De Finétti's Theorem For σ-finite Measures on ${\Bbb R}^\infty$ \ ${0}$

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DE FINÉTTI'S THEOREM FOR σ-FINITE MEASURES ON $\mathbb{R}^\infty \backslash \{\mathbf{0}\}$ *

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Abstract

In this paper, we extend Shiga-Tanaka's result on de Finétti's theorem for σ-finite measures on $\mathbb{R}^\infty \backslash \{\mathbf{0}\}$. Their result is restricted to measures with a moment condition. We remove this restriction.

1 Introduction

For a measure P on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, we denote by P^∞ the measure on $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$ generated by the infinite product of P. Topological space of probability measures on $\mathbb{R}$ endowed with the topology defined by the weak convergence is denoted by $\mathcal{P}(\mathbb{R})$. The following de Finétti's theorem is well known (refer [1]).

Theorem 1.1 *Let μ be a probability measure on $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$. μ is exchangeable if and only if there is a probability measure ν on $(\mathcal{P}(\mathbb{R}), \mathcal{B}(\mathcal{P}(\mathbb{R})))$ such that*

$$\mu(A) = \int_{\mathcal{P}(\mathbb{R})} P^\infty(A)\nu(dP)$$

for $A \in \mathcal{B}(\mathbb{R}^\infty)$. The measure ν is uniquely determined by μ.

We call the measure ν mixing measure of μ. Exchangeability means the invariance under any permutation. Precise definition of the exchangeability will be given in the next section. De Finétti's theorem is easily extended to finite measures.

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In [2], Shiga and Tanaka proved de Finétti's theorem for a class of σ-finite measures on $\mathbb{R}^\infty\backslash\{\mathbf{0}\}$. We denote $\mathbb{R}^\infty\backslash\{\mathbf{0}\}$ by E. Their statement is the following.

Theorem 1.2 *Let μ be a measure on $(E, \mathcal{B}(E))$ satisfying*

$$\int_E (|x_1|^2 \wedge 1)\mu(dx) < \infty, \tag{1}$$

where x_1 is the first component of $x \in E$. Then μ is exchangeable if and only if there are measures ν on $(\mathcal{P}(\mathbb{R}), \mathcal{B}(\mathcal{P}(\mathbb{R})))$ and ρ on $(\mathbb{R}\backslash\{0\}, \mathcal{B}(\mathbb{R}\backslash\{0\}))$ satisfying $\int_{\mathcal{P}(\mathbb{R})} \Big(\int_{\mathbb{R}\backslash\{0\}} (|x|^2 \wedge 1) P(dx)\Big)\nu(dP) < \infty$ and $\int_{\mathbb{R}\backslash\{0\}} (|x|^2 \wedge 1)\rho(dx) < \infty$, respectively, such that

$$\begin{aligned} \mu(A) &= \int_{\mathcal{P}(\mathbb{R})} P^\infty(A)\nu(dP) \\ &\qquad + \int_{\mathbb{R}\backslash\{0\}} \sum_{j=1}^{\infty} (\overbrace{\delta_0 \cdots \delta_0 \delta_x}^{j} \delta_0 \cdots)(A)\rho(dx). \end{aligned} \tag{2}$$

for $A \in \mathcal{B}(E)$.

The above result is interesting because there appears the additional second term in the right hand side of (2). The proof of the above theorem is based on a choice of a special exchangeable function on $\mathbb{R}^\infty$ and the completeness of L^2 space. We extend their result to general σ-finite measures on $(E, \mathcal{B}(E))$ without the moment condition (1). Although the main line of our proof follows Shiga-Tanaka's proof, we state the full proof for completeness. The main point of our proof is a use of a specific positive L^2 function (Lemma 3.1).

2 Main results

Definition 2.1 *We say that a measure μ on $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$ is exchangeable if for any positive integer n and for any permutation σ of n letters $\mu(\sigma(A)) = \mu(A)$ for $A \in \mathcal{B}(\mathbb{R}^\infty)$, where*

$$\sigma(A) = \{(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, x_{n+1}, \ldots) : (x_1, \ldots, x_n, x_{n+1}, \ldots) \in A\}.$$

The following is an extension of Shiga-Tanaka's result.

Theorem 2.1 *Let μ be a measure on $(E, \mathcal{B}(E))$ satisfying*

$$\mu(\{x \in \mathbb{R} : |x| > \epsilon\} \times \mathbb{R}^\infty) < \infty \tag{3}$$

for any $\epsilon > 0$. Then μ is exchangeable if and only if there are measures ν on $(\mathcal{P}(\mathbb{R}), \mathcal{B}(\mathcal{P}(\mathbb{R})))$ and ρ on $(\mathbb{R}\backslash\{0\}, \mathcal{B}(\mathbb{R}\backslash\{0\}))$ satisfying

$$\int_{\mathcal{P}(\mathbb{R})} P(\{x \in \mathbb{R} : |x| > \epsilon\})\nu(dP) < \infty \tag{4}$$

and

$$\rho(\{x \in \mathbb{R} : |x| > \epsilon\}) < \infty, \tag{5}$$

respectively, for any $\epsilon > 0$ such that

$$\begin{aligned} \mu(A) &= \int_{\mathcal{P}(\mathbb{R})} P^\infty(A)\nu(dP) \\ &\quad + \int_{\mathbb{R}\backslash\{0\}} \sum_{j=1}^{\infty} (\overbrace{\delta_0 \cdots \delta_0 \delta_x}^{j} \delta_0 \cdots)(A)\rho(dx). \end{aligned} \tag{6}$$

for $A \in \mathcal{B}(E)$. The pair (ν, ρ) is uniquely determined by μ.

We obtain another version of de Finetti's theorem for another class of σ-finite measures.

Theorem 2.2 *Let μ be a measure on $(E, \mathcal{B}(E))$ satisfying*

$$\mu(\{x \in \mathbb{R} : 0 < |x| < M\} \times \mathbb{R}^\infty) < \infty$$

for any $M > 0$. Then μ is exchangeable if and only if there are measures ν on $(\mathcal{P}(\mathbb{R}), \mathcal{B}(\mathcal{P}(\mathbb{R})))$ and ρ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ satisfying

$$\int_{\mathcal{P}(\mathbb{R})} P(\{x \in \mathbb{R} : 0 < |x| < M\})\nu(dP) < \infty$$

and

$$\rho(\{x \in \mathbb{R} : 0 < |x| < M\}) < \infty,$$

respectively, for any $M > 0$, such that

$$\mu(A) = \int_{\mathcal{P}(\mathbb{R})} P^\infty(A)\nu(dP) + \int_{\mathbb{R}\setminus\{0\}} \sum_{j=1}^{\infty} (\overbrace{\delta_0 \cdots \delta_0 \delta_x}^{j} \delta_0 \cdots)(A)\rho(dx). \quad (7)$$

for $A \in \mathcal{B}(E)$. The pair (ν, ρ) is uniquely determined by μ.

3 Proofs of Theorems and Remarks

Lemma 3.1 *Suppose that μ is a measure on $(E, \mathcal{B}(E))$ satisfying* (3) *for any $\epsilon > 0$. Let $\mu_1(dx) = \mu(dx \times \mathbb{R}^\infty)$ and let*

$$\eta(x) = \begin{cases} \mu_1(\{y \in \mathbb{R} : y \geq x\}) & \text{for } x > 0, \\ \mu_1(\{y \in \mathbb{R} : y \leq x\}) & \text{for } x < 0. \end{cases}$$

Then,

$$\int_E e^{-\eta(x_1)}\mu(dx) < \infty.$$

Proof *We have*

$$\begin{aligned} \int_E e^{-\eta(x_1)}\mu(dx) &= \int_{\mathbb{R}\setminus\{0\}} e^{-\eta(x)}\mu_1(dx) \\ &= \int_{\mathbb{R}\setminus\{0\}} e^{-(\eta_c(x)+\eta_d(x))}(\mu_{1c}(dx) + \mu_{1d}(dx)) \\ &\leq \int_{\mathbb{R}\setminus\{0\}} \left(e^{-\eta_c(x)}\mu_{1c}(dx) + e^{-\eta_d(x)}\mu_{1d}(dx)\right) \\ &\leq 2\int_0^\infty e^{-x}dx + \int_{\mathbb{R}\setminus\{0\}} e^{-\eta_d(x)}\mu_{1d}(dx), \end{aligned}$$

where μ_{1c} and μ_{1d} are a continuous part and a discrete part of μ_1, respectively, and

$$\eta_c(x) = \begin{cases} \mu_{1c}(\{y \in \mathbb{R} : y \geq x\}) & \text{for } x > 0, \\ \mu_{1c}(\{y \in \mathbb{R} : y \leq x\}) & \text{for } x < 0, \end{cases}$$

$$\eta_d(x) = \begin{cases} \mu_{1d}(\{y \in \mathbb{R} : y \geq x\}) & \textit{for } x > 0, \\ \mu_{1d}(\{y \in \mathbb{R} : y \leq x\}) & \textit{for } x < 0. \end{cases}$$

We may assume that the support of μ_{1d} is an infinite set. Let $\{a_n\}_{n=-\infty}^{\infty}$, $0 < \cdots < a_n < a_{n+1} \cdots$ be the support of μ_{1d} in $(0,\infty)$. Set $q_n = \sum_{k \geq n} \mu(\{a_n\})$. We have

$$\int_{(0,\infty)} e^{-\eta_d(x)} \mu_{1d}(dx) = \sum_{-\infty}^{\infty} e^{-q_n}(q_n - q_{n+1}) \leq \int_0^{\infty} e^{-x} dx.$$

In the same way, we have $\int_{(-\infty,0)} e^{\eta_d(x)} \mu_{1d}(dx) \leq \int_0^{\infty} e^{-x} dx$. We get the conclusion. □

Proof of Theorem 2.1 *Let*

$$\xi_n(x) = \frac{1}{n} \sum_{k=1}^{n} e^{-\eta(x_k)/2}$$

for $x = (x_1, x_2, \dots)$. Then $\xi_n \in L^2(\mu)$. We denote by $\|\cdot\|$ the norm in $L^2(\mu)$. We have that

$$\|\xi_m - \xi_n\|^2 = |m-n|(mn)^{-1}(a-b) \to 0$$

as $m, n \to \infty$. Here, $a = \|\xi_1\|^2$ and $b = \int e^{-(\eta(x_1)+\eta(x_2))/2} \mu(dx)$. Hence $\{\xi_n\}$ is a Cauchy sequence in $L^2(\mu)$. There is $\xi \in L^2(\mu)$ such that $\|\xi_n - \xi\| \to 0$ as $n \to \infty$. Since ξ_n is n-exchangeble, ξ is infinite exchangeable. Define μ_n by

$$\mu_n(\cdot) = \begin{cases} \mu(\cdot \cap \xi^{-1}(\{0\})) & \textit{for } n = 0, \\ \mu(\cdot \cap \xi^{-1}((\frac{1}{n}, \frac{1}{n-1}])) & \textit{for } n \geq 1. \end{cases}$$

Then since $\frac{1}{n}\mu_n(E) \leq \int \xi(x)\mu(dx) < \infty$, μ_n is a finite measure. Since ξ is exchangeable, μ_n is an exchangeable measure. Hence by classical de Finétti's Theorem, there is a finite measure ν_n on $(\mathcal{P}(\mathbb{R}), \mathcal{B}(\mathcal{P}(\mathbb{R})))$ such that

$$\mu_n = \int_{\mathcal{P}(\mathbb{R})} P^{\infty} \nu_n(dP).$$

Let $\nu = \sum_{n=1}^{\infty} \nu_n$. *We have*

$$\begin{aligned} & \int_{\mathcal{P}(\mathbb{R})} P(\{y \in \mathbb{R} : |y| > \epsilon\})\nu(dP) \\ = & \sum_{n=1}^{\infty} \mu_n(\{y \in \mathbb{R} : |y| > \epsilon\} \times \mathbb{R}^{\infty}) \\ \leq & \mu(\{y \in \mathbb{R} : |y| > \epsilon\} \times \mathbb{R}^{\infty}) \\ < & \infty. \end{aligned}$$

Hence (4) *holds for any* $\epsilon > 0$. *We have*

$$\lim_{n\to\infty} \int \xi_n(x)^2 \mu_0(dx) = \frac{1}{n}(a_0 + (n-1)b_0) \to b_0$$

where $a_0 = \int e^{-\eta(x_1)}\mu_0(dx)$ *and* $b_0 = \int e^{-(\eta(x_1)+\eta(x_2))/2}\mu_0(dx)$. *On the otherhand,* $\lim_{n\to\infty} \int \xi_n(x)^2\mu_0(dx) = \int \xi(x)^2\mu_0(dx) = 0$. *Hence* $b_0 = 0$ *and we have*

$$\mu_0(\{\mathbf{x} \in \mathbb{R}^{\infty}\backslash\{0\} : |x_i| \neq 0, |x_j| \neq 0\}) = 0 \tag{8}$$

for $i \neq j$. *Set* $\rho(A) = \mu_0(A \times \mathbb{R}^{\infty})$ *for* $A \in \mathcal{B}(\mathbb{R})$, *then* (5) *holds for any* $\epsilon > 0$ *since*

$$\rho(\{y \in \mathbb{R} : |y| > \epsilon\}) \leq \mu(\{y \in \mathbb{R} : |y| > \epsilon\} \times \mathbb{R}^{\infty}) < \infty.$$

For integers $k, n \geq 1$, *let*

$$U_k^n = \begin{cases} \mathbb{R}\backslash\{0\} & \text{for } k = n, \\ \{0\} & \text{otherwise} \end{cases}$$

and $V_n = \prod_{k=1}^{\infty} U_k^n$. *Moreover, let* $W = \cup_{n=1}^{\infty} V_n$, $S = (\mathbb{R}\backslash\{0\})^{\infty}$ *and* $T = E\backslash(S \cup W)$. *By* (8),

$$\mu_0(S \cup T) = 0. \tag{9}$$

By exchangeability, the measure μ_0 *on* W *is represented as* $\sum_{n=1}^{\infty} \rho_n$ *using* $\rho_n = \prod_{n=1}^{\infty} \rho_k^n$, *where*

$$\rho_k^n = \begin{cases} \rho & \text{for } k = n \\ \delta_0 & \text{otherwise.} \end{cases}$$

Note that

$$\sum_{n=1}^{\infty} \mu_n(T \cup W) = 0. \tag{10}$$

We get the uniqueness of the pair (ν, ρ) *by* (9), (10) *and the uniqueness of the mixing measure in the classical de Finetti's theorem. Hence we get the conclusion.* □

The proof of Theorem 2.2 is parallel to the proof of Theorem 2.1 by using

$$\tilde{\eta}(x) = \begin{cases} \mu_1(\{y \in \mathbb{R} : 0 \le y \le x\}) & \text{for } x > 0, \\ \mu_1(\{y \in \mathbb{R} : x \le y \le 0\}) & \text{for } x < 0 \end{cases}$$

instead of η. So, we omit the proof.

Remark 3.1 *Note that* $\mu(W) = \sum_{n=1}^{\infty} \mu(V_n)$. *Hence* $\mu(W) = \infty$ *provided that* $\mu(V_1) > 0$. *If* μ *is a finite measure, then* $\mu(W) = 0$. *This is consistent with the classical de Finétti's theorem.*

Remark 3.2 *Under each assumption in Theorems 2.1 and 2.2,* $\mu(T) = 0$ *by the proof of Theorem 2.1. This fact is explained by the following: Assume* (3). *We may assume that there is a positive measure* ρ *on* $(\mathbb{R}\backslash\{0\}, \mathcal{B}(\mathbb{R}\backslash\{0\}))$ *such that*

$$\mu(\cdot) - \sum_{i \ne j} \int_{(\mathbb{R}\backslash\{0\})^2} (\underbrace{\overbrace{\delta_0 \cdots \delta_0 \delta_{x_i} \delta_0 \cdots \delta_0 \delta_{x_j}}^{j}}_{i} \delta_0 \cdots)(\cdot)\rho(dx_i)\rho(dx_j)$$

is a nonnegative measure. Let $A = \{x_1 \in \mathbb{R} : |x_1| > \epsilon\} \times \mathbb{R}^{\infty}$. *We have*

$$\begin{aligned} \infty &> \mu(A) \\ &\ge \sum_{j=2}^{\infty} \int_{(\mathbb{R}\backslash\{0\})^2} (\overbrace{\delta_{x_1} \delta_0 \cdots \delta_0 \delta_{x_j}}^{j} \delta_0 \cdots)(A)\rho(dx_1)\rho(dx_j) \\ &= \sum_{j=1}^{\infty} \rho(\{x_1 \in \mathbb{R} : |x_1| > \epsilon\})\rho(\mathbb{R}) = \infty. \end{aligned}$$

This is absurd.

Remark 3.3 *If an excchangeable measure on* $(\mathbb{R}^{\infty}, \mathcal{B}(\mathbb{R}^{\infty}))$ *has a point mass at* $\mathbf{0} = (0, 0, \dots)$, *then the second term in* (6) (*and also in* (7)) *disappear.*

References

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