrow \tilde{\pi}(\omega)$$

from the set of quasicharacters of F^\times to the set of the isomorphism classes of distinguished ϵ -representations of \tilde{G}_n . We can calculate that in the following cases.

- (1) If ω is a quasicharacter of F^\times such that $\omega(\mu_n(F)) = 1$, we can construct a distinguished ϵ -representation $\tilde{\pi}(\omega)$ as an image of an intertwining operator

of certain principal series representations of \tilde{G}_n such that $\omega(\tilde{\pi}(\omega)) = \omega$ (see [2]).

- (2) If n is prime to the residual characteristic of F and if ω is a quasicharacter of F^\times of which restriction to $\mu_n(F)$ is injective, then we can construct a supercuspidal distinguished ϵ -representation $\tilde{\pi}(\omega)$ of \tilde{G}_n inducing from a modulo center compact subgroup such that $\omega(\tilde{\pi}(\omega)) = \omega$ (see [1]).

The advantage of our mapping is that the automorphic property is almost trivial. Let k be a global field containing a primitive root of unity and A the ring of its adeles. Let $\tilde{G}_r(A)$ be the global metaplectic group which is a central extension of $G_r(A) = GL(r, A)$ by $\mu_n(k)$ and contains a discrete subgroup $G_r^*(k)$ isomorphic to $G_r(k) = GL(r, k)$. Let $\epsilon : \mu_n(k) \rightarrow C^\times$ be an injective character. We consider irreducible admissible ϵ -representations π_A of $\tilde{G}_r(A)$ of which the restriction to the central subgroup isomorphic to $\mu_n(k)$ is given by ϵ . Then we have a "tensor decomposition"

$$\pi_A \cong \otimes_v \pi_v$$

in the sense of [2] where v runs over all places of k and π_v is an irreducible admissible ϵ -representation of $\tilde{G}_r(k_v)$. We call π_A distinguished if all π_v are distinguished, and automorphic if it has a realization on the space of automorphic functions on $\tilde{G}_r(A)$ with respect to $G_r^*(k)$. If π_A is distinguished, we define

$$\omega(\pi_A) = \prod_v \omega(\pi_v)$$

as a quasicharacter of $A^\times = \prod_v k_v^\times$.

In the last section we prove

Theorem. *If a distinguished ϵ -representation π_A of $\tilde{G}_{nt}(A)$ is automorphic, then $\omega(\pi_A)$ is also automorphic.*

2. The subgroup \tilde{Z}

Proof of Proposition 1. Let $\tilde{\pi}$ be a distinguished representation of \tilde{G}_r . Then by definition $\dim_{\mathbb{C}} Wh(\tilde{\pi}) = 1$. We obtain an irreducible 1-dimensional \tilde{Z} -module $Wh(\tilde{\pi})$. Here the central subgroup $\mathfrak{i}(\mu_n(F))$ of \tilde{Z} acts on $Wh(\tilde{\pi})$ by ϵ . If \tilde{Z} is not abelian, one has $\dim_{\mathbb{C}} Wh(\tilde{\pi}) > 1$. Therefore \tilde{Z} is abelian.

Proof of Proposition 2. For $\tilde{h}, \tilde{h}' \in \tilde{H} = \mathfrak{p}^{-1}(H)$ with $\mathfrak{p}(\tilde{h}) = \text{diag}(h_i)$ and $\mathfrak{p}(\tilde{h}') = \text{diag}(h'_i)$ we have

$$\tilde{h}\tilde{h}'\tilde{h}^{-1}\tilde{h}'^{-1} = \mathfrak{i}\left(\prod_{i=1}^r (h_i, h'_i)_F^{-1} \cdot (\det.p(\tilde{h}), \det.p(\tilde{h}'))_F^{1+2c}\right).$$

If $\tilde{z}, \tilde{z}' \in \tilde{Z}$, then $\mathfrak{p}(\tilde{z}) = z \cdot I$ and $\mathfrak{p}(\tilde{z}') = z' \cdot I$ with $z, z' \in F^\times$. Hence

$$\begin{aligned} \tilde{z}\tilde{z}'\tilde{z}^{-1}\tilde{z}'^{-1} &= \mathfrak{i}\left((z, z')_F^{-r} \cdot (z^r, z'^r)_F^{1+2c}\right) \\ &= \mathfrak{i}\left((z, z')_F^{-r+r^2(1+2c)}\right) \\ &= \mathfrak{i}\left((z, z')_F^{r(r-1+2cr)}\right). \end{aligned}$$

This proves the proposition.

3. The quasi-character $\tilde{\omega}(\pi)$

Let (π, V) be a distinguished representation of $\tilde{G}_{n\ell}$. Then \tilde{Z} is abelian and $Wh(\pi)$ is a 1-dimensional \tilde{Z} -module. Hence there exists a quasi-character $\tilde{\omega}(\pi)$ of the abelian group \tilde{Z} such that

$$\lambda(\pi(x)f) = (\tilde{\omega}(\pi))(x)\lambda(f)$$

for all $\lambda \in Wh(\pi)$, $f \in V$ and $x \in \tilde{Z}$. For a quasi-character χ of F^\times , we write $\pi \otimes (\chi \circ \det)$ for the representation of $\tilde{G}_{n\ell}$ defined by

$$g \rightarrow \pi(g)\chi(\det.\mathbf{p}(g))$$

for $g \in \tilde{G}_{n\ell}$. Then $\pi \otimes (\chi \circ \det)$ is also distinguished because

$$Wh(\pi) \cong Wh(\pi \otimes (\chi \circ \det)).$$

Proposition 3. *One has*

$$\tilde{\omega}(\pi \otimes (\chi \circ \det)) = \tilde{\omega}(\pi)\chi^{n\ell}.$$

Here $\chi^{n\ell}$ is considered a quasicharacter of \tilde{Z} by $\mathbf{p} : \tilde{Z} \rightarrow Z \cong F^\times$.

Proof. Clear.

4. From $\tilde{\omega}(\pi)$ to $\omega(\pi)$

If n is odd, then $\tilde{Z} = \mathbf{i}(\mu_n(F)) \cdot \underline{\mathfrak{z}}(Z)$ is canonically isomorphic to $\mu_n(F) \times F^\times$; hence $\tilde{\omega}(\pi)$ is expressed as

$$\epsilon \times \omega(\pi)$$

with a uniquely determined quasi-character $\omega(\pi)$ of F^\times . If n is even, then \tilde{Z} is isomorphic to the fiber product

$$\mu_n(F) \times_{\mu_2(F)} \tilde{F}^\times$$

where \tilde{F}^\times is the unique nontrivial extension

$$I \rightarrow \mu_2(F) \rightarrow \tilde{F}^\times \rightarrow F^\times \rightarrow I.$$

There exists a genuine character ψ of \tilde{F}^\times derived from Weil representation (see [1]). Let $\tilde{\epsilon}$ be the character of \tilde{Z} associated to ϵ and ψ . Then $\tilde{\omega}(\pi)$ is expressed as

$$\tilde{\epsilon} \times \omega(\pi)$$

with a uniquely determined quasi-character $\omega(\pi)$ of F^\times .

Thus, to any distinguished representation π of \tilde{G}_n , we can associate the quasi-character $\omega(\pi)$ of F^\times :

$$\pi \rightarrow \omega(\pi).$$

If χ is any quasicharacter of F^\times , then

$$\omega(\pi \otimes (\chi \circ \det)) = \omega(\pi)\chi^n.$$

4. The theta representation

Each metaplectic group $\tilde{G}_r^{(c)}$ has an interesting ϵ -representation; which corresponds to the trivial representation of a reductive group over a local field. This representation, which we call the theta representation, is constructed as the image of an intertwining operator of reducible principal series representations (see [2]).

For the sake of simplicity, we assume that n is prime to the residual characteristic of F and that the center $Z(\tilde{G}_r^{(c)})$ is equal to \tilde{Z}^n . The integer ring R_F of F is maximal isotropic with respect to the n -th Hilbert symbol. The metaplectic cover $\mathbf{p} : \tilde{G}_r \rightarrow G_r = GL(r, F)$ splits over $K = GL(r, R_F)$; \tilde{G}_r has a subgroup K^* which is isomorphic to K . If we denote by W the group of permutation matrices in K , we use the same symbol W for the corresponding subgroup of K^* . Let H_* be the subgroup of H defined by

$$\{h = \text{diag}(h_i) : h_i \in (F^\times)^n R_F^\times\}$$

where R_F is the integer ring of F . Then the subgroup

$$\tilde{H}_* = \mathbf{p}^{-1}(H_*) \subset \tilde{H}$$

is maximal abelian. For a quasi-character ω of \tilde{H}_* such that $\omega_\theta \circ \mathbf{i} = \epsilon$ we define $V(\omega)$ to be the space of smooth functions f on $\tilde{G}_r^{(c)}$ satisfying

$$f(nhg) = \omega(h)\mu(\mathbf{p}(h))f(g) \quad \text{for } h \in \tilde{H}_*, n \in N$$

where μ is the modulus character i.e. $\mu(h) = \prod_{i < j} |h_i/h_j|_F^{1/2}$ for $h = \text{diag}(h_i) \in H$. Then we obtain an admissible representation $(\pi, V(\omega))$ by right translation. If $\eta \in \tilde{H}$, we define an element $\lambda_\eta \in Wh(V(\omega))$ by

$$\langle \lambda_\eta, f \rangle = \mu(\mathbf{p}(\eta))^{-1} \int_N f(\eta w_r n) \bar{e}(n) dn \quad \text{for } f \in V(\omega)$$

where w_r is the longest element in W . Then

$$\lambda_{h\eta} = \omega(h)\lambda_\eta \quad \text{for } h \in \tilde{H}_*.$$

It is proved (see 2]) that the vector space $Wh(V(\omega))$ over \mathbb{C} has a basis

$$\{\lambda_\eta : \eta \in \tilde{H}_* \setminus \tilde{H}\}.$$

Let ω_θ (resp. ω'_θ) be a quasicharacter of \tilde{H}_* defined by

$$\begin{aligned} \omega_\theta(\underline{s}(h)) &= \prod_{i < j} |h_i/h_j|_F \quad \text{for } h \in H_* \\ (\text{resp. } \omega'_\theta(\underline{s}(h)) &= \prod_{i < j} |h_i/h_j|_F^{-1} \quad \text{for } h \in H_*) \end{aligned}$$

Then we have an intertwining operator

$$I(w_r) : V(\omega_\theta) \rightarrow V(\omega'_\theta).$$

where

$$(I(w_r)f)(g) = \int_N f(w_r n) dn.$$

The theta representation $(\pi_\theta, V(\theta))$ is defined by

$$V(\theta) = \text{Im}(I(w_r) : V(\omega_\theta) \rightarrow V(\omega'_\theta)).$$

This operator induces a linear map

$$T_\theta : Wh(V(\omega'_\theta)) \rightarrow Wh(V(\omega_\theta))$$

whose matrix coefficients $\tau_r(\eta', \eta)$ for $\eta', \eta \in \tilde{H}_* \backslash \tilde{H}$ are defined by

$$T_\theta \lambda_{\eta'} = \sum_{\eta \in \tilde{H}_* \backslash \tilde{H}} \tau_r(\eta', \eta) \lambda_\eta.$$

The space $Wh(V(\theta))$ is identified with $\text{Im}(T_\theta)$; in particular

$$\dim_{\mathbb{C}} Wh(V(\theta)) = \text{rank } T_\theta.$$

Any element in $Wh(V(\theta))$ is written as

$$\sum_{\eta \in \tilde{H}_* \backslash \tilde{H}} \mathbf{c}(\eta) \lambda_\eta \quad (\lambda_\eta \in Wh(V(\omega_\theta)))$$

where \mathbf{c} is a function \tilde{H} satisfying

$$\mathbf{c}(h\eta) = \omega_\theta(h)^{-1} \mathbf{c}(\eta) \quad \text{for } h \in \tilde{H}_*$$

and the following condition.

For $\mathbf{f} = (f_1, f_2, \dots, f_r) \in \mathbb{Z}^r$ we set

$$\pi^{\mathbf{f}} = \text{diag}(\pi^{f_i}) \in H.$$

Let $\alpha = (i, i+1)$ with $1 \leq i < r$. We set $\mathbf{f}^\alpha = f_i - f_{i+1}$ and for $x \in F^\times$

$$h_\alpha(x) = \underline{g}(\text{diag}(1, \dots, x, x^{-1}, \dots, 1)) \in \tilde{H}$$

where x is in the i -th position. We define "Gauss sum" $g(\pi^\ell)$ for $1 \leq \ell < n$ by

$$g(\pi^\ell) = \int_{R_F^\times} \epsilon(\pi^\ell, x)_F e\left(\frac{x}{\pi}\right) dx.$$

Then the condition on \mathbf{c} is that

- (1) if $\mathbf{f}^\alpha = n - 1$, then $\mathbf{c}(\underline{g}(\pi^{\mathbf{f}})) = 0$,
- (2) if $0 \leq \mathbf{f}^\alpha < n - 1$, then

$$\mathbf{c}(h_\alpha(\pi^{-\mathbf{f}^\alpha - 1}) \cdot \underline{g}(\pi^{\mathbf{f}})) = g(\pi^{\mathbf{f}^\alpha}) \mathbf{c}(\underline{g}(\pi^{\mathbf{f}})).$$

Proposition 4. *If $r = n$, then \mathbf{c} is determined up to constant multiple by the above conditions, hence*

$$\dim_{\mathbb{C}} \text{Wh}(V(\theta)) = 1.$$

In particular we have

$$\mathbf{c}(\underline{g}(\pi^{(1,1,\dots,1)})) = \begin{cases} \mathbf{c}(\underline{g}(I)) & \text{if } n \text{ is odd,} \\ \epsilon(\pi, -1)_4 \frac{g(\pi^2)}{|g(\pi^2)|} \mathbf{c}(\underline{g}(I)) & \text{if } n \text{ is even.} \end{cases}$$

Proof. The first part is one of main results of [2] (see [2]). We prove the second part. In \tilde{H} we have

$$\begin{aligned} \underline{g}(\pi^{(-1,-1,\dots,-1)}) &= \\ \underline{g}(\pi^{(0,\dots,0,-n)}) h_{(n-1,n)}(\pi^{-n+1}) \cdots h_{(23)}(\pi^{-2}) h_{(12)}(\pi^{-1}). \end{aligned}$$

Therefore

$$\begin{aligned} \mathbf{c}(\underline{g}(\pi^{(-1,-1,\dots,-1)})) &= \omega_{\theta}(\underline{g}(\pi^{(0,\dots,0,-n)})) g(\pi^{n-1}) \cdots g(\pi^2) g(\pi) \mathbf{c}(\underline{g}(I)) \\ &= q_F^{\frac{n-1}{2}} g(\pi^{n-1}) \cdots g(\pi^2) g(\pi) \mathbf{c}(\underline{g}(I)) \end{aligned}$$

Since $g(\pi^{\ell})g(\pi^{n-\ell}) = \epsilon(\pi^{\ell}, -1)_F q_F^{-1}$, the assertion follows.

5. Automorphic distinguished representations

Proof of Theorem. Let V_A be a representation space of π_A consisting of automorphic functions on

$$G_{\ell n}^* \backslash \tilde{G}_{\ell n}(A).$$

Let e_A be a non-trivial character of the adelic unipotent group $N(A)$ which is trivial on the subgroup $N(k)$ of rational points. The metaplectic group $\tilde{G}_{\ell n}(A)$ has a subgroup which is canonically isomorphic to $N(A)$; we denote it by $N(A)$ too. If $\phi \in V_A$, then we set

$$\Lambda(\phi) = \int_{N(A)/N(k)} \phi(n) \bar{e}_A(n) dn.$$

Then Λ is a Whittaker functional on V_A with respect to e_A .

It is easy to see that

$$\Lambda(\pi_A(x)\phi) = \Lambda(\phi) \quad \text{for } x \in \tilde{Z}_A \cap G_{\ell n}^*(k).$$

On the other hand, since

$$(\pi_A, V_A) \cong \otimes_v (\pi_v, V_v),$$

we have

$$\Lambda(\pi_A(y)\phi) = \tilde{\omega}(\pi_A)(y) \Lambda(\phi) \quad \text{for } y \in \tilde{Z}_A.$$

Thus $\tilde{\omega}(\pi_A)$ is trivial on $\tilde{Z}_A \cap G_{\ell n}^*(k)$.

If n is odd, then the quasicharacter $\omega(\pi_A)$ on A^\times is trivial on the subgroup k^\times .

If n is even, then the same conclusion holds by the automorphic property of the Weil representation.

REFERENCES

1. S.Gelbart and I.I.Piatetski-Shapiro, *Distinguished Representations and Modular Forms of Half-integral Weight*, *Inventiones math.* **59** (1980), 145–188.
2. D.Kazhdan and S.Patterson, *Metaplectic Forms*, *Publ. Math. IHES* **59** (1984), 35–142.
3. T.Suzuki, *On Supercuspidal Representations of a Metaplectic Group*, *Ryukyu Math. J.* **17** (2004), 75–87.

Department of Mathematical Sciences
Faculty of Science
University of the Ryukyus
Nishihara-cho, Okinawa 903-0213
JAPAN