On the intersection graph of random caps on a sphere

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Abstract

Drop N spherical caps, each of area $4\pi \cdot p(N)$, at random on the surface of a unit sphere, and let G_p denote the intersection graphs of these random caps. Among others, we prove the following: (1) If $N(Np)^{n-1} \to 0$ as $N \to \infty$, then $\Pr(G_p$ has no component of order $\geq n) \to 1$, while if $N(Np)^{n-1} \to \infty$ then $\Pr(G_p$ has an n-clique$) \to 1$ as $N \to \infty$. (2) If $p < \frac{1-\varepsilon}{4N} \log N$, $\varepsilon > 0$ then $\Pr(\delta = 0) \to 1$, while if $p > \frac{1+\varepsilon}{4N} \log N$ then for any positive integer n, $\Pr(\delta \geq n) \to 1$ as $N \to \infty$, where δ denotes the minimum degree of G_p. (3) If $p = \frac{1}{4N}(\log N + x)$ then the number of isolated vertices of G_p is asymptotically ($N \to \infty$) distributed according to Poisson distribution with mean e^{-x}. (4) If $p > \frac{1+\varepsilon}{2N} \log N$, then $\Pr(G_p$ is 2-connected$) \to 1$ as $N \to \infty$.

1 Introduction

On the surface of a unit sphere in 3-space, place at random N spherical caps $C_1, C_2, \ldots, C_N$, each of area $4\pi p(N)$. We suppose that the centers of these caps are independently and uniformly distributed over the surface of the sphere. Then, what is the probability that the sphere is completely covered? This problem is called the coverage problem (see, e.g. Kendall and Moran [3], Santaló [9], Solomon [10]), and it seems difficult to give an exact answer. The following asymptotic answer was given by Maehara [5]: If $p(N) < \frac{1-\varepsilon}{N} \log N, \varepsilon > 0$ then the probability that the surface is completely covered by N caps tends to 0 as $N \to \infty$, while if $p > \frac{1+\varepsilon}{N} \log N$ then for any integer $n > 0$, the probability that every point on the sphere is covered more than n times tends to 1 as $N \to \infty$.

In this paper, we are going to study the asymptotic behavior of the family of N caps for $\frac{1}{N} < p(N) < \frac{1}{N} \log N$. Let v_i denote the center of the cap C_i. Then $v_1, v_2, \ldots, v_N$ are independently and uniformly distributed on the unit sphere. Let $G_p = \Omega(C_i \mid 1 \leq i \leq N)$ be the intersection graph of the random caps $C_1, C_2, \ldots, C_N$, that is, the vertices of G_p are $v_1, v_2, \ldots, v_N$, and two vertices v_i, v_j are adjacent if and only if $C_i \cap C_j \neq \emptyset$.

Among others, we prove the following:

(1) Let n be a fixed positive integer. If $p \ll N^{-n/(n-1)}$, then

$$\Pr(G_p \text{ has no component of order } \geq n) \to 1,$$

while if $p \gg N^{-n/(n-1)}$, then

$$\Pr(G_p \text{ has an } n\text{-clique}) \to 1$$

as $N \to \infty$. (Notation: $f \ll g \Leftrightarrow f/g \to 0\,(N \to \infty)$.)

(2) If $p < \frac{1-\varepsilon}{4N} \log N$, $\varepsilon > 0$, then $\Pr(\delta = 0) \to 1$, while if $p > \frac{1+\varepsilon}{4N} \log N$ then, for any positive integer k, $\Pr(\delta \geq k) \to 1$ as $N \to \infty$, where δ denotes the minimum degree of G_p.

(3) If $p = \frac{1}{4N}(\log N + x)$ then

$$\Pr(\# \text{ of isolated vertices of } G_p = j) \to \frac{e^{\mu}\mu^j}{j!} \text{ as } N \to \infty,$$

where $\mu = e^{-x}$.

(4) If $p > \frac{1+\varepsilon}{2N} \log N$ then $\Pr(G_p \text{ is 2-connected}) \to 1$ as $N \to \infty$.

In the one dimensional case (circle case), existence of the cyclic ordering of the caps (arcs) enable us much detail study, see Maehara [4].

Problem 1 *Find a constant c (if exists) such that $p < \frac{c-\varepsilon}{N} \log N$ implies that $\Pr(G_p$ is connected$) \to 0$, and $p > \frac{c+\varepsilon}{N} \log N$ implies that $\Pr(G_p$ is connected$) \to 1$. (By (2) and (4), such c must lie between $\frac{1}{4}$ and $\frac{1}{2}$. On the analogy of the one-dimensional case, $c = \frac{1}{2}$ is highly probable.)*

2 Number of edges

Let r be the angular radius of the cap of area $4\pi p$, and let D_i denote the cap of angular radius $2r$ with center v_i. Then $C_i \cap C_j \neq \emptyset \Leftrightarrow v_j \in D_i$. Provided that $p(N) = o(1)$, the area of D_i is equal to $(1 + o(1))4 \cdot 4\pi p$. Hence,

$$\Pr(C_i \cap C_j \neq \emptyset) \sim 4p.$$

(Notation: $f \sim g \Leftrightarrow f = (1 + o(1))g$.) Similarly,

$$\Pr(D_i \cap D_j \neq \emptyset) \sim 16p.$$

Let $\mathcal{E}$ denote the set of edges of G_p, and $|\mathcal{E}|$ be its cardinality.

Theorem 1 *If $N^2p \to 0$ as $N \to \infty$, then* $\Pr(|\mathcal{E}| = 0) \to 1$ *as $N \to \infty$, and if $N^2p \to \infty$ then for any $\varepsilon > 0$,*

$$\Pr\left(\left|\frac{|\mathcal{E}|}{2N^2p} - 1\right| < \varepsilon\right) \to 1 \textit{ as } N \to \infty.$$

Proof. For each pair $i, j\,(1 \le i < j \le N)$, let ξ_{ij} denote the random variable such that $\xi_{ij} = 1$ if $C_i \cap C_j \neq \emptyset$, and $\xi_{ij} = 0$ otherwise. Then $|\mathcal{E}| = \sum \xi_{ij}$, where the summation is taken over $\binom{N}{2}$ pairs. Since the expected value $E(\xi_{ij})$ of ξ_{ij} is equal to $4p$, we have

$$E(|\mathcal{E}|) \sim \binom{N}{2} 4p \sim 2N^2p.$$

If $N^2p \to 0$ as $N \to \infty$, then

$$\Pr(|\mathcal{E}| \ge 1) < E(|\mathcal{E}|) \to 0 \text{ as } N \to \infty.$$

Hence $\Pr(|\mathcal{E}| = 0) \to 1$ as $N \to \infty$.

Suppose now $N^2p \to \infty$. If i, j, k, ℓ are all different, then

$$E(\xi_{ij}\xi_{k\ell}) = E(\xi_{ij}\xi_{ik}) \sim (4p)^2, \text{ and } E(\xi_{ij}^2) \sim 4p.$$

Hence

$$\begin{aligned} E(|\mathcal{E}|^2) &= E((\textstyle\sum \xi_{ij})^2) \sim \binom{N}{2} 4p + 2\binom{\binom{N}{2}}{2}(4p)^2 \\ &\sim 2N^2p + (2N^2p)^2 \sim (2N^2p)^2 \sim E(|\mathcal{E}|)^2. \end{aligned}$$

Now, applying Chebyshev's inequality, we have

$$\Pr\left(\left|\frac{|\mathcal{E}|}{2N^2p} - 1\right| \ge \varepsilon\right) = \Pr(||\mathcal{E}| - 2N^2p| \ge \varepsilon 2N^2p) \le \frac{E(|\mathcal{E}|^2) - E(|\mathcal{E}|)^2}{\varepsilon^2 E(|\mathcal{E}|)^2} \to 0.$$

∎

Let $\Delta = \Delta(G_p)$ denote the maximum degree of G_p.

Corollary 1 *If $N^2p \to \infty (N \to \infty)$, then for any $\varepsilon > 0$,*

$$\Pr(\Delta > 4(1-\varepsilon)Np) \to 1 \textit{ and } \Pr(\delta < 4(1+\varepsilon)Np) \to 1 \textit{ as } N \to \infty.$$

Proof. By Theorem 1, the probability that

$$2(1-\varepsilon)N^2p < |\mathcal{E}| < 2(1+\varepsilon)N^2p$$

tends to 1 as $N \to \infty$. Since $N\Delta \ge 2|\mathcal{E}|, N\delta \le 2|\mathcal{E}|$, the corollary follows. ∎

The following theorem was proved in Maehara [6, Theorem 1].

Theorem 2 *If $8N^2p = x$, then* $\Pr(|\mathcal{E}| = k) \to \dfrac{x^k e^{-x}}{k!}$ *as $N \to \infty$.* ∎

3 Components of order n

The next lemma will be used frequently.

Lemma 1 *Let $f = f(N), g = g(N)$ be two nonnegative functions and suppose $f = o(1)$ $(N \to \infty)$. Then $(1-f)^g < e^{-f\cdot g}$ holds for sufficiently large N. Furthermore if $f^2 \cdot g = o(1)$ then $(1-f)^g = (1+o(1))e^{-f\cdot g}$.*

Proof will follow easily from Maclaurin expansion of $\log(1-t)$, $0 < t < 1$:

$$\log(1-t) = -t - \frac{t^2}{2(1-\lambda t)^2}, \ 0 < \lambda < 1.$$

∎

Lemma 2 *Let $A \subset \{1, 2, \ldots, N\}$ be a nonempty subset of fixed size $n = |A|$, and let $H_A = \Omega(C_i \mid i \in A)$ denote the intersection graph of $\{C_i \mid i \in A\}$. If $p = o(1)$, then*

$$\Pr(H_A \textit{ is connected}) \le (4(n-1)p)^{n-1}.$$

Proof. Since the assertion is trivial for $n = 1$, we consider the case $n \ge 2$. We may suppose that $A = \{1, 2, \ldots, n\}$. If H_A is connected, then each angular distance $\widehat{v_1 v_i}$ $(i = 2, \ldots, n)$ is at most $2(n-1)r$. (For otherwise, H_A cannot be connected.) Hence the connectedness of H_A implies that each $v_i (i = 2, \ldots, n)$ must fall in the spherical cap of angular radius $2(n-1)r$ centered at v_1. Since

$$\pi(2(n-1)r)^2 = 4(n-1)^2 \pi r^2 \sim 4(n-1)^2(4\pi p),$$

we have $\Pr(H_A \text{ is connected}) \le (4(n-1)^2 p)^{n-1}$. ∎

Theorem 3 *Let $n > 1$ be a fixed integer.*
(i) If $N(Np)^{n-1} \to 0$ as $N \to \infty$, then

$$\Pr(G_p \textit{ has no component of order } \ge n) \to 1 \textit{ as } N \to \infty.$$

(ii) If $N(Np)^{n-1} \to \infty$, then

$$\Pr(G_p \textit{ has an } n\textit{-clique}) \to 1 \textit{ as } N \to \infty.$$

Proof. (*i*) Suppose that $N(Np)^{n-1} \to 0$. For each subset $A \subset \{1, 2, \ldots, N\}$ of size n, let X_A be the random variable such that $X_A = 1$ if the subgraph $H_A = \Omega(C_i \mid i \in A)$ is connected, and $X_A = 0$ otherwise. Then $\Pr(X_A = 1) \le (4(n-1)p)^{n-1}$ by Lemma 2. Put $X = \sum_{|A|=n} X_A$. If G_p has a component of order $\ge n$, then $X \ge 1$. Hence,

$$\begin{aligned}
&\Pr(G_p \text{ has a component of order } \ge n) \\
&\quad \le \Pr(X \ge 1) \le E(X) = \sum_{|A|=n} \Pr(X_A = 1) \\
&\quad \le \binom{N}{n}(4(n-1)p)^{n-1} < \frac{(4(n-1))^{n-1}}{n!} N(Np)^{n-1} \to 0.
\end{aligned}$$

(ii) Suppose that $N(Np)^{n-1} \to \infty$. First we show that $\Pr(\Delta(G_p) \geq n-1) \to 1$. Since $\Pr(\Delta(G_p) \geq n-1)$ is clearly monotone increasing in p (that is, $p > p' \Rightarrow \Pr(\Delta(G_p) \geq n-1) \geq \Pr(\Delta(G_{p'}) \geq n-1)$), we consider the case $Np = o(1)$ and $N(Np)^{n-1} \to \infty$. Let Y_i be the random variable such that $Y_i = 1$ if $\deg v_i = n-1$ and $Y_i = 0$ otherwise. The expected value of $Y = Y_1 + Y_2 \ldots + Y_N$ is

$$\begin{aligned} E(Y) &= NE(Y_1) = N\binom{N-1}{n-1}(4p)^{n-1}(1-4p)^{N-n} \\ &\sim \frac{N}{(n-1)!}(4Np)^{n-1}e^{-4Np} \sim \frac{N}{(n-1)!}(4Np)^{n-1}. \end{aligned}$$

Next, we estimate the expected value $E(Y_iY_j)$, $i \neq j$ by considering the three exclusive cases e_1, e_2, e_3;

$e_1 : D_i \cap D_j = \emptyset,$

$e_2 : D_i \cap D_j \neq \emptyset$ and $C_i \cap C_j = \emptyset,$

$e_3 : C_i \cap C_j \neq \emptyset.$

Notice that $\Pr(e_1) \sim (1-16p)$, $\Pr(e_2) \sim 12p$, $\Pr(e_3) \sim 4p$.

$$\begin{aligned} &\Pr(e_1 \text{ and } Y_iY_j = 1) \\ &\quad = \Pr(e_1)\Pr(Y_iY_j = 1 \mid e_1) \\ &\quad \sim (1-16p)\binom{N-2}{n-1}\binom{N-n-1}{n-1}(4p)^{2(n-1)}(1-8p)^{N-2n} \\ &\quad < \left(\frac{1}{(n-1)!}(4Np)^{n-1}\right)^2. \end{aligned}$$

In the cases e_2, e_3, some v_ks may fall in $D_i \cap D_j$.

$$\begin{aligned} &\Pr(e_2 \text{ and } Y_iY_j = 1) \\ &\quad = \Pr(e_2)\Pr(Y_1Y_2 = 1 \mid e_2) \\ &\quad < 12p\sum_{\nu=0}^{n-1}\binom{N-2}{\nu}\binom{N-2-\nu}{n-1-\nu}\binom{N-1-n}{n-1-\nu}(4p)^{\nu}(4p)^{2(n-1-\nu)} \\ &\quad < 12p\sum_{\nu=0}^{n-1}N^{2(n-1)-\nu}(4p)^{2(n-1)-\nu} < 12pn(4Np)^{2(n-1)-(n-1)} = 12pn(4Np)^{n-1}. \end{aligned}$$

$$\begin{aligned} &\Pr(e_3 \text{ and } Y_iY_j = 1) \\ &\quad = \Pr(e_3)\Pr(Y_iY_j = 1 \mid e_3) \\ &\quad < 4p\sum_{\nu=0}^{n-2}\binom{N-2}{\nu}\binom{N-2-\nu}{n-2-\nu}\binom{N-2-n}{n-2-\nu}(4p)^{\nu}(4p)^{2(n-2-\nu)} \\ &\quad < 4p\sum_{\nu=0}^{n-2}N^{2(n-2)-\nu}(4p)^{2(n-2)-\nu} < 4p(n-1)(4Np)^{n-2}. \end{aligned}$$

Thus,

$$\begin{aligned} E(Y_iY_j) &= \Pr(Y_iY_j = 1) \\ &= \Pr(e_1 \text{ and } Y_iY_j = 1) + \Pr(e_2 \text{ and } Y_iY_j = 1) + \Pr(e_3 \text{ and } Y_iY_j = 1) \\ &< \left(\frac{1}{(n-1)!}(4Np)^{n-1}\right)^2 + 12pn(4Np)^{n-1} + 4p(n-1)(4Np)^{n-2}. \end{aligned}$$

Hence

$$\begin{aligned} E(Y^2) &= \sum_{i,j} E(Y_iY_j) = \sum_i E(Y_i^2) + \sum_{i\neq j} E(Y_iY_j) \\ &< NE(Y_1) + N^2\left(\frac{1}{(n-1)!}(4Np)^{n-1}\right)^2 \\ &+ N^2 \cdot 12pn(4Np)^{n-1} + N^2 \cdot 4p(n-1)(4Np)^{n-2} \\ &\sim E(Y) + E(Y)^2 + 12n!NpE(Y) + (n-1)\cdot(n-1)!E(Y) \\ &= E(Y)^2\left(\frac{1}{E(Y)} + 1 + \frac{12n!Np}{E(Y)} + \frac{(n-1)\cdot(n-1)!}{E(Y)}\right) \sim E(Y)^2. \end{aligned}$$

Since $E(Y^2) \geq E(Y)^2$ holds generally, we have $E(Y^2) \sim E(Y)^2$. Now, applying Chebychev's inequality,

$$\Pr(Y=0) \leq \Pr(|Y - E(Y)| \geq E(Y)) < \frac{E(Y^2) - E(Y)^2}{E(Y)^2} \to 0.$$

Hence $\Pr(Y \geq 1) \to 1$ as $N \to \infty$, and hence $\Pr(\Delta(G_p) \geq n-1) \to 1$ as $N \to \infty$.

Suppose that the intersection graph $\Omega(C_i \mid i = 1,2,\ldots,N)$ of the N caps $C_1,\ldots,C_N$ has a vertex of degree $\geq n-1$. To clarify the argument, assume that C_1 intersects C_i, $i = 2,3,\ldots,n$. Then D_1 contains v_i, $i = 2,3,\ldots,n$. In this case, v_1 is contained in $D_1 \cap D_2 \cap \ldots \cap D_n$. This implies that $\{v_1, v_2, \ldots, v_n\}$ forms a clique in the intersection graph $\Omega(D_i \mid i = 1,2,\ldots,N)$. Thus, if G_p has a vertex of degree $\geq n-1$, Then G_{4p} has a clique of order n. Since $N(Np)^{n-1} \to \infty$ implies $N(Np/4)^{n-1} \to \infty$, the probability that the maximum degree of $G_{p/4}$ is $\geq n-1$ tends to 1 as $N \to \infty$. Hence $\Pr(G_p$ has an n-clique$) \to 1$ as $N \to \infty$. ▮

4 Minimum degree

Theorem 4 *Let $p = p(N) = \frac{c}{N}\log N$. If $0 < c < \frac{1}{4}$, then $\Pr(\delta = 0) \to 1$, while if $c > \frac{1}{4}$, then for any positive integer n, $\Pr(\delta > n) \to 1$ as $N \to \infty$.*

Remark. It is possible to prove that if $c > \frac{1}{4}$, then

$$\Pr\left(\delta > \frac{(4c-1)\log N}{\log\log N} - 1\right) \to 1 \text{ as } N \to \infty.$$

Proof. (*i*) First, suppose that $c > \frac{1}{4}$. For a fixed integer $n \geq 1$, let W denote the number of those vertices in G_p that have degree at most n. Then $\delta \leq n \Leftrightarrow W \geq 1$. For each i, $1 \leq i \leq N$, let W_i denote the random variable such that $W_i = 1$ if $\deg v_i \leq n$, and $W_i = 0$ otherwise. Then $W = W_1 + W_2 + \ldots + W_N$. Let r be the angular radius of the cap of area $4\pi p$, and let D_i be the cap of angular radius $2r$ with center v_i, as in Section 2. Then $\deg v_i \leq n$ if and only if D_i contains at most n vertices v_j, $j \neq i$. Hence the expected value of W_i is

$$\begin{aligned} E(W_i) &= \Pr(W_i = 1) \sim \sum_{\nu=0}^{n} \binom{N-1}{\nu} (4p)^{\nu} (1-4p)^{N-1-\nu} \\ &\sim \sum_{\nu=0}^{n} \frac{(4Np)^{\nu}}{\nu!} e^{-4pN} < \sum_{\nu=0}^{n} \frac{1}{\nu!} (4c \log N)^n e^{-4c \log N} \sim e(4c \log N)^n N^{-4c}. \end{aligned}$$

Thus the expected value of $W = W_1 + \ldots + W_N$ is

$$E(W) \sim N E(W_1) \sim e(4c \log N)^n N^{1-4c}.$$

Since $c > \frac{1}{4}$, we have $E(W) \to 0$ as $N \to \infty$. Hence

$$\Pr(\delta \leq n) = \Pr(W \geq 1) \leq E(W) \to 0 \text{ as } N \to \infty.$$

(*ii*) Now, suppose that $c < \frac{1}{4}$. Let Z denote the number of isolated vertices in G_p. Then $\delta = 0 \Leftrightarrow Z \geq 1$. For each i, $1 \leq i \leq N$, let Z_i denote the random variable such that $Z_i = 1$ if $\deg v_i = 0$, and $Z_i = 0$ otherwise. Then $Z = Z_1 + Z_2 + \ldots + Z_N$. Since $\deg v_i = 0$ if and only if D_i contains no v_j, $j \neq i$, the expected value of Z_i is

$$\begin{aligned} E(Z_i) &= \Pr(Z_i = 1) \sim (1-4p)^{N-1} \\ &\sim e^{-4pN} = e^{-4c \log N} = N^{-4c}. \end{aligned}$$

Thus the expected value of $Z = Z_1 + \ldots + Z_N$ is

$$E(Z) \sim N E(Z_1) \sim N^{1-4c}.$$

Next, we consider the expected value $E(Z_i Z_j)$, $i \neq j$.

$$E(Z_i Z_j) < \Pr(D_i \cap D_j = \emptyset)(1-8p)^{N-2} + \Pr(D_i \cap D_j \neq \emptyset)(1-4p)^{N-2}.$$

Since $\Pr(D_i \cap D_j \neq \emptyset) \sim 16p$, we have

$$\begin{aligned} E(Z_i Z_j) &< (1-16p)(1-8p)^{N-2} + 16p(1-4p)^{N-2} \\ &\sim e^{-8pN} + 16pe^{-4pN} \\ &\sim N^{-8c} + (16c \log N) N^{-(1+4c)}. \end{aligned}$$

Hence

$$
\begin{aligned}
E(Z^2) &= \sum_{i,j} E(Z_iZ_j) = \sum_i E(Z_i^2) + \sum_{i\neq j} E(Z_iZ_j) \\
&< N^{1-4c} + N(N-1)\left(N^{-8c} + (16c\log N)N^{-(1+4c)}\right) \\
&\sim N^{1-4c} + N^{2-8c} + (16c\log N)N^{1-4c} \\
&= N^{2(1-4c)}\left(\frac{1}{N^{1-4c}} + 1 + \frac{16c\log N}{N^{1-4c}}\right) \\
&\sim N^{2(1-4c)} \sim E(Z)^2.
\end{aligned}
$$

Since $E(Z^2) \geq E(Z)^2$ holds generally, we have $E(Z^2) \sim E(Z)^2$. Now, applying Chebyshev's inequality,

$$\Pr(Z=0) \leq \Pr(|Z - E(Z)| \geq E(Z)) < \frac{E(Z^2) - E(Z)^2}{E(Z)^2} \to 0.$$

Hence

$$\Pr(\delta = 0) = \Pr(Z \geq 1) \to 1 \text{ as } N \to \infty.$$

∎

5 # of isolated vertices at the threshold

Similarly to Lemma 2, the next holds.

Lemma 3 *Let $A \subset \{1,2,\ldots,N\}$ be a nonempty subset of size $a = |A|$, and let $H = \Omega(D_i; i \in A)$ denote the intersection graph of $\{D_i \mid i \in A\}$. If $p = o(1)$, then*

$$\Pr(H \text{ is connected}) \leq (16(a-1)p)^{a-1}.$$

∎

Let $k > 1$ be a fixed integer less than N. For a partition

$$\mathcal{P} = \{A_1, A_2, \ldots, A_m\}$$

of the set $\{1,2,\ldots,k\}$ into m nonempty subsets, let $\langle\mathcal{P}\rangle$ denote the event that (1) $C_1, C_2, \ldots, C_k$ are mutually disjoint, and (2) the intersection graph $\Omega(D_i \mid i = 1,2,\ldots,k)$ of $D_1, \ldots, D_k$ has m connected components $H_i = \Omega(D_j \mid j \in A_i)$, $i = 1,2,\ldots,m$. Let Z_i denote the random variable such that $Z_i = 1$ if and only if $\deg v_i = 0$ in $\Omega(C_i \mid 0 \leq i \leq N)$, as in the previous section.

Lemma 4 *Let $p = \frac{1}{4N}(\log N + x)$. Let $k > 1$ be a fixed integer and let $\mathcal{P}$ be a partition of $\{1,2,\ldots,k\}$ into m subsets. If $m < k$, then*

$$\Pr(\langle\mathcal{P}\rangle \text{ and } Z_1Z_2\ldots Z_k = 1) = o(N^{-k}).$$

Proof. Let $\mathcal{P} = \{A_1, \ldots, A_m\}$, and let $a_i = |A_i|$ be the size of A_i. Since $m < k$, some a_i is greater than 1. By Lemma 3, $\Pr(H_i \text{ is connected}) \le (16(a_i - 1)p)^{a_i - 1}$, for each $i = 1, 2, \ldots, m$. Since $\sum_{i=1}^{m}(a_i - 1) = k - m$, we have

$$\begin{aligned}\Pr(\langle\mathcal{P}\rangle) &< \prod_{i=1}^{m}(16(a_i-1)p)^{a_i-1} \\ &< K \cdot p^{k-m} = \frac{K\cdot(\frac{1}{4}\log N + \frac{x}{4})^{k-m}}{N^{k-m}},\end{aligned}$$

where $K = \prod_{i=1}^{m}(16a_i - 16)^{a_i-1}$. On the other hand, since $C_1, C_2, \ldots, C_k$ are mutually disjoint, $\text{area}(D_{j'} - D_j) > \frac{1}{2}\text{area}(D_{j'})$ for $1 \le j < j' \le k$. Hence, for each A_i of size ≥ 2,

$$\text{area}(\bigcup_{j\in A_i} D_j) \ge \text{area}(D_j) + \frac{1}{2}\text{area}(D_{j'}) > 4\pi(4p + 2p) = 4\pi \cdot 6p.$$

Let ℓ be the number of those a_is that are greater than 1. Then $\ell \ge 1$ and

$$\begin{aligned}\Pr(Z_1 \ldots Z_k = 1 \mid \langle\mathcal{P}\rangle) &< (1 - (m-\ell)4p - \ell\cdot 6p)^{N-k} \\ &= (1 - (4m + 2\ell)p)^{N-k} \sim e^{-(4m+2\ell)pN} \\ &\sim N^{-(m+\ell/2)}e^{-x(m+\ell/2)}.\end{aligned}$$

Hence

$$\Pr(\langle\mathcal{P}\rangle \text{ and } Z_1Z_2\ldots Z_k = 1) < K\cdot(\frac{1}{4}\log N + \frac{x}{4})^{k-m}N^{-(k+\ell/2)}e^{-x(m+\ell/2)} = o(N^{-k}).$$

∎

Let $U_1, U_2, \ldots U_N$ be random variables which may take two values 0 and 1 only, and let $U = U_1 + U_2 + \ldots + U_N$. Then, for a positive integer k, the *k-th binomial moment* M_k of U is defined by

$$M_k = \sum E(M_{i_1}M_{i_2}\ldots M_{i_k}) = E\binom{U}{k},$$

where the summation is over all $1 \le i_1 < i_2 < \ldots < i_k \le N$, that is, over all k-subsets of $\{1, 2, \ldots, N\}$. Then the following lemma holds. For a proof, see Palmer [8] pp. 139-141.

Lemma 5 *Suppose that for each* $k = 1, 2, 3, \ldots$,

$$\lim_{N\to\infty} M_k = \frac{\mu^k}{k!},\quad 0 < \mu < \infty.$$

Then for each integer $j \ge 0$,

$$\lim_{N\to\infty}\Pr(U = j) = \frac{e^{-\mu}\mu^j}{j!}.$$

∎

Let Z denote the number of isolated vertices of G_p.

Theorem 5 *Let* $p = \frac{1}{4N}(\log N + x)$. *Then*

$$\Pr(Z = j) \to \frac{e^{-\mu}\mu^j}{j!} \quad \textit{as } N \to \infty,$$

where $\mu = e^{-x}$.

Proof. By Lemma 5, it is enough to show that for any integer $k \geq 1$, the k-th binomial moment of Z tends to $\mu^k/k!$ as $N \to \infty$, that is,

$$\binom{N}{k} E(Z_1 Z_2 \ldots Z_k) \to \frac{\mu^k}{k!} \text{ as } N \to \infty.$$

Let $\mathcal{P}_0$ denote the partition of $\{1, 2, \ldots, k\}$ into k singleton sets, that is, $\mathcal{P}_0 = \{\{1\}, \{2\}, \ldots, \{k\}\}$. Then

$$\begin{aligned} &E(Z_1 Z_2 \ldots Z_k) \\ &\quad = \Pr(Z_1 Z_2 \ldots Z_k = 1) \\ &\quad = \Pr(\langle \mathcal{P}_0 \rangle \text{ and } Z_1 Z_2 \ldots Z_k = 1) + \sum_{\mathcal{P} \neq \mathcal{P}_0} \Pr(\langle \mathcal{P} \rangle \text{ and } Z_1 Z_2 \ldots Z_k = 1). \end{aligned}$$

Since, for any partition $\mathcal{P} \neq \mathcal{P}_0$,

$$\Pr(\langle \mathcal{P} \rangle \text{ and } Z_1 Z_2 \ldots Z_k = 1) = o(N^{-k})$$

by Lemma 4, and since the number of distinct partitions of $\{1, 2, \ldots, k\}$ (known as Bell number) is clearly less than k^k, we have

$$\sum_{\mathcal{P} \neq \mathcal{P}_0} \Pr(\langle \mathcal{P} \rangle \text{ and } Z_1 Z_2 \ldots Z_k = 1) = o(N^{-k}).$$

On the other hand,

$$\Pr(\langle \mathcal{P}_0 \rangle \text{ and } Z_1 Z_2 \ldots Z_k = 1) = \Pr(\langle \mathcal{P}_0 \rangle) \Pr(Z_1 \ldots Z_k = 1 \mid \langle \mathcal{P}_0 \rangle),$$

$\Pr(\langle \mathcal{P}_0 \rangle) \to 1$ as $N \to \infty$, and

$$\Pr(Z_1 \ldots Z_k = 1 \mid \langle \mathcal{P}_0 \rangle) \sim (1 - 4kp)^{N-k} \sim e^{-4kpN} = (e^{-x}/N)^k.$$

Hence $E(Z_1 Z_2 \ldots Z_k) \sim \mu^k/N^k$ as $N \to \infty$, and hence

$$\binom{N}{k} E(Z_1 \ldots Z_k) \sim \frac{N^k}{k!} \frac{\mu^k}{N^k} = \frac{\mu^k}{k!}.$$

∎

6 Two-connectedness

In a family of caps $C_1, C_2, \ldots, C_N$ with centers $v_1, v_2, \ldots, v_N$, a cap C_i is called an *extremal cap* if there is a great circle through v_i such that all the neighbors of v_i in $\Omega(C_i \mid 0 \le i \le N)$ lie in the same side of the great circle, allowing some of them lie on the great circle.

The following theorem is obtained in [7].

Theorem 6 *Let $\mathcal{C} = \{C_1, C_2, \ldots, C_N\}$ be a family of caps on a sphere, all of the same size smaller than a hemisphere, and let $G = \Omega(\mathcal{C})$ be their intersection graph. If $\mathcal{C}$ has no extremal cap, then G is 2-connected.* ∎

The assumption that all caps are of the same size is not necessary, see [7]. It is also known that the analogous assertion is no longer true in higher dimension.

Theorem 7 *Suppose that $p = \frac{c}{N}\log N$ with $c > \frac{1}{2}$. Then*

$$\Pr(G_p \textit{ is 2-connected}) \to 1 \textit{ as } N \to \infty.$$

Proof. If $v_j \in D_i$, then the geodesic line through v_i, v_j divides D_i into two half caps, and the half cap at the left side of $\overrightarrow{v_i v_j}$ is denoted by $L(\overrightarrow{v_i v_j})$. Let us call $L(\overrightarrow{v_i v_j})$ *empty* if its interior contains no vertex.

For each $i = 1, 2, \ldots, N$, let V_i be the random variable such that $V_i = 1$ if C_i is an extremal cap, and $V_i = 0$ otherwise. Then $V_i = 1$ implies that either D_i contains no $v_j, j \neq i$, or there is a $v_j \in D_i$, $j \neq i$ such that $L(\overrightarrow{v_i v_j})$ is empty. Hence

$$\begin{aligned} E(V_i) &= \Pr(V_i = 1) \\ &\le (1-4p)^{N-1} + \sum_{j \neq i} \Pr(v_j \in D_i \text{ and } L(\overrightarrow{v_i v_j}) \text{ is empty}) \\ &= (1-4p)^{N-1} + (N-1)(4p)(1-2p)^{N-2} \sim e^{-4pN} + 4pNe^{-2pN} \\ &\sim N^{-4c} + (4c\log N)N^{-2c}. \end{aligned}$$

Let $V = V_1 + V_2 + \ldots + V_N$. Then $V \ge 1$ is equivalent to the existence of an extremal cap C_i, and hence

$$G_p \text{ is not 2-connected } \Rightarrow V \ge 1.$$

Now,

$$E(V) = NE(V_1) < N(N^{-4c} + (4c\log N)N^{-2c}) \sim (4c\log N)N^{1-2c} \sim o(1).$$

Therefore

$$\Pr(G_p \text{ is not 2-connected}) \le \Pr(V \ge 1) \le E(V) = o(1).$$

∎

Theorem 8 *If* $p = \frac{c}{N} \log N$*,* $c > 2$*, then*

$$\Pr(G_p \textit{ is Hamiltonian}) \to 1 \textit{ as } N \to \infty.$$

Proof. Let $C_1, \ldots, C_N$ be N caps of angular radius r, and G be their intersection graph. Denote by D_i the cap of angular radius $2r$ concentric with C_i. Then the intersection graph of $D_1, \ldots, D_N$ contains the square G^2 of G. Hence G_p contains the square of $G_{p/4}$. If $p = \frac{c}{N} \log N, c > 2$, then $\frac{p}{4} > \frac{1/2+\varepsilon}{N} \log N$ for some $\varepsilon > 0$. Hence $\Pr(G_{p/4}$ is 2-connected) $\to 1$ as $N \to \infty$, by Theorem 7. Now, by Fleischner's theorem ([2] or see [1]), the square of every 2-connected graph is Hamiltonian. Therefore, $\Pr(G_p$ is Hamiltonian) $\to 1$ as $N \to \infty$. ▮

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