Plane graphs with straight edges whose bounded faces are acute triangles

H. Maehara
Ryukyu University, Nishihara, Okinawa, Japan

Abstract

Let T_n denote a graph obtained as a triangulation of an n-gon in the plane. A cycle of T_n is called an enclosing cycle if at least one vertex lies inside the cycle. It is proved that a T_n admits a straight-line embedding in the plane whose bounded faces are all acute triangles if and only if T_n has no enclosing cycle of length ≤ 4. Those T_n that admit straight-line embeddings in the plane without obtuse triangle are also characterized.

1 Introduction

By a triangulation of a polygon we mean a subdivision of the polygon into nonoverlapping triangles in such a way that any two distinct triangles are either disjoint, have a vertex in common, or have one entire edge in common. In 1960, Martin Gardner [4, pp.39-42] proposed a problem to ask how many acute triangles are necessary for an acute triangulation of an obtuse triangle. Wallace Manheimer [9] gave a solution that the number is seven. Cassidy and Lord [3] showed that a square can be triangulated into eight acute triangles, eight is the minimum number of acute triangles for a square. Maehara [7] showed that every quadrilateral can be triangulated into at most 10 acute triangles, and there is a concave quadrilateral that requires 10 acute triangles. Now, in the first place, does every polygon admit an acute triangulation? This is answered affirmatively by Maehara [8].

As a variation of acute triangulation problem, let us consider the following problem. Given a triangulated polygon, when is it possible to deform the triangulation into an acute triangulation of a polygon? In other words, given a graph G obtained as a triangulation of a polygon, when does G admits a straight-line embedding in the plane whose bounded faces are all acute triangles? A *straight-line embedding* of a graph is an embedding in which all edges are straight line segments.

It was proved by Kaneko, Maehara and Watanabe [5] that any maximal planar graph (embedded in the plane) with $m+1$ faces admits a straight-line embedding in which at least $\lceil m/3 \rceil$ faces are acute triangles, and this bound cannot be improved generally. It was also proved that every maximal outer-planar graph admits a straight-line embedding in the plane whose bounded faces are all acute triangles.

In this paper, we characterize those graphs that admit straight-line embeddings in the plane whose bounded faces are all acute triangles. From now on, by a *triangulation* of an n-gon, we mean a 2-connected simple graph embedded in the plane such that the boundary of the outer face is an n-cycle, and all other faces are triangles. A triangulation of an n-gon is denoted by T_n. Vertices not lying on the outer n-cycle of T_n are called *inner vertices.*

A cycle of a graph G embedded in the plane is called an *enclosing cycle* if at least one vertex of G lies inside the cycle. A *separating cycle* of G is a cycle whose both sides (the inside and the outside) contains at least one vertex of G.

Theorem 1 *A triangulation T_n admits a straight-line embedding in the plane whose bounded faces are all acute triangles if and only if T_n has no enclosing cycle of length ≤ 4.*

Let $I_4 = I_4(T_n)$ denote the set of those inner vertices of T_n that have degree 4. A *non-obtuse* triangle is an acute triangle or a right triangle.

Theorem 2 *A triangulation T_n with more than 5 vertices admits a straight-line embedding whose bounded faces are all non-obtuse triangles if and only if T_n satisfies the following three conditions:*
(1) $n \geq 5$,
(2) I_4 *contains no adjacent pair,*
(3) $T_n - I_4$ *has no separating cycle of length* ≤ 4.

It will be easy to see that a triangulation T_n with at most 5 vertices admits a straight-line embedding in the plane whose bounded faces are non-obtuse triangles whenever T_n is one of the graphs in Figure 1.

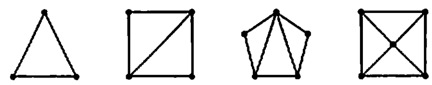

Figure 1: Five graphs of order at most 5

Let $G = (V, E)$ be a planar graph. A *proper weighting* of G is a map $f : V \to [0, \infty)$ such that there is a straight-line embedding of G in the plane whose every edge uv has length $\sqrt{f(u) + f(v)}$. This embedding of G is called the straight-line embedding *associated* with f.

To prove the above theorems, we use the following theorem.

Theorem 3 *Let T_n be a triangulation of an n-gon that satisfies the conditions* (1) $\sim$ (3) *of Theorem* 2. *Then there is a proper weighting f of T_n such that $f(v) > 0$ for every vertex v lying on the outer n-cycle of T_n.*

This theorem can be derived from the *weak orthogonal cap labeling* theorem by Kotlov, Lovász and Vempala [6] which is an extension of a theorem due to Andre'ev [1] and Thurston [11]. We present here a direct proof along the line in the proof by Brightwell and Scheinerman [2] and in the proof of Koebe's theorem by Pach and Agarwal [10].

2 Proof of Theorems 1 and 2

Assuming Theorem 3, we prove Theorems 1 and 2. Theorem 3 will be proved later.

Lemma 1 *Let $ABCD$ be a quadrilateral in the plane and let*

$$ABX, BCY, CDZ, DAW$$

be mutually non-overlapping 4 *triangles contained in $ABCD$ (see Figure 2), where the points X, Y, Z, W are not necessarily all different. If the* 4 *triangles are all non-obtuse triangles, then the* 4 *triangles are all right triangles, and* $\angle XBY = \angle YCZ = \angle ZDW = \angle WAX = 0$.

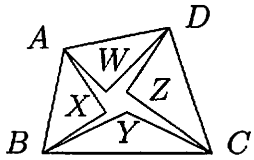

Figure 2: Lemma 1

Proof. From the assumption it follows that the sum of all interior angles of the 4 triangles ($= 4\pi$) is less than or equal to

$$(\angle A + \angle B + \angle C + \angle D) + (\angle AXB + \angle BYC + \angle CZD + \angle DWA),$$

which is $\leq 2\pi + 2\pi$. Hence $\angle AXB + \angle BYC + \angle CZD + \angle DWA = 2\pi$, that is, $\angle AXB = \angle BYC = \angle CZD = \angle DWA = \pi/2$. This implies that $\angle XAB + \angle XBA$, $\angle YBC + \angle YCB$, $\angle ZCD + \angle ZDC$, $\angle WDA + \angle WAD$ are all equal to $\pi/2$. Therefore,

$$\begin{aligned} 2\pi &= \angle XAB + \angle XBA + \angle YBC + \angle YCB \\ &\quad + \angle ZCD + \angle ZDC + \angle WDA + \angle WAD, \end{aligned}$$

and $\angle XBY = \angle YCZ = \angle ZDW = \angle WAX = 0$. □

Similarly, the following lemma holds.

Lemma 2 *Let ABC be a triangle in the plane, and let ABX, BCY, CAZ be mutually non-overlapping 3 triangles contained in ABC. Then at least two of the 3 triangles are obtuse triangles.* □

Lemma 3 *Let T_n be a triangulation with a proper weighting f, and let xyz be a face (triangle) of the straight-line embedding of T_n associated with f. Then*

- $\angle xyz \leq \pi/2$ *with equality only when* $f(y) = 0$ *(hence every triangle is a nonobtuse triangle),*
- $\deg y \geq 5 \Longrightarrow f(y) > 0$, *and*
- $f(v) = 0$ *for every* $v \in I_4$.

Proof. Since

$$\overline{xy} = \sqrt{f(x) + f(y)},\ \overline{yz} = \sqrt{f(y) + f(z)},\ \overline{zx} = \sqrt{f(z) + f(x)},$$

it follows from the cosine law that

$$\cos \angle y = \frac{f(y)}{\sqrt{f(y) + f(x)}\sqrt{f(y) + f(z)}} \geq 0.$$

(Since f has the associated straight-line embedding, $\sqrt{f(u) + f(v)} > 0$ for any adjacent pair u, v.) Hence $\angle y \leq \pi/2$, with equality only when $f(y) = 0$. If $f(y) = 0$, then in any triangle with vertex y, its interior angle at y is $\pi/2$. In this case, y is never the common vertex of more than 4 triangles. Thus, $\deg y \geq 5$ implies that $f(y) > 0$. Let v be an inner vertex of degree 4. Then, since the sum of the 4 angles at v must be equal to 2π, it follows that $f(v) = 0$. □

Proof of Theorem 1. Suppose that T_n admits a straight-line embedding whose faces are all acute triangles except the outer face. Then, by Lemmas 1 and 2, T_n contains no enclosing cycle of length ≤ 4.

Now the converse. If $n \leq 4$ and T_n has no enclosing cycle, then clearly T_n admits a straight-line embedding in the plane whose bounded faces are acute triangles. So, suppose $n \geq 5$. Since T_n contains no enclosing cycle of length ≤ 4, every inner vertex has degree ≥ 5. Thus $I_4 = \emptyset$, and $T_n - I_4 = T_n$ has no separating cycle of length ≤ 4. Hence T_n satisfies the conditions (1) $\sim$ (3) of Theorem 2 with $I_4 = \emptyset$. Therefore, there is a proper weighting f of T_n such that $f(v) > 0$ for every vertex on the outer n-cycle. Since all inner vertices have degree ≥ 5, all bounded faces of the straight-line embedding of T_n associated with f are acute triangles by Lemma 3. □

Proof of Theorem 2. Suppose that T_n is embedded in the plane with straight-line edges whose bounded faces are all non-obtuse triangles. Then, by Lemmas 1 and 2, it follows that $n > 4$. Suppose that two vertices $x, y \in I_4$ are adjacent, and let xyz be a triangle of T_n with side xy. Since the sum of four non-obtuse angles at vertex x is equal to 2π, we have $\angle yxz = \pi/2$. Similarly, $\angle xyz = \pi/2$. Then, two angles of a triangle are equal to $\pi/2$, a contradiction. Hence no two vertices in I_4 are adjacent. Suppose that $T_n - I_4$ has a separating 4-cycle $abcd$. Then the quadrilateral $abcd$ contains four non-obtuse triangles abx, bcy, cdz, daw, where the vertices x, y, z, w are not necessarily all different. By Lemma 1, the rays ax and aw coincide. Hence $w = x$, for otherwise, one of the faces abx, daw contains 4 vertices on its boundary. Thus, $w = x = y = z$, and no other vertex can lie inside $abcd$. This contradicts that $abcd$ is a separating 4-cycle of $T_n - I_4$. Similarly, $T_n - I_4$ has no separating 3-cycle.

The converse follows similarly to the corresponding part of the proof of Theorem 1. □

3 Existence of a proper weighting

Lemma 4 *Let $G = T_3$ be a triangulation of a triangle which has no separating cycle of length ≤ 4, and let v be a vertex of degree ≥ 5. Let $H = (U, E_U)$ be a subgraph of $G - v$, and let τ be the number of 3-cycles in H. If $|U| \geq 4$ and H is not a 4-cycle, then $|E_U| - \tau/2 < 2|U| - 4$.*

Proof. If $|E_U| \leq 3$, then the lemma is clearly true. Suppose that $|E_U| \geq 4$, and let r be the number of faces of H. Applying Euler's formula, we have $4 \leq 2|U| - 2|E_U| + 2r$ (the equality holds if H is connected). Now, prepare $2|E_U|$ pebbles, and for every edge e of H, place one pebble on each side of e. Then, since G has no separating cycle of length ≤ 4, every 3-cycle of H bounds a face of H, and the face of H containing the vertex v of G contains more than 4 pebbles. Hence $2|E_U| > 4(r - \tau) + 3\tau = 4r - \tau$, that is, $2r < |E_U| + \tau/2$. Therefore

$$4 < 2|U| - 2|E_U| + (|E_U| + \tau/2) = 2|U| - (|E_U| - \tau/2),$$

and we have the lemma. □

Theorem 4 *Let $G = (V, E)$ be a triangulation of a triangle with more than six vertices, and let V_4 be the set of those vertices of G with degree 4. Suppose that (a) V_4 contains no adjacent pair, and (b) the graph $G - V_4$ has no separating cycle of length ≤ 4. Then, for any vertex w of degree ≥ 5, $G - w$ has a proper weighting.*

Proof. Reserving the letter x for non-negative real numbers, let us denote the vertices of G by $1, 2, 3, \ldots, m$, where $w = m$. Let $N(w)$ denote the set of neighbors of w, and let $k = |N(w) - V_4|$, $\ell = |N(w) \cap V_4|$. Then $\deg w = k + \ell$. Since V_4 has no adjacent pair, we have $\ell \leq k$, and since $k + \ell \geq 5$, we have $k \geq 3$. We may suppose that

$$1, 2, 3, \dots, k + \ell$$

are the neighbors of w, in the cyclic order around w. Notice that the cycle $123 \dots (k + \ell)$ has no chord that connects two vertices in $N(w) - V_4$, for otherwise this chord and the two edges from w to the endpoints of this chord would form a separating 3-cycle in $G - V_4$. Therefore, if $k = 3$, then $\ell \geq 2$ and at most one edge can lie between the three vertices in $N(w) - V_4$. Similarly, if $k = 4$ then at most 3 edges are possible between four vertices in $N(w) - V_4$.

Assign to each vertex $i = 1, 2, \dots, m$, a weight $x_i \in [0, \infty)$ such that $x_i = 0$ for $i \in V_4$, and $x_j > 0$ for $j \in V - V_4$. For each triangle ijk of G, make a cardboard triangle $p_i p_j p_k$ with side-lengths

$$\overline{p_i p_j} = \sqrt{x_i + x_j},\ \overline{p_j p_k} = \sqrt{x_j + x_k},\ \overline{p_k p_i} = \sqrt{x_k + x_i}.$$

Note that this is indeed possible since V_4 contains no adjacent pair, and

$$\sqrt{x_i + x_j} + \sqrt{x_j + x_k} > \sqrt{x_k + x_i}.$$

Try to glue together the cardboard triangles corresponding to the triangles of $G - w$ along their edges in the same way as they are connected in $G - w$. If, by chance, these cardboard triangles perfectly fit together in the plane without any gaps and without any overlapping, then we have a straight-line embedding of $G - w$ associated with the weighting $V - \{w\} \ni v_i \mapsto x_i \in [0, \infty)$, and the weighting turns out to be a proper weighting of $G - w$. Let us represent the weighting of the vertices of G by the vector $\mathbf{x} = (x_1, x_2, \dots, x_m)$, and let $\theta_i(\mathbf{x})$ denote the sum of the angles at p_i of all cardboard triangles that have p_i as one of their vertices. If $\theta_i(\mathbf{x}) = 2\pi$ for $k + \ell + 1 \leq i \leq m - 1$, then the cardboard triangles corresponding to the triangles of $G - w$ perfectly fit together. Therefore, it is sufficient to show that there is a weighting $\mathbf{x} = (x_1, \dots, x_m)$ with $x_i = 0$ for $i \in V_4$ and $x_j > 0$ for $j \in V - V_4$ such that

$$(\theta_{k+\ell+1}(\mathbf{x}), \theta_{k+\ell+2}(\mathbf{x}), \dots, \theta_{m-1}(\mathbf{x})) = (2\pi, 2\pi, \dots 2\pi)$$

To this end, we prove the following stronger assertion: There is a weighting $\mathbf{x} = (x_1, \dots, x_m)$ with $x_i = 0$ for $i \in V_4$ and $x_j > 0$ for $j \in V - V_4$ such that

$$(\theta_1(\mathbf{x}), \dots, \theta_m(\mathbf{x})) = (\eta_1, \eta_2, \dots, \eta_{k+\ell}, 2\pi, 2\pi, \dots 2\pi), \tag{1}$$

where

$$\eta_i = \begin{cases} 2\pi & \text{if } i \in N(w) \cap V_4 \\ \frac{2(k-2)\pi}{k} & \text{if } i \in N(w) - V_4. \end{cases}$$

If $i \in V_4$, then in any triangle $p_i p_j p_k$, we have $\angle p_i = \pi/2$. Hence,

$$\theta_i(\mathbf{x}) = 2\pi \text{ for every } i \in V_4. \tag{2}$$

Denote the right hand side of (1) by $(\eta_1, \ldots, \eta_m)$. Since G has $2m-4$ triangles (as easily seen), we have

$$\sum_{i=1}^{m} \theta_i(\mathbf{x}) = (2m-4)\pi = 2\pi(m-k) + k \times \frac{2(k-2)\pi}{k} = \sum_{i=1}^{m} \eta_i \tag{3}$$

From now on, let us regard $x_1, \ldots, x_m$ as variables. Since the angle of a triangle $p_i p_j p_k$ at p_i is given by

$$\angle p_i = \cos^{-1}\left(\frac{x_i}{\sqrt{x_i + x_j}\sqrt{x_i + x_k}}\right), \tag{4}$$

it is a continuous function of $x_1, \ldots, x_m$, which is (strictly) monotone decreasing in x_i, monotone non-decreasing in other variables, and $\angle p_i \to 0$ as $x_i \to \infty$. Hence $\theta_i(\mathbf{x})$ is continuous, monotone decreasing in x_i, monotone non-decreasing in other variables, and $\theta_i(\mathbf{x}) \to 0$ as $x_i \to \infty$.

Now starting from $\mathbf{x}^{(1)} = (x_1^{(1)}, \ldots, x_m^{(1)})$ where

$$x_i^{(1)} = \begin{cases} 0 & \text{if } i \in V_4 \\ 1 & \text{if } i \in V - V_4 \end{cases}$$

we define a sequence $\mathbf{x}^{(2)}, \mathbf{x}^{(3)}, \mathbf{x}^{(4)}, \ldots$, recursively in the following way: Suppose that $\mathbf{x}^{(\nu)}$ is already defined. Then, for each i, define $x_i^{(\nu+1)}$ by

$$x_i^{(\nu+1)} = \begin{cases} x_i^{(\nu)} & \text{if } \theta_i(\mathbf{x}^{(\nu)}) \le \eta_i \\ x_i^{(\nu)} + \xi_i & \text{if } \theta_i(\mathbf{x}^{(\nu)}) > \eta_i \end{cases}$$

where ξ_i is the unique value satisfying

$$\theta_i(x_1^{(\nu)}, \ldots x_{i-1}^{(\nu)}, x_i^{(\nu)} + \xi_i, x_{i+1}^{(\nu)}, \ldots, x_m^{(\nu)}) = \eta_i.$$

Note that $x_i^{(\nu+1)} \geq x_i^{(\nu)}$ for each i, and if $i \in V_4$, then by (2), we have $0 = x_i^{(1)} = x_i^{(2)} = x_i^{(3)} = \ldots$. Furthermore, since $\theta_i(\mathbf{x})$ is monotone non-decreasing in x_j, $j \neq i$, it follows that

$$\theta_i(\mathbf{x}^{(\nu)}) \geq \eta_i \Longrightarrow \theta_i(\mathbf{x}^{(\nu+1)}) \geq \eta_i. \tag{5}$$

Let

$$U = \{i : \lim_{\nu\to\infty} x_i^{(\nu)} = \infty\}.$$

If $U = \emptyset$ then each sequence $x_i^{(1)}, x_i^{(2)}, x_i^{(3)}, \ldots$, converges to a definite value x_i^*, and from the definition of $\mathbf{x}^{(\nu)}$, we have

$$\theta_i(x_1^*, \ldots, x_m^*) = \lim_{\nu\to\infty} \theta_i(x_1^{(\nu)}, \ldots x_{i-1}^{(\nu)}, x_i^{(\nu+1)}, x_{i+1}^{(\nu)}, \ldots, x_m^{(\nu)}) \leq \eta_i.$$

Hence, by (3), we have (1). So, let us prove that $U = \emptyset$.

Suppose that $U \neq \emptyset$. Then $U \cap V_4 = \emptyset$. If ν is sufficiently large, then by (5), $\theta_i(\mathbf{x}^{(\nu)}) \geq \eta_i$ holds for all $i \in U$. This implies that U is a proper subset of $V - V_4$, for otherwise, from (2) it follows that $\theta_i(\mathbf{x}^{(\nu)}) \geq \eta_i$ for sufficiently large ν and for every $i = 1, 2, \ldots, m$, and (3) implies that $\theta_i(\mathbf{x}^{(\nu)}) = \eta_i$, which implies $U = \emptyset$, a contradiction. From (4), in a triangle $p_i p_j p_k$, we have

$$\begin{aligned} i \in U,\ j \notin U,\ k \notin U &\Longrightarrow \lim_{\nu\to\infty} \angle p_i = 0, \\ i, j \in U,\ k \notin U &\Longrightarrow \lim_{\nu\to\infty} (\angle p_i + \angle p_j) = \frac{\pi}{2}, \\ i, j, k \in U &\Longrightarrow \lim_{\nu\to\infty} (\angle p_i + \angle p_j + \angle k) = \pi = \left(\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2}\right) - \frac{\pi}{2}. \end{aligned}$$

Let $H = (U, E_U)$ be the subgraph of G induced by U. Then, it follows from the first of the above implications, that H contains no isolated vertex. For each edge e of H, place one black pebble on each side of e, and in each triangular face of H, place one red pebble. Then, from the second and the third of the above implications, we see that each black pebble contributes $\pi/2$ and each red pebble contributes $-\pi/2$ to $\sum_{i\in U} \lim_{\nu\to\infty} \theta_i(\mathbf{x}^{(\nu)})$. Thus, for sufficiently large ν,

$$\sum_{i\in U} \theta_i(\mathbf{x}^{(\nu)}) \approx |E_U|\pi - \tau\pi/2,$$

where τ is the number of triangles in H.

Since H has at least two vertices, $\sum_{i\in U}\eta_i \geq 2\times\frac{2(k-2)\pi}{k} \geq \frac{4}{3}\pi$, and hence H contains at least two edges, and $|U| \geq 3$. Recall that if $k=3$ then at most one edge is possible between the three vertices in $N(w)-V_4$. Hence, if $U\subset N(w)$ then $k\geq 4$, and hence $\sum_{i\in U}\eta_i \geq \min\{3\times\frac{2(4-2)\pi}{4}, 2\times\frac{2(3-2)\pi}{3}+2\pi\} = 3\pi$. Thus H contains at least three edges that do *not* form a triangle. (If H is a triangle, the three edges contribute $3\times\pi$ and one triangle contributes $-\frac{\pi}{2}$ to $\sum_{i\in U}\lim\theta_i(\mathbf{x}^{(\nu)})$, with total $3\pi-\frac{\pi}{2}$, which is less than 3π.) Therefore $|U|\geq 4$. Is it possible that H is a 4-cycle? If so, $U\not\subset N(w)$, and $\sum_{i\in U}\eta_i \geq 3\times\frac{2(4-2)\pi}{4}+2\pi \geq 5\pi$. This implies that H must contain at least 5 edges. Hence H is not a 4-cycle.

Now, by Lemma 4, we have $|E_U|-\tau/2 < 2|U|-4$, that is, $|E_U|-\tau/2 \leq 2|U|-9/2$. Let $s=|U\cap N(w)|$. Then, since

$$2\pi(|U|-k)+\frac{2\pi k(k-2)}{k} \leq 2\pi(|U|-s)+\frac{2\pi s(k-2)}{k},$$

we have

$$\begin{aligned}(2|U|-4)\pi &= 2\pi(|U|-k)+2\pi(k-2) \leq 2\pi(|U|-s)+\frac{2\pi s(k-2)}{k}\\ &= \sum_{i\in U}\eta_i \leq \sum_{i\in U}\theta_i(\mathbf{x}^{(\nu)})\\ &\approx (|E_U|-\tau/2)\pi \leq (2|U|-9/2)\pi,\end{aligned}$$

a contradiction. Hence $U=\emptyset$. □

Proof of Theorem 3. Let $v_1v_2\ldots v_n$ be the outer n-cycle of T_n. Let $W_{2n}=C_{2n}+K_1$ be a wheel (the join of $2n$-cycle C_{2n} and K_1) with center w. Let $y_1z_1y_2z_2\ldots y_nz_n$ be the $2n$-cycle of the wheel. Let G be the graph obtained from the disjoint union $T_n\cup W_{2n}$ by attaching $3n$ edges $v_iz_{i-1}, v_iy_i, v_iz_i$, $i=1,2,\ldots,n$, where $z_0=z_n$. Then, G is a maximal planar graph. From the assumption on T_n it follows that G satisfies the conditions (a) of Theorem 4, and in any planar embedding of G, there is no separating cycle of length ≤ 4, satisfying the condition (b) of Theorem 4. Hence there is a proper weighting f of $G-w$. Since the vertices $v_1,v_2,\ldots,v_n$ have degree greater than 4 in G, restricting this weighting to T_n we have a desired proper weighting of T_n. □

Acknowledgement I wish to thank the referee for his comments and suggestions. The statement of Theorem 1 becomes much simpler by his suggestion.

References

[1] E. Andre'ev, On convex polyhedra in Lobachevsky spaces, *Mat. Sbornik* Ser. 81(1970) 445–478.

[2] G. R. Brightwell and E. R. Scheinerman, Representations of planar graphs, *SIAM. Discrete Math.* 6(1993) 214–229.

[3] C. Cassidy and G. Lord, A square acutely triangulated, *J. Rec. Math.*, 13(1980/1981) 263-268.

[4] M. Gardner, *New Mathematical Diversions*, Mathematical Association of America, Washington, D.C., 1995.

[5] A. Kaneko, H. Maehara and M. Watanabe, On the number of acute triangles in a straight-line embedding of a maximal planar graph, *J. Combin. Theory B* 75(1999) 110–115.

[6] A. Kotlov, L. Lovász and S. Vempala, The Colin de Verdière number and sphere representations of a graph, *Combinatorica* 17(1997) 483–521.

[7] H. Maehara, On acute triangulation of quadrilaterals, *Proc. JCDCG 2000* (LNCS 2098), 2001, 237-354.

[8] H. Maehara, Acute triangulations of polygons, *Europ. J. Combinatorics* 23(2002) 45-55.

[9] W. Manheimer, Solution to problem E1406: dissecting an obtuse triangle into acute triangles, *Amer. Math. Monthly*, 67(1960) 923.

[10] J. Pach and P. K. Agarwal, *Combinatorial Geometry*, Wiley, New York, 1995.

[11] W. Thurston, *Three-dimensional Geometry and Topology*, MSRI, Berkeley 1991.