Gauss Sums Related to the Hilbert Symbol

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GAUSS SUMS RELATED TO THE HILBERT SYMBOL

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1. Introduction

Let F be a finite extension of $\mathbb{Q}_p$ such that $Card.\mu_n(F) = n$ where

$$\mu_n(F) = \{x \in F : x^n = 1\}.$$

Let $(\ ,\ )_F : F^\times \times F^\times \to \mu_n(F)$ be the n-th order Hilbert symbol of F. In [S2] we showed (under certain conditions) the existence of a maximal isotropic subring R_F of F with respect to $(\ ,\ )_F$;

$$\{x \in F^\times : (x, y)_F = 1 \quad \text{for all } y \in R_F^\times\} = (F^\times)^n R_F^\times.$$

In this paper we consider "Gauss Sums" on R_F.

Let $|\ |_F$ be a normalized valuation on F and π be a uniformizer of F. For $x \in F^\times$ define $ord_F x \in \mathbb{Z}$ by

$$|x|_F = |\pi|_F^{ord_F x}.$$

If we write $n = n_0 p^m$ where $(n_0, p) = 1$, then the ramification index $ord_F p$ of F is given by

$$e = e_1 p^{m-1}(p-1).$$

For the sake of simplicity we assume that

(1) $p \neq 2$,

(2) e_1 is odd.

We choose and fix an injective character ϵ of $\mu_n(F)$ and a non-trivial character e_F of F (which we specify in (2.2)) ; then we define

$$\Gamma(\xi, x) = \int_{R_F^\times} \epsilon(\xi, y)_F e_F(xy) dy$$

for $\xi, x \in F^\times$. Here dy is a Haar measure on F so that

$$\int_{R_F} dy = 1.$$

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Let P_F be the maximal ideal of the integer ring O_F and set $q_F = Card.O_F/P_F$. Let δ the integer so that

$$1+P_F^{\delta+1} \subset (1+P_F)^n \quad \text{and} \quad 1+P_F^{\delta} \not\subset (1+P_F)^n.$$

We see that $\delta = e_1 p^m + e(m-1)$; this number relates to the structure of R_F. The maximal ideal I_F of R_F contains $P_F^{\frac{\delta+1}{2}}$. We have

$$R_F^\times = M \cdot (1+I_F)$$

where $M = \mu_{q_F-1}(F)$ the group of $q_F - 1$-th roots of unity in O_F; and

$$[F^\times : (F^\times)^n R_F^\times] = n[P_F : I_F] = nq_F^{e/2}.$$

Let $\hat{R}_F, \hat{I}_F$ and $(P_F^{\delta+1})\hat{}$ be the dual lattice of R_F, I_F and $P_F^{\delta+1}$ in F with respect to e_F.
It is easy to see that

(1) $\Gamma(\xi\xi', xx') = \epsilon(\xi, x')^{-1}\Gamma(\xi, x)$ for $\xi' \in (F^\times)^n R_F^\times$ and $x' \in R_F^\times$.
(2) If $x \notin (P_F^{\delta+1})\hat{}$, then $\Gamma(\xi, x) = 0$ for any $\xi \in F^\times$.
(3) If $x \in \hat{R}_F$, then $\Gamma(\xi, x) = 0$ unless $\xi \in (F^\times)^n R_F^\times$ in which case we have

$$\Gamma(\xi, x) = 1 - q_F^{-1}.$$

(4) If $x \in \hat{I}_F - \hat{R}_F$, then $\Gamma(\xi, x) = 0$ unless $\xi \in (F^\times)^{p^m} R_F^\times$ in which case we have

$$\Gamma(\xi, x) = \begin{cases} \sum_{\alpha \in M} \epsilon(\xi, \alpha)_F e_F(\alpha x) q_F^{-1} & \text{if } \xi \notin (F^\times)^n R_F^\times, \\ -q_F^{-1} & \text{if } \xi \in (F^\times)^n R_F^\times. \end{cases}$$

(5) If $x \in (P_F^{\delta+1})\hat{} - \hat{I}_F$ and $\xi \in (F^\times)^{p^m} R_F^\times$, then $\Gamma(\xi, x) = 0$.

Hence the calculation of $\Gamma(\xi, x)$ is reduced to the case where

$$\xi \notin (F^\times)^{p^m} R_F^\times.$$

For such ξ we can define an ideal $I_F(\xi)$ of R_F which satisfies

$$\epsilon(\xi, 1-t)_F = e_F(\frac{1}{\xi}\frac{d\xi}{d\pi}t) \text{ for } t \in I_F(\xi)$$

where $\frac{d\xi}{d\pi}$ is a formal derivative of ξ with respect to π .

Theorem 1. *Let $\xi \notin (F^\times)^{p^m} R_F^\times$. Then $\Gamma(\xi, x) \neq 0$ if and only if there exists $\alpha \in R_F^\times$ such that*

$$\xi^{-1}\frac{d\xi}{d\pi} \equiv \alpha x \mod \hat{I}_F(\xi)$$

in which case

$$\Gamma(\xi, x) = \epsilon(\xi, \alpha)_F e_F(\alpha x)|I_F(\xi)|.$$

Here $|I_F(\xi)| = Card.R_F/I_F(\xi)$.

The next problem, which is motivated from the theory of Whittaker models of metaplectic forms, is that: for a given $\nu \in F^\times$, determine the set

$$D(\nu) = \{\xi \in F^\times/R_F^\times : \Gamma(\xi, \xi^{-1}\nu) \neq 0, \xi^{-1}\nu \notin \hat{R}_F\}.$$

Note that $D(\nu)$ essentially depends only the class $\nu(F^\times)^n R_F^\times$. We can classify elements $\nu \in F^\times/(F^\times)^n R_F^\times$ into three types : singular, p-singular and regular ones. Then we obtain the followings.

(1) If ν is singular, $D(\nu)$ consists a single element $\xi \in (F^\times)^n R_F^\times$ such that

$$\Gamma(\xi, \xi^{-1}\nu) = -q_F^{-1}.$$

(2) If ν is p-singular, $D(\nu)$ consists a single element

$$\xi \in (F^\times)^{p^m} R_F^\times - (F^\times)^n R_F^\times$$

such that $|\Gamma(\xi, \xi^{-1}\nu)|^2 = q_F^{-1}$.

Theorem 2 describes the structure of $D(\nu)$ for all other regular elements ν. As a collorary we obtain that if ν is not singular

$$\sum_{\xi \in D(\nu)} |\Gamma(\xi, \frac{\nu}{\xi})|^2 = q_F^{-1}.$$

2. The ring R_F

Let F_T be the maximal unramified extension in F and O_T be its integer ring. Then O_F is identified with the ring $O_T[[\pi]]$ which consists of power series

$$\sum_{i=0}^{\infty} a_i \pi^i$$

where $a_i \in O_T$ (or $a_i \in M$). The subring $O_T[[\pi^{p^\ell}]]$ $(1 \leq \ell \leq m)$ also plays a role. We define

$$\begin{aligned} I_F(0) &= \{x \in O_F : \delta/2 < ord_F x\}, \\ I_F(\ell) &= \{x \in O_T[[\pi^{p^\ell}]] : \frac{\delta - \ell e}{2} < ord_F x\} + I_F(\ell - 1) \\ &\qquad (1 \leq \ell \leq m-1) \quad \text{and} \\ I_F(m) &= \{x \in O_T[[\pi^{p^m}]] : 0 < ord_F x\} + I_F(m-1). \end{aligned}$$

We set

(2 1) $$R_F = O_T + I_F$$

with $I_F = I_F(m)$. Then R_F is a local ring with a maximal ideal I_F such that

$$R_F/I_F \cong O_F/P_F.$$

We note that the mapping

$$t \to \epsilon(\pi, 1-t)_F$$

defines a character on $I_F(0)$. We take e_F so that

$$\epsilon(\pi, 1-t)_F = e_F(\frac{t}{\pi}) \quad \text{for all} \quad t \in I_F(0). \tag{2 2}$$

By the definition of δ we see that

$$e_F(P_F^{\delta}) = 1 \ \text{ and } \ e_F(P_F^{\delta-1}) \neq 1.$$

If A is a O_T-lattice of F, define the dual lattice $\hat{A}$ by

$$\hat{A} = \{x \in F : e_F(xy) = 1 \quad \text{for} \quad y \in A\}. \tag{2 3}$$

Proposition 1.

(1) $\hat{I}_F(0) = \pi^{-1} I_F(0)$.

(2) $\hat{I}_F(\ell-1) = O_T < ip^{\ell}\pi^{ip^{\ell}-1}; \frac{\delta-\ell e}{2} < ip^{\ell} < \frac{\delta-\ell e+e}{2}, (i,p) = 1 > +\hat{I}_F(\ell)$ $(1 \le \ell \le m-1)$.

(3) $\hat{I}_F(m-1) = O_T < \pi^{\delta-ip^m-1}; 0 < ip^m < \frac{\delta-(m-1)e}{2}, (i,p) = 1 > +\hat{I}_F$.

(4) $\hat{I}_F = O_T < \pi^{\delta-1} > +\hat{R}_F$.

Here $O_T < \pi^{\alpha} >$ denote the O_T-submodule generated by π^{α}.

Proof. Since $e_F(P_F^{\delta}) = 1$ and $e_F(P_F^{\delta-1}) \neq 1$ we have

$$(P_F^{\delta})\hat{} = O_F.$$

Hence

$$\begin{aligned} \hat{I}_F(0) &= (P_F^{\frac{\delta+1}{2}})\hat{} \\ &= P_F^{\frac{\delta-1}{2}} (P_F^{\delta})\hat{} \\ &= \pi^{-1} I_F(0). \end{aligned}$$

By definition we have

$$I_F(\ell) = O_T < \pi^{ip^{\ell}}; \frac{\delta-\ell e}{2} < ip^{\ell} < \frac{\delta-\ell e+e}{2}, (i,p) = 1 > +I_F(\ell-1).$$

Since

$$e_F(\pi^{\delta-ip^{\ell}-1} \cdot I_F(\ell-1)) = 1 \quad \text{and} \quad e_F(\pi^{\delta-ip^{\ell}-1} \cdot \pi^{ip^{\ell}}) \neq 1$$

we have $\pi^{\delta-ip^{\ell}-1} \in \hat{I}_F(\ell-1) - \hat{I}_F(\ell)$. Considering the dimensions of $I_F(\ell)/I_F(\ell-1)$ and $\hat{I}_F(\ell-1)/\hat{I}_F(\ell)$ over $O_T/(p)$, we obtain the result.

Corollary.

(1) *If* $t_1, t_2 \in I_F(\ell)$, *then* $e_F(\frac{p^\ell}{\pi} t_1 t_2) = 1$.

(2) *If* $t_1 \in \vee_T[[\pi^{p^\ell}]]$ *and* $e_F(\frac{p^\ell}{\pi} t_1 t) = 1$ *for all* $t \in I_F(\ell)$, *then* $t_1 \in I_F(\ell)$.

Proof. If $t_1 \in I_F(\ell)$, then $\frac{p^\ell}{\pi} t_1 \in \hat{I}_F(\ell)$ by Proposition 1. Hence

$$e_F(\frac{p^\ell}{\pi} t_1 t_2) = 1 \quad \text{for all} \quad t_2 \in I_F(\ell).$$

Conversely, if $e_F(\frac{p^\ell}{\pi} t_1 t) = 1$ for all $t \in I_F(\ell)$, then

$$\frac{p^\ell}{\pi} t_1 \in \hat{I}_F(\ell).$$

Since $t_1 \in O_T[[\pi^{p^\ell}]]$ we have $t_1 \in I_F(\ell)$.

3. The ideal $I_F(\xi)$

Lemma 1. *Let* i *be an integer prime to* p *and* $0 \le \ell \le m-1$.

(1) *If* $t \in I_F(\ell)$, *then*

$$\epsilon(\pi^{ip^\ell}, 1-t)_F = e_F(\frac{ip^\ell}{\pi} t).$$

(2) *If* $t\pi^{ip^\ell} \in I_F(\ell)$ *and* $t \in I_F$, *then*

$$\epsilon(1-\pi^{ip^\ell}, 1-t)_F = e_F(\frac{-ip^\ell \pi^{ip^\ell - 1}}{1-\pi^{ip^\ell}} t).$$

Proof. By (1.2) we have

$$\epsilon(\pi^i, 1-t)_F = e_F(\frac{i}{\pi} t) \quad \text{for} t \in I_F(0).$$

We prove (1) by the induction on ℓ. If $\ell = 0$, we have done. Assume that if $t \in I_F(\ell - 1)$, then

$$\epsilon(\pi^{ip^{\ell-1}}, 1-t)_F = e_F(\frac{ip^{\ell-1}}{\pi} t).$$

If $t \in I_F(\ell) - I_F(\ell-1)$, then

$$pt \in I_F(\ell-1) \quad \text{and} \quad e_F(\frac{p^\ell}{\pi} t^2) = 1$$

by Corollary to Proposition 1 ; hence

$$\begin{aligned}
\epsilon(\pi^{ip^\ell}, 1-t)_F &= \epsilon(\pi^{ip^{\ell-1}}, (1-t)^p)_F \\
&= \epsilon(\pi^{ip^{\ell-1}}, 1-pt+\binom{p}{2}t^2+\cdots)_F \\
&= e_F(\frac{ip^{\ell-1}}{\pi}(pt-\binom{p}{2}t^2+\cdots)) \\
&= e_F(\frac{ip^\ell}{\pi}t)
\end{aligned}$$

To prove (2) we need the important property of the Hilbert symbol

$$(1-x, 1-y)_F = (1-x, 1-xy)_F(x, 1-xy)_F(1-xy, 1-y)_F.$$

In fact we have

$$\begin{aligned}
\epsilon(1-\pi^{ip^\ell}, 1-t)_F &= \epsilon(1-\pi^{ip^\ell}, 1-\pi^{ip^\ell}t)_F\epsilon(\pi^{ip^\ell}, 1-\pi^{ip^\ell}t)_F \\
&\quad \cdot \epsilon(1-\pi^{ip^\ell}t, 1-t)_F \\
&= \epsilon(1-\pi^{ip^\ell}, 1-\pi^{ip^\ell}t)_F e_F(-ip^\ell\pi^{ip^\ell-1}t)
\end{aligned}$$

because $(1+I_F(\ell), 1+I_F)_F = 1$. Replacing t by $\pi^{ip^\ell}t$ we obtain

$$\epsilon(1-\pi^{ip^\ell}, 1-\pi^{ip^\ell}t)_F = \epsilon(1-\pi^{ip^\ell}, 1-\pi^{2ip^\ell}t)_F e_F(-ip^\ell\pi^{ip^\ell-1}\pi^{ip^\ell}t).$$

Thus we have

$$\begin{aligned}
\epsilon(1-\pi^{ip^\ell}, 1-t)_F &= \epsilon(1-\pi^{ip^\ell}, 1-\pi^{ip^\ell}t)_F e_F(-ip^\ell\pi^{ip^\ell-1}t) \\
&= \epsilon(1-\pi^{ip^\ell}, 1-\pi^{2ip^\ell}t)_F e_F(-ip^\ell\pi^{ip^\ell-1}(1+\pi^{ip^\ell})t) \\
&= \cdots \\
&= e_F((-ip^\ell\pi^{ip^\ell-1}(1+\pi^{ip^\ell}+\pi^{2ip^\ell}+\cdots)t) \\
&= e_F(\frac{-ip^\ell\pi^\ell-1}{1-\pi^{ip^\ell}}t).
\end{aligned}$$

Now let us define $I_F(\xi)$ for $\xi \in F^\times$ with $0 \leq ord_F\xi < p^m$ and $\xi \notin R_F^\times$. We write

$$\xi = \xi_0 + \xi_1 + \cdots + \xi_m$$

with

$$\xi_\ell = \sum_{(i,p)=1} a_{ip^\ell}\pi^{ip^\ell} \quad (0 \leq \ell \leq m)$$

where ($a_{ip^\ell} \in M \cup \{0\}$). We call ξ to be of type $(i'p^{k'}, ip^k)$ if

$$(3\ 1) \qquad ord_F \xi = i'p^{k'},$$

$$(3\ 2) \qquad \min_{0\le \ell \le m-1} ord_F \pi^{\frac{\ell e}{2}} \xi_\ell = \frac{ke}{2} + ip^k.$$

We see in this case that

$$ord_F \xi_\ell > ip^k + \frac{ke - \ell e}{2} \quad \text{for } \ell < k;$$

and $\xi I_F(k) \subset I_F(k)$.

If ξ is of type $(i'p^{k'}, ip^k)$, we define

$$(3\ 3) \qquad I_F(\xi) = I_F \cap \pi^{-ip^k + i'p^{k'}} I_F(k).$$

Lemma 2. *If $ord_F \xi = 0$, then*

$$I_F(\xi) = I_F \cap \xi I_F;$$

therefore

$$I_F(\xi^{-1}) = I_F(\xi) \quad \textit{and} \quad \xi I_F(\xi) = I_F(\xi).$$

Proof. If ξ is of type $(0, ip^k)$, then by (3.3)

$$I_F(\xi) = I_F \cap \pi^{-ip^k} I_F(k).$$

If $t \in I_F(k)$, then $\xi^{-1} t \in I_F(k) \subset I_F$ because ξ^{-1} is also of type $(0, ip^k)$; hence $I_F(k) \subset I_F \cap \xi I_F$. For $t \in I_F - I_F(k)$ with $t \in O_T[[\pi^{p^k}]]$, we have

$$\xi t \in I_F \quad \text{if and only if} \quad \pi^{ip^k} t \in I_F(k).$$

Hence the assertion follows.

Proposition 2. *Let ξ be of type $(i'p^{k'}, ip^k)$. If $t \in I_F(\xi)$, then*

$$\epsilon(\xi, 1-t)_F = e_F\left(\frac{1}{\xi}\frac{d\xi}{d\pi} t\right).$$

Furthermore, if $t_1 \in R_F$ satisfies

$$e_F\left(\frac{1}{\xi}\frac{d\xi}{d\pi} t_1 t\right) = 1 \quad \textit{for all} \quad t \in I_F(k),$$

then $t_1 \in I_F(\xi)$.

Proof. We may assume that ξ is of type $(i'p^{k'}, ip^k)$. We write

$$\xi = a\pi^{i'p^{k'}} \prod (1 - a_{j,\ell}\pi^{jp^\ell})$$

If $t \in I_F(k')$, then

$$\epsilon(\pi^{i'p^{k'}}, 1-t)_F = e_F(\frac{i'p^{k'}\pi^{i'p^{k'}-1}}{\pi^{i'p^{k'}}}t)$$

; if $\pi^{jp^\ell}t \in I_F(\ell)$ and $t \in I_F$), then

$$\epsilon(1 - a_{j,\ell}\pi^{jp^\ell}, 1-t)_F = e_F(\frac{-a_{j,\ell}jp^\ell\pi^{jp^\ell-1}}{1 - a_{j,\ell}\pi^{jp^\ell}}t).$$

By definition we have

$$I_F(\xi) = I_F(k') \cap \bigcap_{j,\ell} \pi^{-jp^\ell} I_F(\ell).$$

Thus we obtain that if $t \in I_F(\xi)$ then

$$\begin{aligned}\epsilon(\xi, 1-t)_F &= e_F(\frac{i'p^{k'}\pi^{i'p^{k'}-1}}{\pi^{i'p^{k'}}}t + \sum_{j,\ell} \frac{-a_{j,\ell}jp^\ell\pi^{jp^\ell-1}}{1 - a_{j,\ell}\pi^{jp^\ell}}t) \\ &= e_F(\frac{1}{\xi}\frac{d\xi}{d\pi}t)\end{aligned}$$

Let us prove the latter part. We assume that $t_1 \in R_F - I_F(k)$;hence $t_1 \in O_T[[\pi^{p^k}]]$. By definition we have

$$\frac{1}{\xi}\frac{d\xi}{d\pi}t_1 \in \hat{I}_F(k).$$

Since ξ is of $(i'p^{k'}, ip^k)$ we obtain

$$p^k \frac{\pi^{ip^k - i'p^{k'} - 1}}{1 - \pi^{ip^k - i'p^{k'}}} t_1 \in \hat{I}_F(k) \quad \text{i.e.} \quad \pi^{ip^k - i'p^{k'}} t_1 \in I_F(k).$$

Hence $t_1 \in I_F \cap \pi^{-ip^k + i'p^{k'}} I_F(k)$.

Now we can state the first result.

Theorem 1. *Let ξ be as above. For any $x \in F^\times$, one has*

$$\Gamma(\xi, x) \neq 0$$

if and only if there exists $\alpha \in R_F^\times$ such that

$$\frac{1}{\xi}\frac{d\xi}{d\pi} \equiv \alpha x \mod \hat{I}_F(\xi)$$

in which case we have

$$\Gamma(\xi,x)=\epsilon(\xi,\alpha)_F e_F(\alpha x)|I_F(\xi)|.$$

Proof. We have

$$\begin{aligned}\Gamma(\xi,x)&=\int_{R_F^\times}\epsilon(\xi,y)_F e_F(xy)dy\\ &=\sum_{\alpha\in R_F^\times/(1+I_F(\xi))}\int_{I_F(\xi)}\epsilon(\xi,\alpha(1-t))_F e_F(\alpha x(1-t))dt\\ &=\sum_{\alpha\in R_F^\times/(1+I_F(\xi))}\epsilon(\xi,\alpha)e_F(\alpha x)\int_{I_F(\xi)}\epsilon(\xi,1-t)_F e_F(-\alpha xt)dt\\ &=\sum_{\alpha\in R_F^\times/(1+I_F(\xi))}\epsilon(\xi,\alpha)e_F(\alpha x)\int_{I_F(\xi)}e_F((\frac{1}{\xi}\frac{d\xi}{d\pi}-\alpha x)t)dt\end{aligned}$$

It is enough to see that there exists at most one such $\alpha \mod (1+I_F(\xi))$ that

$$\frac{1}{\xi}\frac{d\xi}{d\pi}\equiv\alpha x \mod \hat{I}_F(\xi).$$

If α and α' satisfy the above relation, then

$$(\alpha^{-1}-\alpha'^{-1})\frac{1}{\xi}\frac{d\xi}{d\pi}\equiv 0 \mod \hat{I}_F(k).$$

Then by Proposition 2 we have

$$\alpha^{-1}-\alpha'^{-1}\in I_F(\xi).$$

4. The set $D(\nu)$

Recall that for $\nu\in F^\times$ we have defined

$$D(\nu)=\{\xi\in F^\times/R_F^\times:\Gamma(\xi,\xi^{-1}\nu)\neq 0,\xi^{-1}\nu\notin\hat{R}_F\}. \tag{4 1}$$

Lemma 3. *If $x\in\hat{I}_F-\hat{R}_F$, then $\Gamma(\xi,x)=0$ unless $\xi\in(F^\times)^{p^m}R_F^\times$ in which case we have*

$$\Gamma(\xi,x)=\begin{cases}\sum_{\alpha\in M}\epsilon(\xi,\alpha)_F e_F(\alpha x)q_F^{-1} & \textit{if}\quad \xi\notin(F^\times)^n R_F^\times\\ -q_F^{-1} & \textit{if}\quad \xi\in(F^\times)^n R_F^\times\end{cases}$$

Proof. If $x\in\hat{I}_F-\hat{R}_F$, then

$$e_F(xI_F)=1 \quad\text{and}\quad e_F(xR_F)\neq 1.$$

Since $M \cup \{0\}$ is a representative set of R_F/I_F, we have

$$\sum_{\alpha \in M \cup \{0\}} e_F(\alpha x) = 0;$$

therefore

$$\sum_{\alpha \in M} e_F(\alpha x) = -1.$$

Then

$$\begin{aligned}\Gamma(\xi, x) &= \sum_{\alpha \in M} \epsilon(\xi, \alpha)_F e_F(\alpha x) \int_{I_F} \epsilon(\xi, 1+t)_F e_F(\alpha t x) dt \\ &= \sum_{\alpha \in M} \epsilon(\xi, \alpha)_F e_F(\alpha x) \int_{I_F} \epsilon(\xi, 1+t)_F dt.\end{aligned}$$

If $\xi \notin (F^\times)^{p^m} R_F^\times$, then

$$\int_{I_F} \epsilon(\xi, 1+t)_F dt = 0 \quad \text{and} \quad \Gamma(\xi, x) = 0.$$

If $\xi \in (F^\times)^{p^m} R_F^\times$, then

$$\begin{aligned}\Gamma(\xi, x) &= \sum_{\alpha \in M} \epsilon(\xi, \alpha)_F e_F(\alpha x) |I_F| \\ &= \begin{cases} \sum_{\alpha \in M} \epsilon(\xi, \alpha)_F e_F(\alpha x) q_F^{-1} & \text{if} \quad \xi \notin (F^\times)^n R_F^\times \\ -q_F^{-1} & \text{if} \quad \xi \in (F^\times)^n R_F^\times. \end{cases}\end{aligned}$$

Lemma 4. *Let ξ be of type $(i'p^{k'}, ip^k)$. Then*

$$\Gamma(\xi, \frac{\nu}{\xi}) \neq 0$$

if

$$\frac{d\xi}{d\pi} \equiv \nu \mod \pi^{ip^k} \hat{I}_F(k).$$

Convsersely, if

$$\Gamma(\xi, \frac{\nu}{\xi}) \neq 0$$

then there exists $\alpha \in R_F^\times$ such that

$$\frac{d\xi}{d\pi} \equiv \alpha\nu \mod \pi^{i'p^{k'}} \hat{I}_F + \pi^{ip^k} \hat{I}_F(k).$$

Proof. Since ξ is of type $(i'p^{k'}, ip^k)$, we have

$$\xi \pi^{-i'p^{k'}} I_F(k) = I_F(k).$$

If

$$\frac{d\xi}{d\pi} \equiv \nu \mod \pi^{ip^k} \hat{I}_F(k)$$

then

$$\frac{1}{\xi}\frac{d\xi}{d\pi} \equiv \frac{1}{\xi}\nu \mod \pi^{-ip^k+\widehat{i'p^{k'}}} I_F(k).$$

Thus

$$\frac{1}{\xi}\frac{d\xi}{d\pi} \equiv \frac{1}{\xi}\nu \mod \hat{I}_F(\xi)$$

because $\pi^{-ip^k+\widehat{i'p^{k'}}} I_F(k) \subset \hat{I}_F(\xi)$.

Let prove the latter part. By Theorem 1 there exists $\alpha \in R_F^\times$ such that

$$\frac{1}{\xi}\frac{d\xi}{d\pi} \equiv \alpha\frac{1}{\xi}\nu \mod \hat{I}_F(\xi).$$

Then

$$\frac{d\xi}{d\pi} \equiv \alpha\nu \mod \hat{\xi}^{-1} I_F(\xi).$$

If $i'p^{k'} \neq ip^k$ then $\xi_0 = \xi\pi^{-i'p^{k'}}$ is of type $(0, ip^k - i'p^{k'})$,and $I_F(\xi) = I_F(\xi_0)$; by Lemma 2

$$\begin{aligned}\xi^{-1} I_F(\xi) &= \pi^{-i'p^{k'}} \xi_0^{-1} I_F(\xi_0)\\ &= \pi^{-i'p^{k'}} I_F(\xi_0)\\ &= \pi^{-i'p^{k'}} I_F \cap \pi^{-ip^k} I_F(k)).\end{aligned}$$

If $i'p^{k'} = ip^k$, then $I_F(\xi) = I_F(k)$ and

$$\xi^{-1} I_F(\xi) = \pi^{-ip^k} I_F(k).$$

Thus the assertion follows.

Proposition 3.

(1) *If $\nu \in \hat{I}_F - \hat{R}_F$, then $D(\nu) = \{1\}$.*

(2) *If $\nu \in \hat{I}_F(m) - \hat{I}_F$, then $D(\nu) = \{\pi^{ip^m}\}$ for some $1 \leq i < n_0$.*

Proof. Let $\nu \in \hat{I}_F - \hat{R}_F$. It follows from Lemma 2 that $1 \in D(\nu)$. If $\nu \in \hat{I}_F(m) - \hat{R}_F$,then there is no $\xi \notin (F^\times)^{p^m} R_F^\times$ such that

$$\frac{d\xi}{d\pi} \equiv \nu \mod \pi^{ip^k} \hat{I}_F(k)$$

because ν contains a non-vanishing term

$$a\pi^{\delta - ip^m - 1} \quad (a \in M, \quad 0 \leq ip^m < \frac{\delta - (m-1)e}{2}).$$

Recall that

$$\begin{aligned}\hat{I}_F(m-1) =& O_T < \pi^{\delta - ip^m - 1}; 0 < ip^m < \frac{\delta - (m-1)e}{2} >\\ &+ O_T < \pi^{\delta - 1} > + \hat{R}_F.\end{aligned}$$

Thus ,if $\nu \in \hat{I}_F(m) - \hat{I}_F$, then $\pi^{ip^m}\nu \in \hat{I}_F - \hat{R}_F$.

Definition 2.

(1) We call ν *sinular* if there exists an integer i such that $\nu\pi^{ni} \in \hat{I}_F - \hat{R}_F$.
(2) We call ν *p-singular* if ν is not singular and if there exists an integer j such that $\nu\pi^{p^m j}$ is singular.
(3) We call any other element in $F^\times$ *regular*.

Example 1. If $F = \mathbb{Q}_p(1^{1/p})$ and $n = p$, then $\delta = p$;

$$R_F = \mathbb{Z}_p + I_F(0), \quad I_F(0) = P_F^{\frac{p+1}{2}};$$

and

$$\begin{aligned}\hat{I}_F &= P_F^{\frac{p-1}{2}} \\ &= \mathbb{Z}_p\pi^{p-1} + \hat{R}_F.\end{aligned}$$

All the elements ν in $F^\times$ such that $\nu \notin \hat{R}_F$ and $\pi^p\nu \in \hat{R}_F$ fall into the following three class :

$$\begin{aligned}&[1] \quad 0 \le ord_F\nu \le \frac{p-3}{2} \\ &[2] \quad \frac{-p-1}{2} \le ord_F\nu \le -2, \quad e_F(\pi^p\nu) = 1 \\ &[3] \quad \frac{p-1}{2} \le ord_F\nu \le p-2, \quad e_F(\nu) \neq 1.\end{aligned}$$

The elements in [1] and [2] are regular; the elements in [3] are singular.

Example 2. If $F = \mathbb{Q}_p(1^{\frac{1}{p^2}})$ and $n = p^2$, then $\delta = 2p^2 - p$;

$$R_F = \mathbb{Z}_p + \mathbb{Z}_p\pi^{\frac{p+1}{2}p} + \cdots + \mathbb{Z}_p\pi^{(p-1)p} + I_F(0), \quad I_F(0) = P_F^{p^2-\frac{p-1}{2}};$$

and

$$\begin{aligned}\hat{I}_F(0) &= P_F^{p^2-\frac{p+1}{2}} \\ &= \mathbb{Z}_p\pi^{2p^2-p-1} + \mathbb{Z}_p\pi^{p^2-1} + \cdots + \mathbb{Z}_p\pi^{\frac{3}{2}p(p-1)-1} + \hat{R}_F.\end{aligned}$$

The regular elements ν in $F^\times$ such that $\nu \notin \hat{R}_F$ and $\pi^p\nu \in \hat{R}_F$ are classified as follows.

$$\begin{aligned}&[1] \quad 0 \le ord_F\nu \le p^2 - \frac{p+1}{2}, \quad (ord_F\nu + 1, p) = 1 \\ &[2] \quad \frac{-p-1}{2} \le ord_F\nu \le -2, \\ &[3] \quad \frac{p^2+p}{2} - 1 \le ord_F\nu \le p(p-1) - 1, \quad p|(ord_F\nu + 1) \\ &[4] \quad p^2 - 1 \le ord_F\nu \le \frac{3}{2}p(p-1) - 1, \quad p|(ord_F\nu + 1).\end{aligned}$$

Let ν be a regular element in $F^\times$ such that $\nu \notin \hat{R}_F$ and $\pi^{p^m}\nu \in \hat{R}_F$. Then $\nu \notin \hat{I}_F(m)$ because ν is regular. We can write

$$\nu = \sum_{\ell}^{m} \nu_\ell$$

with

$$\nu_\ell = \sum_{(i,p)=1} ip^\ell a_{i,\ell}\pi^{ip^{ell}-1} \quad (0 \leq \ell < m)$$
$$\nu_m = \sum_i a_{i,m}\pi^{\delta - ip^m - 1}$$

where $a_{i,\ell} \in M \cup \{0\}$. Since $\nu \notin \hat{I}_F(m)$ we have $\delta < ord_F\nu_m$. If we put

$$ord_F\nu_\ell = e\ell + i_\ell p^\ell - 1 \quad (0 \leq \ell < m),$$

then

$$\frac{\delta + \ell e}{2} - p^m < \ell e + i_\ell p^\ell < \frac{\delta + \ell e}{2}.$$

Since $D(\nu)$ depends only on $\nu R_F^\times$, it is enough to consider $\nu(1 + I_F)$; hence we may assume that $\nu_m = 0$,i.e.

$$\nu = \sum_{\ell=0}^{m-1} \nu_\ell.$$

If $i_\ell p^\ell > 0$ for all ℓ $(0 \leq \ell \leq m-1)$, we call ν *positive.* In the case where $e_1 = 1$, there exist non-positive elements ν such that $\nu \notin \hat{R}_F$ and $\pi^{p^m}\nu \in \hat{R}_F$ (see Example 1 and 2).

Now we construct an element $\xi \notin (F^\times)^{p^m} R_F^\times$ such that

$$\frac{d\xi}{d\pi} \equiv \nu \mod \pi^{ip^k}\hat{I}_F(k).$$

Here k is the number $0 \leq k < m$ so that

$$\min_{0 \leq \ell < m} ord_F(\pi^{-\ell e/2}\nu_\ell) = \frac{ke}{2} + i_k p^k - 1.$$

From the expansion of ν_ℓ we remove the terms of ν_ℓ which are contained in $\pi^{i_k p^k}\hat{I}_F(k)$; then

$$\tilde{\nu}_\ell = \sum_{(i,p)=1} ip^\ell a_{i,\ell}\pi^{ip^\ell - 1} \quad (p^\ell \pi^{ip^\ell - 1} \notin \pi^{i_k p^k}\hat{I}_F(k)).$$

We set

$$\xi_\ell = \sum_{(i,p)=1} a_{i,\ell}\pi^{ip^\ell}$$

where the summation is taken over the same range for $\tilde{\nu}_\ell$. Then the order ip^ℓ of terms which appear in the expansion of ξ_ℓ satisfies

$$\frac{\delta - \ell e}{2} - p^m < ip^\ell < \begin{cases} i_k p^k + \frac{\delta + ke}{2} - \ell e & \text{if } \ell \geq k, \\ i_k p^k + \frac{\delta - \ell e}{2} & \text{if} \ell < k. \end{cases}$$

If we set $\xi = \sum_{\ell=0}^{m-1} \xi_\ell$, then ξ is not in $I_F(m)$ and of type $(i_{k'}p^{k'}, i_k p^k)$ where

$$\begin{aligned} ord_F \xi &= i_{k'}p^{k'} \ (k' \geq k) \\ i_k p^k - i_{k'}p^{k'} &< \frac{e}{2}(k' - k). \end{aligned}$$

By construction it is clear that $\xi \in D(\nu)$.

By Lemma 4, every other element ξ' in $D(\nu)$ is of type $(ip^\ell, i_k p^k)$ with $i_k p^k - \frac{\delta - \ell e}{2} < ip^\ell \leq i_{k'}p^{k'}$ and

$$\frac{d\xi'}{d\pi} \equiv \frac{d\xi}{d\pi} \mod \pi^{i\ell} \hat{I}_F + \pi^{i_k p^k} \hat{I}_F(k).$$

Since

$$\xi' - \xi \in \pi^{i\ell} I_F + \pi^{i_k p^k} I_F(k),$$

we can choose a reprsentative ξ' in $\xi(1 + I_F)$;

$$\xi' = \xi + \sum_{\ell'=k+1}^{m} \sum_{(i',p)=1} a_{i',\ell'} \pi^{ip^\ell}$$

where $p^\ell \pi^{ip^\ell} \in \pi^{i_k p^k} \hat{I}_F(k)$,i.e.$jp^\ell + \ell e > i_k p^k + \frac{\delta + ke}{2}$.

We write $D_1(\nu)$ $(resp.D_0(\nu))$ for the subset of $D(\nu)$ consisting of the elements ξ' with $ord_F \xi' = i_{k'}p^{k'}$ (resp. $ord_F \xi' < i_{k'}p^{k'}$) and D_{0,ip^ℓ} the subset of $D(\nu)$ consisting of the elements ξ' with $ord_F \xi' = ip^\ell < i_{k'}p^{k'}$.

We define

$$\begin{aligned} \Delta_1(\xi) = \{jp^\ell :&(j,p) = 1, k < \ell < m, jp^\ell + \ell e > i_k p^k + \frac{\delta + ke}{2}, \\ &i_{k'}p^{k'} < jp^\ell < i_k p^k + \frac{\delta - ke}{2} - 2(i_k p^k - i_{k'}p^{k'})\} \\ &\cup \{jp^m : i_{k'}p^{k'} < jp^m < i_k p^k + \frac{\delta - ke}{2} - 2(i_k p^k - i_{k'}p^{k'})\} \\ \Delta_0(\xi) = \{jp^\ell :&(j,p) = 1, k < \ell < m, \\ &0 < jp^\ell < i_{k'}p^{k'}, jp^\ell + \ell e > i_k p^k + \frac{\delta + ke}{2}\} \\ &\cup \{jp^m : 0 \leq jp^m < i_{k'}p^{k'}\} \end{aligned}$$

Then we see that

$$\Delta_1(\xi) \cup \Delta_0(\xi) = \{jp^{\ell} :(j,p) = 1, k < \ell < m, jp^{\ell} + \ell e > i_k p^k + \frac{\delta - ke}{2},$$
$$0 < jp^{\ell} < i_k p^k + \frac{\delta - ke}{2} - 2(i_k p^k - i_{k'} p^{k'})\}$$
$$\cup\{jp^m :0 < jp^m < i_k p^k + \frac{\delta - ke}{2} - 2(i_k p^k - i_{k'} p^{k'})\};$$

and

$$q_F^{|\Delta_1(\xi)|+|\Delta_0(\xi)|+1} = |I_F(\xi)|^2 \tag{4.2}$$

where $|\Delta_i(\xi)|$ denotes the cardinality of $\Delta_i(\xi)$.

If $\xi' \in D_{0,ip^{\ell}}$, we define

$$\Delta_1(\xi') = \{jp^{\ell'} :(j,p) = 1, k < \ell' < m, jp^{\ell'} + \ell' e > i_k p^k + \frac{\delta + ke}{2}$$
$$ip^{\ell} < jp^{\ell'} < i_k p^k + \frac{\delta - ke}{2} - 2(i_k p^k - ip^{\ell})\}$$
$$\cup\{jp^m : ip^{\ell} < jp^{\ell'} < i_k p^k + \frac{\delta - ke}{2} - 2(i_k p^k - ip^{\ell})\}$$
$$\Delta_0(\xi') = \{jp^{\ell'} :(j,p) = 1, k < \ell' < m,$$
$$0 < jp^{\ell'} < ip^{\ell}, jp^{\ell'} + \ell' e > i_k p^k + \frac{\delta + ke}{2}\}$$
$$\cup\{jp^m : 0 \le jp^m < ip^{\ell}\}.$$

Then we see that

$$q_F^{|\Delta_1(\xi')|+|\Delta_0(\xi)|+1} = |I_F(\xi')|^2. \tag{4 3}$$

Theorem 2. *Let ν be a regular element in $F^{\times}$ such that $\nu \notin \hat{R}_F$ and $\pi^{p^m}\nu \in \hat{R}_F$. With the notations as above, one has*

$$D(\nu) = D_1(\nu) \cup D_2(\nu)$$

where

$$D_1(\nu) = \{\xi + \sum_{jp^{\ell} \in \Delta_1(\xi)} a_{jp^{\ell}} \pi^{jp^{\ell}} : a_{jp^{\ell}} \in M \cup \{0\}\ \}$$

and

$$D_0(\nu) = \bigsqcup_{jp^{\ell} \in \Delta_0(\xi)} D_{0,jp^{\ell}}(\nu)$$
$$D_{0,jp^{\ell}}(\nu) = \{a_{jp^{\ell}} \pi^{jp^{\ell}} + \xi + \sum_{j'p^{\ell'} \in \Delta_1(a_{jp^{\ell}} \pi^{jp^{\ell}} + \xi)} a_{j'p^{\ell'}} \pi^{j'p^{\ell'}} :$$
$$a_{jp^{\ell}} \in M \quad \text{and} \quad a_{j'p^{\ell'}} \in M \cup \{0\}\}.$$

Corollary. *If ν is not singular, then*

$$\sum_{\xi\in D(\nu)} |\Gamma(\xi, \frac{\nu}{\xi})|^2 = q_F^{-1}.$$

Proof. It is clear when ν is p-singular. We prove that when ν is regular. We assume that $\nu \notin \hat{R}_F$ and $\pi^{p^m}\nu \in \hat{R}_F$. Then by Theorem 2 we have

$$\begin{aligned}\sum_{\xi'\in D(\nu)} |\Gamma(\xi', \frac{\nu}{\xi'})|^2 &= \sum_{\xi'\in D_1(\nu)} |\Gamma(\xi', \frac{\nu}{\xi'})|^2 + \sum_{\xi'\in D_0(\nu)} |\Gamma(\xi', \frac{\nu}{\xi'})|^2 \\ &= \sum_{\xi'\in D_1(\nu)} |I_F(\xi)|^{-2} + \sum_{ip^\ell\in\Delta_0(\xi)} \sum_{\xi'\in D_{0,ip^\ell}} |I_F(\pi^{ip^\ell}+\xi)|^{-2}.\end{aligned}$$

By (4.2) we have

$$\begin{aligned}\sum_{\xi'\in D_1(\nu)} |I_F(\xi)|^{-2} &= |D_1(\nu)||I_F(\xi)|^{-2} \\ &= q_F^{\Delta_1(\xi)} q_F^{-\Delta_1(\xi)-\Delta_0(\xi)-1} \\ &= q_F^{-\Delta_0(\xi)-1}.\end{aligned}$$

Similarly by (4.3) we have

$$\begin{aligned}\sum_{\xi'\in D_{0,ip^\ell}} |I_F(\pi^{ip^\ell}+\xi)|^{-2} &= |D_{0,ip^\ell}||I_F(\pi^{ip^\ell}+\xi)|^{-2} \\ &=(q_F-1)q_F^{\Delta_1(\pi^{ip^\ell}+\xi)} q_F^{-\Delta_1(\pi^{ip^\ell}+\xi)-\Delta_0(\pi^{ip^\ell}+\xi)-1} \\ &=q_F^{-\Delta_0(\pi^{ip^\ell}+\xi)}(1-q_F^{-1}).\end{aligned}$$

Finally we have

$$\begin{aligned}\sum_{\xi'\in D(\nu)} |\Gamma(\xi', \frac{\nu}{\xi'})|^2 &=q_F^{-\Delta_0(\xi)-1} + \sum_{ip^\ell\in\Delta_0(\xi)} q_F^{-\Delta_0(\pi^{ip^\ell}+\xi)}(1-q_F^{-1}) \\ &=q_F^{-1}.\end{aligned}$$

Appendix

A1. The metaplectic group

So far we have studied "Gauss Sums" over a maximal isotropic ring R_F with respect to the Hilbert symbol of the ground field F. This result is one of preparatory work to develop a comprehensive local theory of metaplectic forms. In [KP] Kazhdan and Patterson invent a very beatiful formulation (local and global) for metaplectic forms. Unfortunately they describe their local theory only for the case where the metaplectic degree n is co-prime to the residual characteristic of F. In that case R_F coincides with the integer ring of F. Therefore we need a local theory of metaplectic forms, which is based on R_F. In this appendix we give some statements in that theory, which follow immediately from our results.

Let us recall several definitions related to the local metaplectic group (see [KP]). Let

$$\begin{aligned} G =& GL(2,F) \\ H =& \{h = diag(h_1,h_2) \in G : h_1,h_2 \in F^\times\} \\ N =& \{n = \begin{pmatrix} 1 & n_{12} \\ 0 & 1 \end{pmatrix} \in G : n_{12} \in F\} \\ N_- =& \{n = \begin{pmatrix} 1 & 0 \\ n_{21} & 1 \end{pmatrix} \in G : n_{21} \in F\} \\ W =& \{I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\} \\ K_0 =& GL(2, O_F) \\ K =& GL(2, R_F) \end{aligned}$$

For $g_1, g_2 \in G$ we set

$$\sigma(g_1,g_2) = \left(\frac{\chi(g_1g_2)}{\chi(g_1)}, \frac{\chi(g_1g_2)}{\chi(g_2)}\right)_F \left(det(g_1), \frac{\chi(g_1g_2)}{\chi(g_1)}\right)_F$$

where

$$\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c = 0. \end{cases}$$

We see that

(A 1)

$$\sigma(h,h') = (h_1,h_2')_F \quad \text{for } h = diag(h_1,h_2),\ h' = diag(h_1',h_2') \in H$$

(A 2)

$$\sigma(ng, g'n') = \sigma(g,g') \quad \text{for } n,\ n' \in N$$

We define the metaplectic group

$$\tilde{G} = \{(g,\zeta) : g \in G,\ \zeta \in \mu_n(F)\}$$

with multiplicaion law

$$(g,\zeta)(g',\zeta') = (gg', \sigma(g,g')\zeta\zeta');$$

and related mappings

$$\begin{aligned} \mathbf{i} :& \mu_n(F) \to \tilde{G} \quad \text{by} \quad \mathbf{i}(\zeta) = (I,\zeta) \\ \mathbf{p} :& \tilde{G} \to G \quad \text{by} \quad \mathbf{p}(g,\zeta) = g \\ \mathbf{s} :& G \to \tilde{G} \quad \text{by} \quad \mathbf{s}(g) = (g,1) \\ h_{12} :& F^\times \to \tilde{G} \quad \text{by} \quad h_{12}(x) = (diag(x,x^{-1}),1) \end{aligned}$$

By (A,2) , the restriction of $\mathbf{s}$ to N is a homomorphism ; we write N for the subgroup $\mathbf{s}(N)$ of $\tilde{G}$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0$ set

$$\kappa(g) = \begin{cases} (c, d/det(g))_F & \text{if } 0 < |c|_F < 1 \\ 1 & \text{if } |c|_F = 0, 1 \end{cases}$$

Then if $g_1, g_2 \in K$ we obtain

$$\sigma(g_1, g_2) = \frac{\kappa(g_1 g_2)}{\kappa(g_1)\kappa(g_2)};$$

if we define

$$\kappa* : K \to \tilde{G} \quad \text{by} \quad \kappa * (g) = (g, \kappa(g))$$

then $\kappa*$ is a homomorphism. We write $K*$ for the subgroup $\kappa * (K)$ of $\tilde{G}$. Since $W subset K$, we write also W for the subgroup $\kappa(W)$ of $K*$; we call $s \in W$ *the simple reflection.* By (A.1) the center of $\tilde{H} = \mathbf{p}^{-1}(H)$ is equal to

$$\tilde{H}^n = \mathbf{p}^{-1}\{h^n : h \in H\}$$

and a maximal abelian subgroup of its is given by

$$\tilde{H}^n(\tilde{H} \cap K*)$$

which we denote by $\tilde{H}*$. We see that if $h \in \tilde{H}*$ then

$$h^s = s^{-1}hs \in \tilde{H} * .$$

Recall the injective character $\epsilon : \mu_n(F) \to \mathbb{C}^\times$. Let ω be a quasi-character of $\tilde{H}_*$ such that $\omega \circ \mathbf{i} = \epsilon$. Let $V(\omega)$ be a space of smooth functions $f : \tilde{G} \to \mathbb{C}$ satisfying

$$f(nhg) = (\omega\mu)(h)f(g) \quad \text{for all } n \in N \text{ and all } h \in \tilde{H} * .$$

Here $\mu : \tilde{H} \to \mathbb{C}^\times$ is defined by

$$\mu(h) = |h_1/h_2|_F \quad \text{where} \quad \mathbf{p}(h) = diag(h_1, h_2).$$

Acting $\tilde{G}$ on $V(\omega)$ by right translation, we obtain an admissible representation $(\pi, V(\omega))$. We assume that ω is trivial on $\tilde{H} \cap K*$. Then the subspace $V(\omega)_{K*}$ consisting of $K*$-invariant functions has dimension one. Let $v(\omega)$ be the element in $V(\omega)_{K*}$ such that $v(\omega)(\mathbf{s}(I)) = 1$. We set

$$\omega_{12}(x) = \omega(h_{12}(x^n)) \quad \text{for } x \in F^\times$$

then ω_{12} is a unramified quasi-character of $F^\times$. We set

$$^s\omega(h) = \omega(h^s) \quad \text{for} \quad h \in \tilde{H}*$$

then ${}^s\omega$ is also a quasi-character of $\tilde{H}*$ satisfying the above two conditions. There exists an intertwining operator

$$I_s : V(\omega) \to V({}^s\omega)$$

which is defined by the integrations

$$(I_s f)(g) = \int_F f(ss\left(\begin{pmatrix} 1 & x \\ 1 & 0 \end{pmatrix}\right) g)dx \quad \text{for } f \in V(\omega)$$

and another technique (analytic continuation or regularization). Since $I_s v(\omega)$ is a $K*$-invariant element in $V({}^s\omega)$, there exists a constat c such that

$$I_s v(\omega) = cv({}^s\omega).$$

Evaluating each side at $g = \mathrm{s}(I)$ we obtain

$$c = \int_F v(\omega)(ss\left(\begin{pmatrix} 1 & x \\ 1 & 0 \end{pmatrix}\right))dx.$$

This integral can be easily caculated when $|\omega_{12}(\pi)| < 1$;thus we have

$$I_s v(\omega) = \frac{1 - q_F^{-1}\omega_{12}(\pi)}{1 - \omega_{12}(\pi)} v({}^s\omega).$$

We define a normalized intertwining operator $I'_s : V(\omega) \to V({}^s\omega)$ by

$$I'_s = \frac{1 - \omega_{12}(\pi)}{1 - q_F^{-1}\omega_{12}(\pi)} \cdot I_s.$$

We denote by $I"_s$ the normalized intertwining operator from $V({}^s\omega)$ to $V(\omega)$ obtained by the same way. Then

$$I"_s \circ I'_s = I_{V(\omega)}, \quad I'_s \circ I"_s = I_{V({}^s\omega)}$$

where $I_{V(\omega)}$ is the identity map of $V(\omega)$ (see Theorem I.2.6 [KP]). In other words, we have

Proposition A1. *The composition*

$$V(\omega) \xrightarrow{I_s} V({}^s\omega) \xrightarrow{I_s} V(\omega)$$

is equal to

$$\frac{1 - q_F^{-1}\omega_{12}(\pi)}{1 - \omega_{12}(\pi)} \frac{1 - q_F^{-1}\omega_{12}(\pi)^{-1}}{1 - \omega_{12}(\pi)^{-1}} I_{V(\omega)}.$$

A2. The τ-functions

Recall the character e_F of F. We define a non-degenerate character e_N of N by

$$e_N(\mathbf{s}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)) = e_F(x).$$

For a given representation $V(\omega)$ we define *the Whittaker space* $Wh(V(\omega))$ by

$$\{\lambda \in V' :< \lambda, \pi(n)f >= e_N(n) < \lambda, f > \quad \text{for } n \in N, f \in V(\omega)\}$$

where V' is the algebraic dual space of $V(\omega)$ with a bilinear form $<\ ,\ >$. For $\eta \in \tilde{H}$ we can define $\lambda_\eta \in Wh(V(\omega))$ by

$$< \lambda_\eta, f >= \mu(\mathbf{p}(\eta))^{-1} \int_N f(\eta sn)\bar{e}_N(n)dn \quad \text{for} \quad f \in V(\omega).$$

Then

$$\lambda_{h\eta} = \omega(h)\lambda_\eta \quad \text{for} \quad h \in \tilde{H}_*.$$

It is proved (see KP]) that the vecor space $Wh(V(\omega))$ over $\mathbb{C}$ has a basis $\{\lambda_\eta : \eta \in \tilde{H}_* \backslash \tilde{H}\}$.

The intertwining operator $I_s : V(\omega) \to V({}^s\omega)$ induces a linear map

$$T : Wh(V({}^s\omega)) \to Wh(V(\omega))$$

defined by $< T(\lambda), f >=< \lambda, I_s f >$ for $\lambda \in Wh(V({}^s\omega))$ and $f \in V(\omega)$. The matrix coefficients $\tau(\omega, \eta', \eta)$ ($\eta', \eta \in \tilde{H}_* \backslash \tilde{H}$) of T are defined by

$$T(\lambda_{\eta'}) = \sum_{\eta \in \tilde{H}_* \backslash \tilde{H}} \tau(\omega, \eta', \eta)\lambda_\eta \quad (\lambda_{\eta'} \in Wh(V({}^s\omega)),\ \lambda_\eta \in V(\omega)).$$

We have

$$\tau(\omega, h'\eta', h\eta) = {}^s\omega(h')\omega(h)\tau(\omega, \eta', \eta) \quad \text{for} \quad h', h \in \tilde{H}_*.$$

By the same way as that in Lemma I.3.1([KP]) we can compute $\tau(\omega, \eta, \eta')$. Put $\nu = \eta^{(12)}$ where $\eta^{(12)} = \eta_1/\eta_2$ if $\mathbf{p}(\eta) = diag(\eta_1, \eta_2)$. Then

$$\begin{aligned}
\tau(\omega, \eta, \eta') &= \int_{F^\times} \omega(h_{12}(\nu^{-1}z)\eta\eta'^{-1})\bar{e}_F(z)|z|_F^{-1}dz \\
&= \int_{F^\times} \omega(h_{12}(\xi^{-1})\eta\eta'^{-1})\bar{e}_F(\frac{\nu}{\xi})|\xi|_F^{-1}d\xi \\
&= \sum_{\xi \in F^\times/R_F^\times} \int_{R_F^\times} \omega(h_{12}(\xi^{-1}y)\eta\eta'^{-1})e_F(\frac{\nu}{\xi}y)dy \\
&= \sum_{\xi \in F^\times/R_F^\times} \omega(h_{12}(\xi^{-1})\eta\eta'^{-1})\bar{\Gamma}(\xi, \frac{\nu}{\xi})
\end{aligned}$$

where ω is extended to $\tilde{H}$ by

$$\omega(h) = \begin{cases} \omega(h) & \text{if} \quad h \in \tilde{H}* \\ 0 & \text{if} \quad \tilde{H} - \tilde{H}*. \end{cases}$$

Dividing the summation over all $\xi \in F^\times/R_F^\times$ into two parts we set

$$\tau_0(\omega,\eta,\eta') = \sum_{\substack{\xi\in F^\times/R_F^\times \\ \frac{\nu}{\xi}\in\hat{R}_F}} \omega(h_{12}(\xi^{-1})\eta\eta'^{-1})\bar{\Gamma}(\xi,\frac{\nu}{\xi})$$

$$\tau_1(\omega,\eta,\eta') = \sum_{\substack{\xi\in F^\times/R_F^\times \\ \frac{\nu}{\xi}\notin\hat{R}_F}} \omega(h_{12}(\xi^{-1})\eta\eta'^{-1})\bar{\Gamma}(\xi,\frac{\nu}{\xi})$$

By the property of "Gauss sum" (3) (see Introduction)

$$\begin{aligned}\tau_0(\omega,\eta,\eta') &= \sum_{\substack{k\in\mathbb{Z} \\ \nu\pi^{kn}\in\hat{R}_F}} \omega(h_{12}(\pi^{kn})\eta\eta'^{-1})\bar{\Gamma}(\pi^{-kn},\pi^{kn}\nu) \\ &= \omega(\eta\eta'^{-1}) \sum_{\substack{k\in\mathbb{Z} \\ \nu\pi^{kn}\in\hat{R}_F}} \omega_{12}(\pi^k)(1-q_F^{-1}) \\ &= \omega(\eta\eta'^{-1})\frac{1-q_F^{-1}}{1-\omega_{12}(\pi)}\omega_{12}(\pi)^{k_0(\nu)}\end{aligned}$$

where $k_0(\nu) = \min_{\nu\pi^{kn}\in\hat{R}_F}\{k \in \mathbb{Z}\}$;

$$\begin{aligned}\tau_1(\omega,\eta,\eta') &= \sum_{\substack{\xi\in F^\times/R_F^\times \\ \frac{\nu}{\xi}\in P_F^{-1}-\hat{R}_F}} \omega(h_{12}(\xi^{-1})\eta\eta'^{-1})\bar{\Gamma}(\xi,\frac{\nu}{\xi}) \\ &= \sum_{\xi\in D(\nu)} \omega(h_{12}(\xi^{-1})\eta\eta'^{-1})\bar{\Gamma}(\xi,\frac{\nu}{\xi})\end{aligned}$$

Proposition A2. *Let the notation be as above.*

(1) $\tau(\omega,\eta,\eta') = 0$ *unless* $h_{12}(\xi^{-1})\eta\eta'^{-1} \in \tilde{H}_*$ *for* $\xi \in D(\nu)\cup\{1\}$. *In particular if* ν *is singular then*

$$\tau(\omega,\eta,\eta') = 0 \quad \textit{unless} \quad \eta\eta'^{-1} \in \tilde{H}_*.$$

(2) *If* ν *is not singular with* $\nu \notin \hat{R}_F$ *and* $\pi^n\nu \in \hat{R}_F$, *then*

$$\tau(\omega,\eta,\eta) = \frac{1-q_F^{-1}}{1-\omega_{12}(\pi)}\omega_{12}(\pi);$$

and for $\xi \in D(\nu)$

$$\tau(\omega, \eta, h_{12}(\xi^{-1})\eta) = \bar{\Gamma}(\xi, \frac{\nu}{\xi}).$$

(3) *If ν is singular with $\nu \in \hat{I}_F - \hat{R}_F$, then*

$$\tau(\omega, \eta, \eta) = \frac{1 - q_F^{-1}\omega_{12}(\pi)^{-1}}{1 - \omega_{12}(\pi)}\omega_{12}(\pi).$$

Proof. From the above argument (1) and (2) are clear. We prove (3). Since $\nu \in \hat{I}_F - \hat{R}_F$, we have $D(\nu) = \{1\}$ and

$$\begin{aligned}\tau(\omega, \eta, \eta) &= \tau_0(\omega, \eta, \eta) + \tau_1(\omega, \eta, \eta)\\ &= \frac{1 - q_F^{-1}}{1 - \omega_{12}(\pi)}\omega_{12}(\pi) + \bar{\Gamma}(1, \nu)\\ &= \frac{1 - q_F^{-1}}{1 - \omega_{12}(\pi)}\omega_{12}(\pi) + (-q_F^{-1})\\ &= \frac{1 - q_F^{-1}\omega_{12}(\pi)^{-1}}{1 - \omega_{12}(\pi)}\omega_{12}(\pi).\end{aligned}$$

Corollary. *With the notation as above, for $\xi \in D(\nu)$*

$$\tau(\omega, h_{12}(\xi^{-1})\eta, \eta) = \Gamma(\xi, \frac{\nu}{\xi}).$$

Proof. If $\xi \in D(\nu)$,then

$$\begin{aligned}\Gamma(\xi^{-1}, (\xi^{-1})^{-1}\frac{\nu}{\xi^2}) &= \Gamma(\xi^{-1}, -(\xi^{-1})^{-1}\frac{\nu}{\xi^2})\\ &= \bar{\Gamma}(\xi, \frac{\nu}{\xi})\\ &\neq 0,\end{aligned}$$

therefore $\xi^{-1} \in D(\frac{\nu}{\xi^2})$. Since $(h_{12}(\xi^{-1})\eta)^{(12)} = \frac{\nu}{\xi^2}$, we obtain

$$\begin{aligned}\tau(\omega, h_{12}(\xi^{-1})\eta, \eta) &= \tau(\omega, h_{12}(\xi^{-1})\eta, h_{12}((\xi^{-1})^{-1})h_{12}(\xi^{-1})\eta)\\ &= \bar{\Gamma}(\xi^{-1}, (\xi^{-1})^{-1}\frac{\nu}{\xi^2})\\ &= \Gamma(\xi, \frac{\nu}{\xi}).\end{aligned}$$

Proposition A3. *For η, $\eta' \in \tilde{H}$ we have*

$$\sum_{\eta" \in \tilde{H}_* \backslash \tilde{H}} \tau({}^s\omega, \eta, \eta")\tau(\omega, \eta", \eta')$$
$$= \frac{1 - q_F^{-1}\omega_{12}(\pi)}{1 - \omega_{12}(\pi)} \frac{1 - q_F^{-1}\omega_{12}(\pi)^{-1}}{1 - \omega_{12}(\pi)^{-1}} \omega(\eta\eta'^{-1}).$$

If ν is not singular, then

$$\sum_{\xi \in D(\nu)} |\Gamma(\xi, \frac{\nu}{\xi})|^2 = q_F^{-1}.$$

Proof. The first statement is an immediate consequence of Proposition A1. The second statement follows from the first and Proposition A2 (1),(2) by taking $\eta = \eta'$. In fact by Proposition A2 (1),(2) we have

$$\sum_{\eta" \in \tilde{H}_* \backslash \tilde{H}} \tau({}^s\omega, \eta, \eta")\tau(\omega, \eta", \eta)$$
$$= \sum_{\xi \in D(\nu)} \tau({}^s\omega, \eta, h_{12}(\xi^{-1})\eta)\tau(\omega, h_{12}(\xi^{-1})\eta, \eta) + \tau({}^s\omega, \eta, \eta)\tau(\omega, \eta, \eta)$$
$$= \sum_{\xi \in D(\nu)} |\Gamma(\xi, \frac{\nu}{\xi})|^2 + \frac{1 - q_F^{-1}}{1 - \omega_{12}(\pi)} \frac{1 - q_F^{-1}}{1 - \omega_{12}^{-1}(\pi)}.$$

Remark. Thus we obtain two proofs for Corollary to Theorem 2. One is a direct proof using the structure of $D(\nu)$. Another is due to the local functional equation of the intertwining operator (Proposition A1). Recall that the proof in [KP] is not a pure local one, which is half local and half global.

A3. The W-functions

A $\mathbb{C}$-valued function W on $\tilde{G}$ is called *a K_*-invariant Whittaker ϵ-function* if

(1) $W(ng) = e_N(n)W(g)$ for $n \in N$,
(2) $W(\mathbf{i}(\zeta)g) = \epsilon(\zeta)W(g)$ for $\zeta \in \mu_n(F)$ and
(3) $W(gk) = W(g)$ for $k \in K_*$.

Since the decomposition

$$\tilde{G} = \bigsqcup_{\xi \in O_F/R_F} N\tilde{H}\mathrm{s}\begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix} K_*$$

holds, the value $W(g)$ for $g \in \tilde{G}$ is reduced to

$$W\left(h\mathrm{s}\begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix}\right) \quad \text{for} \quad h \in \tilde{H}, \xi \in O_F/R_F.$$

For $h \in \tilde{H}$ with $\mathbf{p}(h) = diag(h_1, h_2)$, put $\nu = \frac{h_1}{h_2}$.

Let us see the relation between ν and ξ which is derived from

$$W\left(h\mathrm{s}\begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix}\mathrm{s}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = W\left(h\mathrm{s}\begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix}\right)$$

for $x \in R_F$.

Lemma A1. *Suppose that* $\xi \in O_F - R_F$. *Then for* $x \in R_F$

$$\begin{aligned} &W\left(h\mathbf{s}\begin{pmatrix}1 & 0\\ \xi & 1\end{pmatrix}\right)\\ =&\gamma(\nu,\xi,x)W\left(h\cdot h_{12}(\frac{1}{1+\xi x})\mathbf{s}\begin{pmatrix}1 & 0\\ \frac{\xi}{1+\xi x} & 1\end{pmatrix}\right)\end{aligned}$$

with

$$\gamma(\nu,\xi,x)=\epsilon(-\xi,1+\xi x)_F^{-1}e_F(\frac{\nu x}{1+\xi x}).$$

Proof. In the group $\tilde{G}$ we have that

$$\begin{aligned} &\mathbf{s}\begin{pmatrix}1 & 0\\ \xi & 1\end{pmatrix}\mathbf{s}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\\ =&\mathbf{i}(-\xi,1+\xi x)_F^{-1}\mathbf{s}\begin{pmatrix}1 & \frac{x}{1+\xi x}\\ 0 & 1\end{pmatrix}\mathbf{s}\begin{pmatrix}\frac{1}{1+\xi x} & 0\\ 0 & 1+\xi x\end{pmatrix}\mathbf{s}\begin{pmatrix}1 & 0\\ \frac{\xi}{1+\xi x} & 1\end{pmatrix}.\end{aligned}$$

Hence the assertion follows from the definition of W.

Proposition A4.

(1) *If* $\xi=0$, *then* $W(h)=0$ *unless* $\nu\in\hat{R}_F$.

(2) *If* $\xi\in O_F-R_F$, *then*

$$W\left(h\mathbf{s}\begin{pmatrix}1 & 0\\ \xi & 1\end{pmatrix}\right)=0$$

unless

$$\nu\equiv\frac{d\xi}{d\pi}\mod \xi\hat{I}_F(\xi).$$

Proof. (1) is cear. (2) follows from the Lemma A1. In fact ,if $\xi x\in I_F$, then $\gamma(\nu,\xi,x)$ must be 1 when

$$W\left(h\mathbf{s}\begin{pmatrix}1 & 0\\ \xi & 1\end{pmatrix}\right)\neq 0.$$

Here we see that $\xi^{-1}I_F\cap R_F=\xi^{-1}I_F(\xi)$ and that

$$e_F(\frac{\nu x}{1+\xi x})=e_F(\nu x)\quad\text{for}\quad x\in\xi^{-1}I_F(\xi).$$

References

[KP] D.A.Kazhdan and S.J.Patterson, *Metaplectic Forms*, Publ. Math. IHES **59** (1984), 35–142.

[S1] T.Suzuki, *On the biquadratic theta series*, J.reine angew. Math. **438** (1993), 31-85.

[S2] T.Suzuki, *A maximal isotropic ring of a local field and spherical functions on a metaplectic group*, Ryukyu Math. J. **15** (2002), 71–100.

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