The problem of thirteen spheres : A proof for undergraduates

The problem of thirteen spheres – A proof for undergraduates

H. Maehara
College of Education, Ryukyu University

Abstract

The purpose of this note is to present an elementary proof of the fact that no more than twelve unit balls can simultaneously touch a unit ball in 3-space, a proof that is accessible for undergraduates.

1 Introduction

How many mutually non-overlapping unit balls can simultaneously touch a ball of the same size in 3-dimensional Euclidean space? Consider a regular icosahedron inscribed in a unit sphere S^2, and suppose that 12 unit balls are put in such a way that each ball is tangent to the sphere at one vertex of the icosahedron. Then, by calculating the distances of the centers of balls, it can be proved that these twelve balls are disjoint. Therefore, twelve unit balls can touch a unit ball. Then, can thirteen unit balls touch a unit ball simultaneously? This is the problem of thirteen spheres, and it is said that this problem was discussed between A. Newton and D. Gregory in 1694. K. Schütte and B.L. van der Waerden [15] proved in 1953 that no more than twelve unit ball can touch a unit ball. In 1956, Leech published two pages paper [9] (see also [16], pp. 10–12) to sketch a proof of the same assertion. But because of many skips, it is difficult to follow completely. A complete proof along the line of Leech's proof is given in [11]. Recently, related to the problem of thirteen spheres, several papers [1,3,4,5,6,11,12,13,14] appeared.

In [12], Oreg R. Musin solved the same type problem in four dimensions; the maximum number of unit balls that can touch a unit ball in four dimensions is 24. Applying a similar method, Musin [13] also gave a new proof of the problem of thirteen spheres.

B. Casselman wrote in [6] that it would be valuable if someone were to publish an account of Leech's proof that made it accessible to an elementary undergraduate course. This gave me a motivation to write this note.

In this note, we basically follow Leech's proof [9] as well as [11], but we use Fejes Tóth's Lemma [7] to make arguments simpler, and improve a few details of [11] to make the proof accessible for undergraduates. Except a few standard formulas, proofs of all assertions are presented. The exceptions are: (1) Girard's formula (the spherical excess formula) for the area of a spherical triangle, (2) the spherical cosine law, and (3) Euler's formula for a connected plane graph. I hope the contents of this note provide suitable materials for elementary undergraduate course.

2 Basic formulas and key lemmas

Let S^2 denote the unit sphere in R^3 centered at the origin O. An arc of a great circle on S^2 that has length less than π is called simply a *segment*. A segment with two end-points A and B, and its length, are denoted by the same notation AB. A subset $W \subset S^2$ is called *convex* if every pair of points $A, B \in W$ can be connected by a segment contained in W. In the following, figures (arcs, triangles, quadrilaterals, caps, etc) imply spherical figures on the unit sphere S^2. A triangle ABC is a convex domain on S^2 bounded by three segments AB, BC, CA. The notation $\Delta(x, y, z)$ stands for a triangle with edge-lengths x, y, z. A quadrilateral $ABCD$ is a (not necessarily convex) domain bounded by four segments AB, BC, CD, DA. A *cap* is a domain bounded by a circle. For a triangle ABC, *cap*(ABC) stands for the cap enclosed by the circum-scribed circle of ABC and containing the triangle ABC. A circular arc with two end-points A, C and passing through B is denoted by $\widehat{ABC}$. The area of a figure is denoted by $|\cdot|$. The following two formulas are well known.

(2.1) **Girard's formula.** $|ABC| = \angle A + \angle B + \angle C - \pi$.

(2.2) **Spherical cosine law.** Let θ be the angle of $\Delta(x, y, z)$ opposite to the edge z. Then $\cos z = \cos x \cos y + \sin x \sin y \cos \theta$.

Proofs of these two formulas are omitted. Since $\sin x \sin y > 0$ in the spherical cosine law, we have the following.

(2.3) The angle θ of $\Delta(x, y, z)$ opposite to z is monotone increasing on z.

(2.4) **Fejes Tóth's lemma** [7,8]. Let d be the length of the shortest edge of a triangle ABC. If the angular radius of $cap(ABC)$ is less than d, then $|ABC| \geq |\Delta(d,d,d)|$.

Suppose that a quadrilateral $ABCD$ is the union of two triangles ABC and ACD, see Figure 1. If D is not an interior point of $cap(ABC)$, then AC is called a *proper diagonal* of $ABCD$.

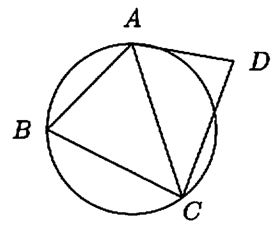

Figure 1: A proper diagonal AC of $ABCD$

(2.5) **Proper diagonal lemma** [4,11]. Let AC be a proper diagonal of a quadrilateral $ABCD$. If we deform $ABCD$ with keeping its edge lengths fixed so that the length of the diagonal AC decreases, then the area $|ABCD|$ decreases.

The above two key lemmas will be proved in Section 5.

A triangle ABC is called a *major triangle* if it contains the center of $cap(ABC)$. Let ABC be a major triangle, and let ADC be the triangle obtained by reflecting ABC with respect to the edge AC. Then AC becomes a proper diagonal of the quadrilateral $ABCD$. By applying the proper diagonal lemma to this quadrilateral $ABCD$ and considering the half area $\frac{1}{2}|ABCD|$, we have the following corollary.

(2.6) If x decreases in a major triangle $\Delta(x,y,z)$, then $|\Delta(x,y,z)|$ decreases.

If P is the center of $cap(ABC)$, then the intersection point of the ray $\overrightarrow{OP}$ and the *plane* ABC is the *circum-center* of the *planar triangle* ABC. Therefore, a triangle ABC is a major triangle if and only if the *planar triangle* ABC is an acute triangle or a right triangle. Using this fact we have the following.

(2.7) For every $x,y,z \in [\frac{\pi}{3}, \frac{\pi}{2}]$, $\Delta(x,y,z)$ is a major triangle.

(2.8) For every $x,y \in [\frac{\pi}{3}, \frac{2\pi}{3}]$, $\Delta(x,y,\frac{\pi}{2})$ is a major triangle.

3 The problem of thirteen spheres

A subset $X \subset S^2$ is called $\frac{\pi}{3}$*-separated* if no two points of X are closer than $\frac{\pi}{3}$ in spherical distance. Then, it is not difficult to see that n mutually non-overlapping unit balls can simultaneously touch S^2 if and only if there is a $\frac{\pi}{3}$-separated point set of cardinality n on S^2.

Suppose a regular icosahedron is inscribed in S^2. Then, by projecting the edges of this icosahedron onto S^2 from the center O of S^2, S^2 is divided into 20 equilateral triangles of area $4\pi/20 \approx 0.628$. Since $|\Delta(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})| \approx 0.551$, we can deduce that the edges of the equilateral triangles on S^2 are longer than $\frac{\pi}{3}$. Hence the 12 vertices of the icosahedron are $\frac{\pi}{3}$-separated. Thus, at least twelve unit balls can simultaneously touch S^2.

Let us introduce the following notations:

$$\hat{a} := \frac{\pi}{3}, \quad \hat{b} := \arccos\frac{1}{7} \approx 1.427, \quad \delta := |\Delta(\hat{a}, \hat{a}, \hat{a})|.$$

By applying spherical cosine formula, we can calculate the following:

$$\begin{array}{rl} \delta := |\Delta(\hat{a}, \hat{a}, \hat{a})| \approx 0.551 & |\Delta(\hat{a}, \hat{a}, \hat{b})| \approx 0.667 \\ |\Delta(\hat{a}, \hat{b}, \hat{b})| \approx 0.892 & |\Delta(\hat{b}, \hat{b}, \hat{b})| \approx 1.194 \\ |\Delta(\hat{a}, \frac{\pi}{2}, \frac{\pi}{2})| \approx 1.047 & |\Delta(\hat{a}, \hat{a}, \frac{\pi}{2})| \approx 0.679 \end{array}$$

Theorem 3.1. *Every* $\frac{\pi}{3}$*-separated set on* S^2 *has at most* 12 *points.*

Proof. Let X be a maximal (with respect to containment) $\frac{\pi}{3}$-separated point set on S^2, and let n be its cardinality. Then the convex hull $\Gamma(X)$ of X, that is, the convex polyhedron spanned by X, contains the center O of S^2. (For otherwise, we can add another point to X without violating $\frac{\pi}{3}$-separated-condition.) Now, by projecting the edges of the polyhedron $\Gamma(X)$ onto S^2 by the central projection from O, let us divide S^2 into spherical polygons. Further, by adding diagonals to these polygons, make a *triangulation* $\mathcal{T}$ of S^2. Then, $\mathcal{T}$ satisfies the following.

1° (By Euler's formula) $\mathcal{T}$ has $2n - 4$ triangles.

2° The interior of the circum-scribed cap of each triangle in $\mathcal{T}$ contains no vertex of $\mathcal{T}$ (because the plane determined by the vertices of a triangle of $\Gamma(X)$ is a supporting plane of $\Gamma(X)$, and the interior of the circum-scribed cap lies opposite to $\Gamma(X)$ with respect to the plane).

3° Hence, each edges of $\mathcal{T}$ is a proper diagonal of the quadrilateral obtained as the union of the two triangles sharing the edge.

4° The radius of the circum-scribed cap of each triangle in $\mathcal{T}$ is less than $\hat{a}$ (for otherwise, we can add another point to X without violating $\frac{\pi}{3}$-separated-condition).

By 4° and Fejes Tóth's lemma, the area of every triangle in $\mathcal{T}$ is greater than or equal to $\delta = |\Delta(\hat{a}, \hat{a}, \hat{a})|$. Hence, $2n - 4 \le 4\pi/\delta \approx 22.8$, and we have $n \le 13$. Thus, to prove the theorem, it is enough to show that $n \ne 13$.

Assertion 1. *If $n = 13$, then at most one edge of $\mathcal{T}$ has length greater than or equal to $\hat{b}$.* (This is proved later.)

Suppose $n = 13$. Then $\mathcal{T}$ has 22 triangles. Let G be the graph obtained from $\mathcal{T}$ by eliminating the edges of length greater than or equal to $\hat{b}$.

In the remaining, we will show that this graph G should satisfy some properties, and then, by showing that no graph can satisfy such properties, we will get a contradiction.

Assertion 2. *Let $\theta = \theta(x, y, z)$ be the angle of $\Delta(x, y, z)$ opposite to the edge z. If $\hat{a} \le x \le y < \hat{b}$ and $\hat{a} \le z$, then $\theta > \frac{\pi}{3}$.*

Proof. By (2.3), $\theta(x, y, z) \ge \theta(x, y, \hat{a})$. Put $f(x, y, z) = \cos\theta$. Then, since $f_y(x, y, z) = (\cos x - \cos y \cos z)/(\sin^2 y \sin x) > 0$ for $\hat{a} \le x \le y < \hat{b}$, we have $f(x, y, \hat{a}) < f(x, \hat{b}, \hat{a})$. Since $f_x(x, \hat{b}, \hat{a}) = \sqrt{3}(2-7\cos x)/(24 \sin^2 x)$, $f(x, \hat{b}, \hat{a})$ takes, in the interval $\hat{a} \le x \le \hat{b}$, its maximum value at $x = \hat{a}$ or at $x = \hat{b}$. Since $f(\hat{a}, \hat{b}, \hat{a}) = \frac{1}{2} > \frac{47}{96} = f(\hat{b}, \hat{b}, \hat{a})$, we have $f(x, y, z) < f(\hat{a}, \hat{b}, \hat{a}) = \frac{1}{2}$. Thus, $\cos\theta < \frac{1}{2}$ and hence, $\theta > \frac{\pi}{3}$. □

By Assertion 2, we can deduce that each vertex of G has degree at most 5. (Indeed, $\hat{b}$ was chosen to guarantee this degree condition.) If $\mathcal{T}$ has no edge of length greater than or equal to $\hat{b}$, then G has $(22 \times 3)/2 = 33$ edges, and the average degree of a vertex becomes $66/13 > 5$, a contradiction. Therefore, $\mathcal{T}$ must have exactly one edge of length at least $\hat{b}$. Note that since no two edges of G cross each other, G is a *planar graph*, that is, G can be represented by a drawing on the plane in which no two edges cross each other. Thus,

(1) G is a planar graph having 32 edges, and 21 faces consisting of one quadrilateral and 20 triangles.

Since the sum of the degrees of the vertices of G is 64,

(2) G has one vertex of degree 4 and 12 vertices of degree 5.

Since $\Delta(x,y,z)$ is a major triangle for $x,y,z \in [\hat{a},\hat{b}]$, the triangle formed by any 3-cycle of G has area less than $|\Delta(\hat{b},\hat{b},\hat{b})|$. Since $3\delta > |\Delta(\hat{b},\hat{b},\hat{b})|$, the triangle formed by a 3-cycle of G can contain no vertex in its interior. Thus,

(3) every 3-cycle of G is the boundary of a (triangular) face.

Now, is there a graph satisfying the above (1)(2)(3)? Let us try to draw such a planar graph, starting from a quadrilateral.

Case (i). The four vertices of the unique quadrilateral are all of degree 5.

From each vertex • of the quadrilateral, 3 edges emanate outward (see Figure 2 left). Their end-vertices ∘ and the vertices • of the quadrilateral are all different by (3). By (2), the vertices ∘ are mutually different. Now, from

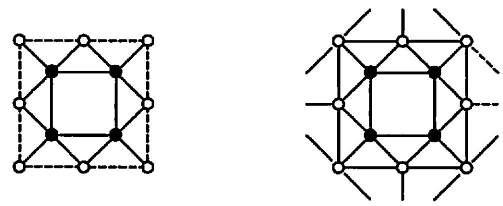

Figure 2: Case (i)

the outermost 8 vertices, 11 edges go outward (see Figure 2 right). Since they are edges of triangles, in consideration of (2), we can deduce that these edges are all different. Then, there appear more than 32 edges. Therefore this case is impossible.

Case (ii). One vertex of the quadrilateral has degree 4.

Similarly to the case (i), it can be verified that to draw a planar graph satisfying (1)(2)(3) is impossible. Therefore, $n \neq 13$. □

Proof of Assertion 1.

Suppose $n = 13$. Then $\mathcal{T}$ has 22 triangles. Suppose that the common edge AC of the triangles ABC and ACD is the longest edge of $\mathcal{T}$, and let e denote the second longest edge (and its length). We are going to show that $e < \hat{b}$.

(i) First, suppose $e > \frac{\pi}{2}$. If we deform the quadrilateral $ABCD$ with keeping its edge lengths so that the length of the (proper) diagonal AC

becomes $\frac{\pi}{2}$, then $|ABCD|$ decreases by the proper diagonal lemma, and since every edge has length less than $\frac{2\pi}{3}$ by the property 4° of $\mathcal{T}$, both triangles ABC, ACD become major triangles by (2.7), (2.8). If the edge e is an edge of $ABCD$, then

$$|ABCD| > |\Delta(\hat{a}, \frac{\pi}{2}, \frac{\pi}{2})| + \delta \approx 1.047 + 0.551$$

and $4\pi > 21\delta + 1.047 \approx 12.624 > 4\pi$, a contradiction.

If e is not an edge of $ABCD$, then $|ABCD| > 2|\Delta(\hat{a}, \hat{a}, \frac{\pi}{2})| \approx 1.359$. Similarly, the sum of the area of the two triangles sharing e in common is at least $2|\Delta(\hat{a}, \hat{a}, \frac{\pi}{2})|$. Therefore, we have

$$4\pi > (22-4)\delta + 2 \times 1.359 \approx 12.642 > 4\pi,$$

a contradiction, too. Hence, $e \leq \frac{\pi}{2}$.

(ii) Next, suppose $\hat{b} \leq e \leq \frac{\pi}{2}$. Then, triangles other than ABC, ACD are all major triangles by (2.7). If e is an edge of $ABCD$, then $|ABCD| > |\Delta(\hat{a}, \hat{b}, \hat{b})| + |\Delta(\hat{a}, \hat{a}, \hat{b})|$, and since there must be another triangle that has e as an edge,

$$4\pi \geq (22-3)\delta + 2|\Delta(\hat{a}, \hat{a}, \hat{b})| + |\Delta(\hat{a}, \hat{b}, \hat{b})| \approx 12.701 > 4\pi,$$

a contradiction. If e is not an edge of $ABCD$, then the both triangle sharing e in common have area at least $|\Delta(\hat{a}, \hat{a}, \hat{b})|$, we have

$$4\pi \geq (22-4)\delta + 4|\Delta(\hat{a}, \hat{a}, \hat{b})| \approx 12.59 > 4\pi,$$

a contradiction. Therefore, $e < \hat{b}$. □

4 Areas and Lexell circle

Theorem 4.1 (Inscribed Angle Theorem). *For a triangle ABC, let P be the center of cap(ABC). Then $\angle C - (\angle A + \angle B) = \pm 2\angle PAB$ holds, where the sign is $(-)$ if $\widehat{ACB}$ is a major arc, and $(+)$ otherwise.*

Proof would be clear from Figure 3. □

Corollary 4.1. (1) *$\angle C = \angle A + \angle B$ if and only if $\widehat{ACB}$ is a semi-circle.* (2) *If $\widehat{ACB}$ is a minor arc, then $\angle ACB > \frac{\pi}{2}$.*

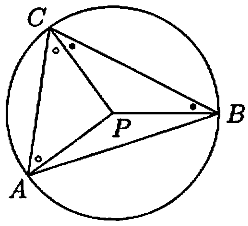

Figure 3: Proof of Theorem 4.1

Proof of (2). Since $\angle ACB > \angle ABC + \angle BAC$, and the sum of three interior angles of a triangle is greater than π, it follows that $\angle ACB > \frac{\pi}{2}$. □

For a point $P \in S^2$, its antipodal point is denoted by P^*.

Lemma 4.1. *For triangles ABC and B^*A^*C, $\angle A + \angle A^* = \angle B + \angle B^* = \pi$ holds. Hence, if $\widehat{ABC}$ is a major arc (resp. semi-circle, minor arc), then $\widehat{B^*A^*C}$ is a major arc (resp. semi-circle, minor arc).* □

Theorem 4.2 (A. J. Lexell 1784). *Let ABD be a triangle. Then, for every $C \in \widehat{A^*DB^*}$, $|ABC| = |ABD|$ holds.*

Proof [10]. Consulting Figure 4, for every point $C \in \widehat{A^*DB^*}$, $\theta - \alpha^* - \beta^*$ is constant by Theorem 4.1, and hence, $\theta - (\pi - \alpha) - (\pi - \beta)$ is constant. Therefore, $\theta + \alpha + \beta$ is constant, and $|ABC| = |ABD|$. □

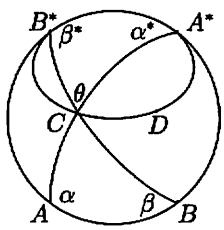

Figure 4: Proof of Theorem 4.2

The circle A^*CB^* and the arc $\widehat{A^*CB^*}$ are called the *Lexell circle* and the *Lexell arc* of the triangle ABC, respectively.

Corollary 4.2. *For a fixed segment AB, the shorter the arc $\widehat{A^*CB^*}$ is, the larger the area $|ABC|$ is.* □

Lemma 4.2 ([4,11]). *Let A, B, C denote the vertices of a $\Delta(a,b,c)$ opposite to a, b, c, respectively. If $\widehat{ABC}$ is a major arc, then $|\Delta(a,b,c)|$ decreases as b decreases. If $\widehat{ABC}$ is a semicircle, then any change of b reduces $|\Delta(a,b,c)|$.*

Proof. Fix the segment AB and consider to move C along the circle Γ with radius a and center B. Consulting Figure 4, if $\widehat{ABC}$ is a major arc, then so is $\widehat{B^*A^*C}$, and the center of the cap $cap(B^*A^*C)$ lies in the same side of the great circle B^*CB as A^*. Therefore, Γ crosses $\widehat{B^*CA^*}$ at C. In this case, if C moves along Γ toward A (so that $b = AC$ decreases), then the arc $\widehat{B^*CA^*}$ becomes longer, and hence $|ABC|$ decreases.

Suppose now $\widehat{ABC}$ is a semicircle. Then, $\widehat{B^*A^*C}$ is also a semicircle and B^*C passes through the center of $cap(B^*CA^*)$. Therefore the circle Γ is tangent to $cap(B^*CA^*)$ at C. In this case, if C moves along Γ, then $\widehat{B^*CA^*}$ always becomes longer, and $|ABC|$ decreases. □

Corollary 4.3. *The area of a triangle ABC with two fixed edge-lengths $AB = c, BC = a$ $(a + c < \pi)$ becomes maximum when $\widehat{ABC}$ is a semicircle.* □

Corollary 4.4. *A convex quadrilateral with three fixed edge-lengths $AB = a, BC = b, CD = c$ $(a+b+c < \pi)$ has maximum area when both $\widehat{ABD}, \widehat{ACD}$ becomes the same semicircle.* □

A quadrilateral that is inscribed in a cap is called a *cyclic quadrilateral.* If the cap is smaller than a hemi-sphere, then a cyclic quadrilateral is convex.

Theorem 4.3 (Isoperimetric Theorem [2]). *If we deform a cyclic convex quadrilateral with keeping its four edge-lengths, then its area decreases.*

Proof [11]. Suppose that $ABCD$ be a cyclic convex quadrilateral, see Figure 5. Put $a = AB, b = BC, c = CD, d = DA$. We may suppose that c is the largest edge and $d \geq b$. Then the diameter AP of the circumscribed cap of $ABCD$ intersects the edge CD, and we have

$$d + DP < \pi, \quad a + b + CP < \pi.$$

Now, deform $ABCD$ into $A'B'CD$ with fixing C, D and lengths a, b, c, d, in the same side of the great circle CD as $ABCD$. Then, by Corollaries 4.3 and 4.4, we have $|A'PD| < |APD|$ and $|A'B'CP| < |ABCP|$. Therefore,

$$\begin{aligned} |A'B'CD| &\leq |A'B'CP| + |A'DP| - |CPD| \\ &< |ABCP| + |PAD| - |CPD| = |ABCD|. \end{aligned}$$

□

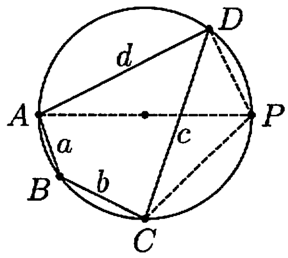

Figure 5: Proof of Theorem 4.3

Lemma 4.3. *In a cyclic convex quadrilateral $Q = ABCD$ with $AD \leq CD$, if we decrease the length of AD with keeping the lengths of AB, BC, CD fixed, then the area $|Q|$ always decreases.*

Proof. Suppose that $Q = ABCD$ is deformed into $Q' = A'B'C'D'$, where $AB = A'B', BC = B'C', CD = C'D'$ and $AD > A'D'$. It is enough to show that $|Q'| < |Q|$ when Q' is convex. Now, in Q', with keeping A', B', C' and the length of $C'D'$ fixed, move D' so that the length of $A'D'$ return to the original length of AD. Let $A'B'C'D''$ be the resulting quadrilateral. Since $A'D'' = AD \leq CD = C'D''$, the arc $\widehat{A'C'D''}$ is a major arc. Hence, $|A'C'D'| < |A'C'D''|$ by Lemma 4.2, and hence, $|A'B'C'D'| < |A'B'C'D''|$. Since $A'B'C'D''$ is obtained from Q by deforming with keeping four edge lengths fixed, its area is less than the area of Q by Theorem 4.3. Therefore, $|Q'| < |Q|$. □

5 Proofs of the key lemmas

Proof of Fejes Tóth's lemma. Suppose $AB = d$. Let ABD be the equilateral triangle lying in the same side of the great circle AB as ABC, and let A', B' be the points as shown in Figure 6. Notice that both $|ABD|, |ABA'|$ are equal to the half area of the convex quadrilateral $ABA'D$. Hence $|ABD| = |ABA'| = |ABB'|$. Therefore, the arc $\widehat{A'DB'}$ is a part of the Lexell arc $\widehat{A^*DB^*}$ by Corollary 4.2. Since $AC \geq d, BC \geq d$ and the radius of $cap(ABC)$ is less than d, the point C is not interior to the circles A, B, and lies inside the circle D of Figure 6. Then, the segment AC intersects $\widehat{A^*DB^*}$, and hence $\widehat{A^*CB^*}$ is not longer than $\widehat{A^*DB^*}$. Therefore $|ABC| \geq |\Delta(d, d, d)|$. □

Proof of the proper diagonal lemma. If neither $\widehat{ABC}, \widehat{ADC}$ are minor

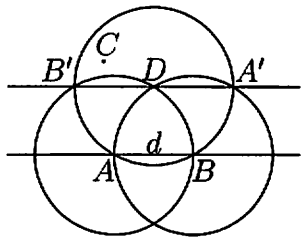

Figure 6: Proof of Fejes Tóth's lemma

arcs, then the lemma follows by applying Lemma 4.2. So, suppose that $\widehat{ADC}$ is a minor arc. Let P be the point on the arc of the circle ABC intercepted by the triangle ACD such that $\angle PDA = \angle PDC$. (Such P exists clearly.) Then, by Corollary 4.1 (2), the arcs $\widehat{PDA}$, $\widehat{PDC}$ are both major arcs. Suppose $AP \leq CP$. Now, deform the quadrilateral $ABCD$ with keeping the four

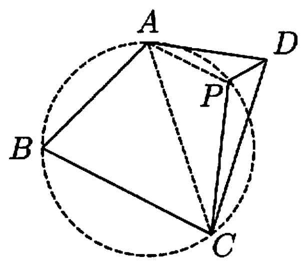

Figure 7: Proof of the proper diagonal lemma

edge-lengths and with attaching the triangle CPD so that AC decreases. Then by (2.3), $\angle ADC$ decreases. Hence $\angle ADP$ decreases, and AP decreases. Then, since $\widehat{ADP}$ is a major arc, the area $|ADP|$ decreases by Lemma 4.2. Since $AP \leq CP$, the area of $ABCP$ decreases by Lemma 4.3. Therefore,

$$|ABCD| = |ABCP| + |APD| + |CPD|$$

decreases. □

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