Distance graphs and rigidity

Hiroshi Maehara
Ryukyu University, Okinawa, Japan

Abstract

We survey many old and new theorems, and open problems related to distance graphs in Euclidean spaces. In the last two sections we present some new results with their proofs. We cover the following topics:

1. Distance graphs
2. Rigidity of graphs
3. Bipartite graphs in the plane
4. Unit-bar-graphs
5. Algebraic-distance graphs
6. Distance set with RC-property
7. Algebraic-distance graphs on circles
8. Integral- and rational-distance graphs
9. $\sqrt{Q}$-distance graphs

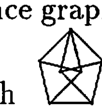
1 Distance graphs

Let R_+ denote the set of positive real numbers, and D be a nonempty subset of R^+. We refer to D as the *distance set.* For a nonempty subset X of a Euclidean space R^n, we define the D*-distance graph* on X, denoted by $X(D)$, as the graph with vertex set X and edge-set $\{xy \mid d(x,y) \in D\}$, where $d(x,y)$ is the Euclidean distance between x and y. Then $X(D)$ is a simple graph.

Suppose that D is a proper subset of R_+ and $\alpha \in D, \beta \in R_+ - D$. Then, for any $\varepsilon > 0$, we can choose two numbers $a, b \geq \min\{\alpha, \beta\}$ such that $a \in D, b \in R_+ - D$ and $|a - b| < \varepsilon$. For any $n > 0$, if ε is sufficiently small, there is an n-dimensional simplex in R^n whose prescribed edges have length a and remaining edges have length b. Any graph of order $n+1$ can be represented by the D-distance graph on the vertex set of such a simplex in R^n.

Theorem 1.1 [39] *If D is a proper subset of R_+, then every finite graph G can be represented by a D-distance graph in some R^n.* ▮

The minimum dimension n for G is called the D-dimension of G and is denoted by $\dim_D G$.

By specifying the distance set D in various ways, we obtain many interesting classes of graphs. For example, by taking $\{1\}$ as the distance set, we have *unit-distance graphs.* We write simply $X(1)$ for the unit-distance graph $X(\{1\})$ on $X \subseteq R^n$. A famous problem on unit-distance graph (see [13,35]) is to ask the chromatic number $\chi(R^2(1))$ of the unit-distance graph $R^2(1)$ on the whole plane R^2. The graph $R^2(1)$ contains a subgraph  called the *Moser's spindle* [57] whose chromatic number is 4. At present it is known that

$$4 \leq \chi(R^2(1)) \leq 7.$$

Chilakamarri [8] proved that the chromatic number of the unit distance graph of any Minkowski plane also lies in $\{4, 5, 6, 7\}$.

Let Q^n denote the set of rational points in R^n. Then $Q^n(1)$ is connected for $n \geq 5$ (Chilakamarri [7]). Though $\chi(R^n(1))$ is not known for any $n \geq 2$, the following is known, see [9]:

$$\chi(Q^1(1)) = \chi(Q^2(1)) = \chi(Q^3(1)) = 2 \text{ and } \chi(Q^4(1)) = 4.$$

Chilakamarri [9] is a nice survey on the chromatic number problem on unit-distance graphs.

Denote by Z_+ and Q_+ the set of positive integers and the set of positive rational numbers, respectively. If we take Z_+ or Q_+ as the distance set, then we have *integral-distance graphs* or *rational-distance graphs.*

Theorem 1.2 (Anning, Erdős [2]) *If X is an infinite set in R^n $(n \geq 2)$ and $X(Z_+)$ is complete (that is, any two vertices in $X(Z_+)$ are adjacent), then the set X lies on a straight line.* ▮

For a proof of the case $n = 2$ using 'hyperbolas', see the books Hadwiger and Debruner [23], Klee and Wagon [35].

If we take the interval $I := \{t : 0 < t \leq 1\}$ as the distance set, then we have *unit neighborhood graphs.* These are also known as the intersection graphs of unit balls. For a finite graph G, $\dim_I G$ is called the *sphericity* of G [21,27,39], and denoted by $sph(G)$. The graphs with sphericity 1 are known as the unit interval graphs, and they are characterized and enumerated [24,38].

Clearly $sph(K_n) = 1$ for $n > 1$. It is, however, usually difficult to determine $sph(G)$ for an arbitrarily given graph G. In fact the following result is known, see [5,6,28].

Theorem 1.3 (Breu, Kirkpatric [6]) *The recognition of the intersection graph of unit-disks in the plane is* NP*-hard.* ∎

It is known (Maehara, Reiterman, Rödl, Šiňajova [54]) that the sphericity of the complement of any tree is at most 3, and there is a tree whose complement has sphericity 3. For complete bipartite graphs, the bounds

$$n < sph(K(n,n)) < 3n/2 \quad \text{for } n \geq 4$$

are known [41]. Moser and Pach [59] contains a brief survey on graph-dimensions.

Eggleton, Erdős, Skilton [15,16,17] studied *prime-distance graphs* in which the distance set is the set P of the prime numbers or a subset of it. Among others, they proved that the chromatic number of $Z(P)$, the prime-distance graph on the integers $Z \subset R^1$, is equal to 4.

There are many papers on colorings of distance graphs $Z(D)$ for $D \subset Z_+$. Voigt and Walther [66] proved that the chromatic number of $Z(\{2,3,u,u+l\})$ is 3 for any $l \geq 10$ and $u = l^2-6l+3$. Deuber and Zhu [14] gave a classification of those $D = \{a,b,c\}$ for which b is a multiple of a and $\chi(Z(D)) = 3$. Kemnitz and Marangio [31,32] determined the chromatic number of $Z(D)$ for 4-element set $D \subset Z_+$ of the form $D = \{x, y, x+y, y-x\}, x < y$ or an arithmetical progression $D = \{a + kd \mid k = 0,1,2,\ldots\}$, and proved that for $D = \{x, 2x, \ldots, nx, y\}, \gcd(x,y) = 1$, the chromatic number of $Z(D)$ is at most $n+2$. Ruzsa, Tuza and Voigt [61] proved that the chromatic number of $Z(\{d_1, d_2, \ldots\})$ is finite whenever $\inf\{d_{i+1}/d_i\} > 1$.

2 Rigidity of graphs

Let us recall here some fundamentals on the rigidity and flexibility of graphs in R^n.

By a graph in R^n we mean a graph whose vertices are points in R^n and whose edges are line-segments connecting vertices. A graph in R^n is also considered as a *representation* of an abstract graph. A graph G in R^n is said to be *flexible*, if it admits a *continuous deformation*, that is, if we can continuously move the vertices of G in R^n in such a way that (1) the distances between adjacent vertices are unchanged, and (2) at least a pair of non-adjacent vertices change their mutual distance. If G admits no continuous deformation, then G is said to be *rigid*. Equivalently, we may define a graph G in R^n with vertex-set $X \subset R^n$ to be rigid if there is a $\delta > 0$ such that any map $\varphi : X \to R^n$ satisfying $d(\varphi(x), x) < \delta$ for all $x \in X$, and $d(\varphi(x), \varphi(y)) = d(x, y)$ for all edges xy of G, is an isometry on X. Here, an isometry on X is a map $\psi : X \to X$ that satisfies $d(x, y) = d(\psi(x), \psi(y))$ for all $x, y \in X$.

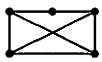
For example, the graph consisting of four vertices and four edges of a square in the plane R^2 is flexible. Indeed, it deforms into a family of rhombi. On the other hand, the graph obtained as the 1-dimensional skeleton of a k-dimensional simplex in R^n ($k \leq n$) is rigid since this is the complete graph K_{k+1}.

A *vector field* f on $X \subset R^n$ is a map $f : X \to R^n$. When we want to show the domain of f explicitly, we use the notation $f|X$. If the values of f are obtained as the velocity vectors of a smooth 'rigid motion' of X in R^n, then f is called *trivial*. An *infinitesimal motion* of a graph G with the vertex-set $X \subset R^n$ is a vector field $f|X$ that satisfies

$$(f(x) - f(y)) \cdot (x - y) = 0$$

for all edges xy of G, where $\cdot$ denotes the inner product. A nontrivial infinitesimal motion of G is called an *infinitesimal deformation* of G. If G admits an infinitesimal deformation, then G is called *infinitesimally flexible*, otherwise, G is called *infinitesimally rigid*.

If a graph in R^n admits a continuous deformation, then it admits a *smooth* deformation, see, e.g., Asimov and Roth [3]. If a graph G in R^n admits a smooth deformation, then the velocity vectors of the vertices at some instant constitute an infinitesimal deformation of G. Hence "flexible" implies "infinitesimally flexible", and "infinitesimally rigid" implies "rigid".

Note that a rigid graph is not always infinitesimally rigid. For example, the graph  in R^2 is rigid but not infinitesimally rigid. (By assigning a vertical nonzero vector to the vertex of degree 2, and zero vectors to all other vertices, we get an infinitesimal deformation.)

For more information on rigidity or flexibility, see, e.g., [3,4,60]

Let us state here some results on the rigidity of a closed polyhedral surface. We regard a closed polyhedral surface as a *hinged-panel-manifold*, that is, a closed two-dimensional manifold in R^3 obtained by attaching rigid panel-polygons along the edges with hinges. Then a question arises naturally: Is there a flexible closed polyhedral surface? Cauchy proved that the polyhedral surface of a compact convex polyhedron is rigid (*Cauchy's rigidity theorem* for a convex polyhedron). Gluck [20] proved that almost all polyhedral surfaces that are homeomorphic to a sphere and whose faces are all triangles, are rigid. In 1976, however, Connelly [11] found a flexible closed polyhedral surface with faces all triangles, homeomorphic to a sphere, and yet flexible. His flexible surface preserves its volume (content) under continuous deformation, that is, the volume of the polyhedron remains constant during continuous deformation. This fact led him to the *Bellow Conjecture.* It asserts that each flexible closed surface in R^3 conserves its volume during continuous deformation. Recently, the affirmative answer to the Bellow Conjecture was obtained for flexible polyhedra in R^3, see [12,62,63].

3 Bipartite graphs in the plane

To construct a rigid graph in the plane, we usually use triangles (3-cycles). So, it would be an interesting fact that most representations of $K(3,3)$ in the plane are rigid.

For two disjoint, nonempty (possibly infinite) sets $X, Y \subset R^2$, let $K(X,Y)$ denote the complete bipartite graph with partite sets X and Y. The size (cardinality) of X is denoted by $|X|$. It will be easy to see that if $|X| \leq 2$, then $K(X,Y)$ is always infinitesimally flexible.

Theorem 3.1 (Bolker, Roth [4]) *Suppose that $X, Y \subset R^2$ are two disjoint sets of size ≥ 3 such that no three points in $X \cup Y$ are collinear. If $K(X,Y)$ admits an infinitesimal deformation, then $X \cup Y$ lies on a conic.*

Proof [50]. Suppose that $f : X \cup Y \to R^2$ is an infinitesimal deformation of $K(X,Y)$, and let p_1, p_2, p_3 be three points in X. Then for any $q \in Y$,

$$(q - p_i) \cdot (f(q) - f(p_i)) = 0 \quad (i = 1, 2, 3). \tag{1}$$

Hence for each $q \in Y$, the value $f(q)$ is uniquely determined by the two values, say, $f(p_1), f(p_2)$, and similarly $f(p)$ ($p \in X - \{p_1, p_2\}$) are also determined by $f(p_1), f(p_2)$ via some two values $f(q), f(q')$ ($q, q' \in Y$). Therefore $f|\{p_1, p_2\}$ must be an infinitesimal deformation, i.e., $(p_1 - p_2) \cdot (f(p_1) - f(p_2)) \neq 0$. (For otherwise, $f : X \cup Y \to R^2$ becomes a trivial motion.) Now, letting $f(q) = (u, v)$, we have

$$(1) \;\Leftrightarrow\; (q - p_i) \cdot f(q) - (q - p_i) \cdot f(p_i) = 0 \;\; (i = 1, 2, 3)$$
$$\Leftrightarrow \begin{pmatrix} q - p_1 & (q - p_1) \cdot f(p_1) \\ q - p_2 & (q - p_2) \cdot f(p_2) \\ q - p_3 & (q - p_3) \cdot f(p_3) \end{pmatrix} \begin{pmatrix} {}^t(u, v) \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \qquad (2)$$

Let $p_i = (a_i, b_i)$ $(i = 1, 2, 3)$, and define a polynomial $F(x, y)$ of x, y by

$$F(\mathrm{x}) = F(x, y) = \begin{vmatrix} x - a_1 & y - b_1 & (x - a_1, y - b_1) \cdot f(p_1) \\ x - a_2 & y - b_2 & (x - a_2, y - b_2) \cdot f(p_2) \\ x - a_3 & y - b_3 & (x - a_3, y - b_3) \cdot f(p_3) \end{vmatrix}.$$

Then, since $(p_1 - p_2) \cdot (f(p_1) - f(p_2)) \neq 0$ and p_1, p_2, p_3 are not collinear it follows that $F((p_1 + p_2)/2) \neq 0$. Hence $F(x, y)$ is a nontrivial polynomial of x, y with degree at most 2. Since $F(q) = 0$ for all $q \in Y$ by (2), and since $F(p_i) = 0$ $(i = 1, 2, 3)$ as verified easily, the set $\{p_1, p_2, p_3\} \cup Y$ lies on the conic $F(x, y) = 0$. Similarly, for any $p \in X - \{p_1, p_2\}$, the set $\{p_1, p_2, p\} \cup Y$ lies on a conic. Since a *proper* conic is determined by five points on it, we can conclude that $X \cup Y$ lies on a conic. ∎

The following precise result was proved by Whitely [67]: For two disjoint sets $X, Y \subset R^2$ ($|X| \geq 3, |Y| \geq 3$) in R^2, $K(X, Y)$ is infinitesimally flexible in the plane if and only if one of the following holds:
(1) X and a point of Y lie on a line.
(2) Y and a point of X lie on a line.
(3) $X \cup Y$ lies on a conic.

When does a representation of $K(m, n)$, $m, n \geq 3$, in R^2 admit a continuous deformation?

Theorem 3.2 (Maehara, Tokushige [56]) *Let $X, Y \subset R^2$ be two disjoint finite sets such that $|X| \geq 3, |Y| \geq 5$. Then $K(X, Y)$ admits a continuous deformation if and only if X lies on a line L and Y lies on a line perpendicular to L.* ∎

To see the *if* part of the proof, suppose that $X = \{p_1, p_2, p_3, \ldots\}$ lies on the x-axis and $Y = \{q_1, q_2, q_3, \ldots\}$ lies on the y-axis, with no q_j on the origin. Then we can put

$$\begin{aligned} p_i &= (\tau_i\sqrt{a_i + t}, 0), \quad i = 1, 2, \ldots, \\ q_j &= (0, \varepsilon_j\sqrt{b_j - t}), \quad j = 1, 2, \ldots, \end{aligned}$$

where $\tau_i, \varepsilon_j = \pm 1$ and $a_i, t \geq 0$, $b_j > 0$. Then the length of the edge p_iq_j is equal to $a_i + b_j$, which is irrelevant to t. Hence by varying t, we can deform $K(X, Y)$.

The *only if* part of the proof is not easy.

We cannot relax the condition $|Y| \geq 5$ in Theorem 3.2 to $|Y| \geq 4$, by the following result (see Wunderlich [68]).

Theorem 3.3 (Bottema) *There is a flexible representation $K(X, Y)$ of $K(4, 4)$ in the plane such that the convex hulls of X and Y are both rectangles.*

Proof. The simultaneous equation on x, y, z containing a parameter t

$$\begin{cases} (x - t)^2 + (y - z)^2 = 4 \\ (x - t)^2 + (y + z)^2 = 6 \\ (x + t)^2 + (y - z)^2 = 8 \\ (x + t)^2 + (y + z)^2 = 10 \end{cases}$$

can be solved easily, and has real solutions $x = x(t), y = y(t), z = z(t)$ that are continuous in some range of t. Let

$$\begin{aligned} &p_1 = (t, z), \quad p_2 = (-t, z), \quad p_3 = (-t, -z), \quad p_4 = (t, -z), \\ &q_1 = (x, y), \quad q_2 = (-x, y), \quad q_3 = (-x, -y), \quad q_4 = (x, -y), \end{aligned}$$

and put $X = \{p_1, p_2, p_3, p_4\}$, $Y = \{q_1, q_2, q_3, q_4\}$. Then, varying t in some range, we have a continuous deformation of $K(X, Y)$. ∎

Problem 3.1 *Characterize the flexible representations of*

$$K(3, 3), \ K(3, 4), \ K(4, 4)$$

in the plane.

Let us call an (abstract) graph *absolutely 2-rigid* if it admits no flexible representation in R^2.

Problem 3.2 *Characterize absolutely 2-rigid graphs.*

It seems that a graph G of order > 2 is absolutely 2-rigid if and only if G can be obtained from K_2 by repeating the following operations: (1) attaching a vertex of degree 2, and (2) adding an edge.

4 Unit-bar-graphs

A subgraph (not necessarily an *induced* subgraph) of the unit-distance graph $R^n(1)$ is called a *unit-bar-graph* in R^n. Thus in a unit-bar-graph, a pair of non-adjacent vertices can have unit distance.

By Theorem 3.1, there is a 'bipartite' graph that is rigid in R^2. How about unit-bar-graphs? Is there a (nontrivial) rigid unit-bar-graph in the plane that has no 3-cycle? A *rigid bipartite unit-bar-graph in the plane* was constructed in [43]. Unfortunately, that graph is not *infinitesimally* rigid. An infinitesimally rigid unit-bar-graph in the plane that has no 3-cycle was given by Maehara and Chinen [52]. Figure 1 shows their graph.

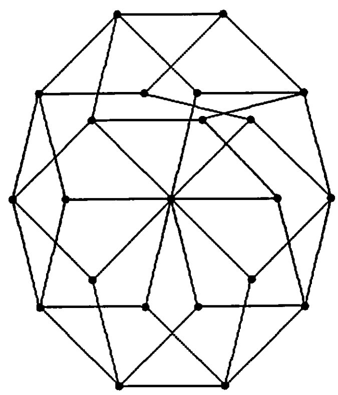


Figure 1: A triangle-free rigid unit-bar-graph in the plane

It is an easy exercise of elementary geometry to show that the unit-bar-graph in Figure 1 is rigid. The infinitesimal rigidity of the graph is shown by calculating the rank of its "rigidity matrix".

Maehara and Tokushige [55] constructed a rigid unit-bar-graph in R^3 that contains no 3-cycle. Their graph consists of 26 vertices and 78 edges (unit-bars). Its infinitesimal rigidity was checked by calculating the rank of rigidity matrix.

Problem 4.1 *Find an infinitesimally rigid bipartite unit-bar-graph in the plane.*

Problem 4.2 *Find a general method to construct a triangle-free, infinitesimally rigid unit-bar-graph in R^n.*

Let G be a flexible unit-bar-graph in R^n. Then, by adding some edges of appropriate lengths, we can always extend G to a rigid graph in R^n. How about when only unit-bars (edges of unit-lengths) are available? Can we always extend G to a rigid unit-bar-graph in R^n? If necessary, we may continuously deform G as far as no two distinct vertices come to the same position. It was proved in [45] that any unit-bar-graph in R^n can be extended to a rigid unit-bar-graph in R^n.

Though K_n ($n \geq 4$) is not isomorphic to a unit-bar-graph in the plane, every finite graph G is 'homeomorphic' to a unit-bar-graph in the plane, that is, by inserting a number of vertices into the edges of G, we can change G into a graph isomorphic to a unit-bar-graph in the plane. The *subdivision number* of G (denoted by $sd(G)$) is defined to be the minimum number of vertices we need to insert to change G into a graph isomorphic to a unit-bar-graph in the plane. If G has m edges, then $sd(G) \leq m$. This can be seen as follows: Put the vertices of G inside a circle of radius < 1 on the plane. For each pair of vertices that should be adjacent in G, connect them by a path consisting of 2 unit-bars. Then we obtain a unit-bar-graph that is homeomorphic to G. Hence $sd(G) \leq m$. Let $t(n)$ denote the maximum number of edges of a graph on n vertices that contains no 4-cycle. It is known that

$$t(n) \leq \frac{1}{4}n\left(1 + \sqrt{4n-3}\right).$$

A nice proof of this inequality is presented in the book by Aigner and Ziegler [1]. Concerning the subdivision number, Gervacio and Maehara [19] proved the following:

$$\binom{n}{2} - t(n) \leq sd(K_n) \leq \frac{1}{2}(n-2)(n-3) + 2,$$

$$sd(K(n,n)) \le (n-2)^2 + \left\lceil \frac{n-2}{2} \right\rceil,$$
$$sd(K(m,n)) = (m-1)(n-m) \quad \text{for } n \ge m(m-1).$$

To close this section, let us mention $\{1\}$-dimensions. We write $\dim_1 G$ for $\dim_{\{1\}} G$. It is clear that $\dim_1 K_n = n-1$ for any complete graph K_n. It was proved by Lenz (see [18]) that $\dim_1 K(m,n) = 4$ for $m,n \ge 4$. The $\{1\}$-dimensions of complete multipartite graphs were determined in [40]:

For $n_1, n_2, \ldots, n_u \ge 3$,

$$\dim_1 K(\underbrace{1,1,\ldots,1}_{s}, \underbrace{2,2,\ldots 2}_{t}, \underbrace{n_1, n_2, \ldots, n_u}_{u}) = \begin{cases} s+t+2u & (t+u \ge 2) \\ s+t+2u-1 & (t+u \le 1) \end{cases}.$$

5 Algebraic-distance graphs

Let us denote by A_+ the set of positive (real) algebraic numbers. For a nonempty subset $X \subset R^n$, $X(A_+)$ is called the *algebraic-distance graph* on X. The following "rigid-complete theorem" relates algebraic-distance graphs to rigidity.

Theorem 5.1 (Homma, Maehara [30]) *For a finite point-set $X \subset R^n$, the algebraic-distance graph $X(A_+)$ is rigid if and only if $X(A_+)$ is a complete graph.*

Following [45], we present here a short proof applying a result by Homma, Kato, Maehara [29]:

Let $f(x_1, x_2, \ldots, x_N)$, $g(x_1, x_2, \ldots, x_N)$ be two polynomials whose coefficients are all real algebraic numbers. Then the maximal values of $f(x_1, x_2, \ldots, x_N)$ under the condition $g(x_1, x_2, \ldots, x_N) = 0$ are all algebraic numbers.

Actually this assertion is true for any 'real algebraic functions' f, g over the field of real algebraic numbers.

Proof of the rigid-complete theorem. Let us consider the case $n = 2$. If $X(A_+)$ is complete, then it is clearly rigid. Suppose that $X(A_+)$ is not complete. Let $X = \{p_1, p_2, \ldots, p_m\}$, and put

$$E = \{ij \mid d(p_i, p_j) \in A_+,\ i < j\}.$$

Since $X(A_+)$ is not complete, we may suppose that $d(p_1, p_2) \notin A_+$. Now consider the polynomials $f(\mathbf{x}), g(\mathbf{x})$ of $2m$ variables $\mathbf{x} = (x_1, y_1, \ldots, x_m, y_m)$:

$$\begin{aligned} f(x_1, y_1, \ldots, x_m, y_m) &= (x_1 - x_2)^2 + (y_1 - y_2)^2, \\ g(x_1, y_1, \ldots, x_m, y_m) &= \sum_{ij \in E} \{(x_i - x_j)^2 + (y_i - y_j)^2 - e_{ij}^2\}^2, \end{aligned}$$

where $e_{ij} = d(p_i, p_j)$. Then $f(\mathbf{x}), g(\mathbf{x})$ are polynomials over the real algebraic numbers, and $g(\mathbf{p}) = 0$, where $\mathbf{p} = (p_1, p_2, \ldots, p_m) \in R^{2m}$. Since $f(\mathbf{p}) = d(p_1, p_2)^2$ is not algebraic, $f(\mathbf{p})$ is not a maximal value of f under the condition $g = 0$. Hence, for any $\delta > 0$, there is a point $\mathbf{q} = (q_1, q_2, \ldots, q_m) \in R^{2m}$ such that $d(p_i, q_i) < \delta$ for all $i = 1, 2, \ldots, m$, $f(\mathbf{q}) > f(\mathbf{p})$, and $g(\mathbf{q}) = 0$. However, $g(\mathbf{q}) = 0$ implies that $d(q_i, q_j) = d(p_i, p_j)$ for all edges $p_i p_j$ of $X(A_+)$. Hence, recalling the second equivalent definition of the rigidity of a graph, we can conclude that $X(A_+)$ is not rigid. ∎

Let us identify R^n with the subset of R^{n+1} consisting of the points whose last coordinates are 0. Let $G = (V, E)$ be a graph with vertex set $V \subset R^n$. A *suspension* of G with *poles*

$$p = (0, 0, \ldots, 0, z),\ q = (0, 0, \ldots, 0, -z) \in R^{n+1}$$

(where $z \neq 0$) is a graph with vertex-set $V \cup \{p, q\}$ and edge-set

$$E \cup \{pv \mid v \in V\} \cup \{qv \mid v \in V\}.$$

If the suspension of G is flexible in R^{n+1} for some $z \neq 0$, then G is said to be *suspension-flexible.*

Corollary 5.1 *Let S_τ be a sphere of transcendental radius $\tau > 0$ in R^n. Then every finite subgraph of $S_\tau(A_+)$ is suspension-flexible.*

Proof. We may suppose that S_τ is centered at the origin O of R^n. Let $G = (V, E)$ be a finite subgraph of $S_\tau(A_+)$. Let $z = \sqrt{k - \tau^2}$ for some integer $k > \tau^2$, and let $p = (0, \ldots, 0, z)$, $q = (0, \ldots, 0, -z) \in R^{n+1}$. Then the algebraic-distance graph on $V \cup \{p, q\}$ is not complete because $d(p, q) = 2\sqrt{k - \tau^2}$ is transcendental. Hence it is flexible by the rigid-complete theorem. This graph contains the suspension of G as a spanning subgraph. Hence G is suspension-flexible. ∎

The *diameter* of a graph G is the minimum integer m such that any two vertices of G can be connected by a path consisting of at most m edges. If a graph is disconnected, then its diameter is ∞.

Theorem 5.2 (Homma, Maehara [30]) *Let S_r be a sphere of radius $r > 0$ in R^n, $n \geq 3$. Then the diameter of $S_r(A_+)$ is 2 if r is algebraic, and 3 if r is transcendental.*

Proof. Since S_r has two points with transcendental distance apart, the diameter of $S_r(A_+)$ is at least 2.

Let x, y be two points that are not antipodal to each other, that is $d(x,y) < 2r$. Then we can choose $z \in S_r$ so that $d(x,z) = d(y,z) \in A_+$. Hence in $S_r(A_+)$, any non-antipodal pair x, y can be connected by a path with two edges.

If $r \in A_+$, then an antipodal pair are adjacent in $S_r(A_+)$. Hence the diameter of $S_r(A_+)$ is 2 in this case.

Suppose now r is transcendental, and let x, y be an antipodal pair. Then x, y cannot be connected by a path with two edges, for otherwise, $(2r)^2$ becomes algebraic by Pythagorian theorem, a contradiction. It will be clear that x, y can be connected by a path with three edges. ∎

For any $m, n \geq 3$, the A_+-dimension of $K(m,n)$ is 2. This can be seen as follows: Let $p_i = (\sqrt{i+\pi}, 0) \in R^2$ $(i = 1, 2, \ldots, m)$, $q_j = (0, \sqrt{3+j-\pi}) \in R^2$ $(j = 1, 2, \ldots, n)$, and $X = \{p_1, \ldots, p_m\}$, $Y = \{q_1, \ldots, q_n\}$. Then $(X \cup Y)(A_+) = K(X,Y)$. Hence, $\dim_{A_+} K(m,n) = 2$. It was also proved in [30] that

$$\dim_{A_+} K(\underbrace{2, 2, \ldots, 2}_{n}) \geq n.$$

Hence, $\dim_{A_+} G$ is also unbounded for finite graphs G.

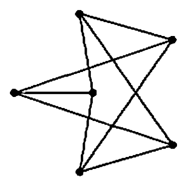
6 Distance sets with RC-property

In this section, we consider those distance sets that contain 1. A distance set D is said to have the *n-rigid-complete-property* (n-RC-property) if, for any finite subset $X \subset R^n$, the rigidity of $X(D)$ implies the completeness of the graph $X(D)$. By the rigid-complete theorem, the set A_+ has the n-RC-property for every $n > 0$.

What is the minimum distance set D that has the n-RC-Property? Since $1 \in D$, it will be clear that Z_+ is the minimum distance set with the 1-RC-property.

A number σ is called a *surd number* if σ can be obtained from 0 and 1 by applying a finite number of arithmetic operations and extractions of square root. It is well known that given a line-segment of unit length in the plane, a line-segment of length $|\sigma|$ can be constructed by ruler and compass if and only if σ is a surd number. See, e.g., Stewart [64].

Now, since our distance sets contain 1, it will be easy to see that any distance set with the 2-RC-property contains the set Σ_+ of positive *surd numbers.* It is known [47] that any rigid Σ_+-distance graph in R^2 with at most five vertices is complete. So, Σ_+ could be a candidate for the minimum distance set with the 2-RC-property. The Σ_+-distance graph on a point-set is called the *surd-distance graphs* on the point-set. It turned out, however, that there is a surd-distance graph in R^2 with six vertices that is rigid but not complete. Figure 2 shows such a surd-distance graph obtained in [49]. It can be proved that none of the distances between non-adjacent pairs of vertices in Figure 2 is a surd number. Hence, it is impossible to construct a congruent copy of this rigid graph by ruler and compass from the data of edge-lengths and the graph-structure of this graph.



short edge = 1, long edge = 2

Figure 2: A rigid surd-distance graph

Employing Kempe's idea [34], it was proved in [45] that for any positive algebraic number α, there is a rigid unit-distance graph G in R^2 that contains two vertices exactly distance α apart. Thus it turned out that A_+ is the minimum distance set with 2-RC-property.

This result can be extended to arbitrary dimension $n > 2$.

Theorem 6.1 [45] *The set of positive algebraic numbers A_+ is the minimum distance set that has n-RC-property for each $n \geq 2$.* ▮

7 Algebraic-distance graphs on circles

Let C be a circle in R^2 with center O and let $K = v_1 v_2 \ldots v_n$ be an 'oriented' cycle *inscribed* in C (that is, $v_1, \ldots, v_n$ lie on C). As a closed polygonal curve, K may have self-intersections. For each 'oriented' edge v_jv_{j+1} not passing through the center O, let us assign a sign $\epsilon_j = \epsilon(v_jv_{j+1})$ as follows: If $O \to v_j \to v_{j+1} \to O$ is counterclockwise, then $\epsilon_j = +1$, otherwise, $\epsilon_j = -1$.

The *winding number* of K (around O) is defined by $wind(K) = 0$ if one of the edges of K passes through the center O of the circle C, and

$$wind(K) = \frac{1}{2\pi}\sum_{j=1}^{n} \epsilon_j \angle v_jOv_{j+1}$$

otherwise, where $v_{n+1} \equiv v_1$ and $\angle v_jOv_{j+1}$ is measured in radians, $0 < \angle v_jOv_{j+1} < \pi$. Note that $wind(K)$ takes only integral values. The *signed area* of K (denoted by $area(K)$) is defined by

$$area(K) = \sum_{j=1}^{n} \epsilon_j |\Delta Ov_jv_{j+1}|,$$

where $|\Delta Ov_jv_{j+1}|$ is the area of the triangle ΔOv_jv_{j+1}.

The following theorem is a special case of Connelly's suspension theorem.

Theorem 7.1 (Connelly [10]) *If an oriented cycle K inscribed in a circle in R^2 is suspension-flexible, then $wind(K) = area(K) = 0$.*

Proof. We use the following fact: If the function

$$f(x) = a_1\cos^{-1}(1 - c_1x) + \ldots + a_m\cos^{-1}(1 - c_mx) \quad (c_1 < c_2 < \ldots < c_m)$$

is constant on an interval of x, then $a_1 = a_2 = \ldots = a_m = 0$.

This can be seen as follows. Suppose that $a_m \neq 0$. Then since $f(x)$ is real analytic on the interval $(0, 2/c_m)$, it is constant on the interval. Hence the derivative $f'(x) = 0$ on the interval. However, $f'(x) \to \infty$ as $x \to 2/c_m$ as easily verified, a contradiction.

Now, let $K = v_1v_2\ldots v_n$ be an oriented cycle inscribed in a circle C with center O. Suppose that a suspension of K is flexible. Then we can continuously change the radius of the circle circumscribed to K. We may also suppose, by changing the radius r if necessary, that no edge of K passes through O. Let $e_j = d(v_j, v_{j+1})$. Then, from the cosine law, we have $\cos\angle v_jOv_{j+1} = 1 - (1/2)(e_j/r)^2$. Letting $x = 1/(2r^2)$, we get $\angle v_jOv_{j+1} = \cos^{-1}(1 - e_j^2x)$. Let $c_1, c_2, \ldots, c_m$ be the distinct numbers in $\{e_1, \ldots, e_n\}$. Then

$$wind(K) = \frac{1}{2\pi}\sum_{1\leq i\leq m}\left(\sum_{e_j=c_i}\epsilon_j\right)\cos^{-1}(1 - c_i^2x).$$

Since $wind(K)$ remains fixed under a small change of the radius r (recall that no edges passes through O), it follows from the fact mentioned in the begining of the proof that

$$\sum_{e_j=c_i} \epsilon_j = 0 \quad for\ i = 1, \ldots, m.$$

Hence $wind(K) = 0$. Since $|\Delta O v_j v_{j+1}| = \frac{1}{2} e_j \sqrt{r^2 - (e_j/2)^2}$, we have

$$area(K) = \frac{1}{2} \sum_{1 \le i \le m} \left(\sum_{e_j=c_i} \epsilon_j \right) c_i \sqrt{r^2 - (c_i/2)^2} = 0.$$

∎

The suspension-flexible oriented cycles inscribed in a circle are characterized in the following way.

Theorem 7.2 [44] *Let K be an oriented cycle inscribed in a circle none of whose edges passes through the center of the circle. Then, K is suspension-flexible if and only if each edge e of K has a partner-edge e' with the same length as e and opposite sign.* ∎

Corollary 7.1 *If a cycle inscribed in a circle is suspension-flexible, then it has even number of edges.* ∎

Let α denote a positive algebraic number, and τ denote a positive transcendental number. A circle with radius r is denoted by C_r. Since any cycle of $C_\tau(A_+)$ is suspension-flexible by Corollary 5.1, $C_\tau(A_+)$ has no odd cycle. Hence $C_\tau(A_+)$ is a bipartite graph. On the other hand, any connected component of $C_\alpha(A_+)$ is a complete graph. This can be seen as follows: Let xyz be a path of $C_\alpha(A_+)$. Then the algebraic-distance graph on $\{O, x, y, z\}$ is clearly rigid in the plane, and hence $d(x,z) \in A_+$. Hence any connected component of $C_\alpha(A_+)$ is complete. Thus, concerning the chromatic number of $C_r(A_+)$, we have the next result.

$$\chi(C_r(A_+)) = \begin{cases} 2 & (r \notin A_+) \\ \infty & (r \in A_+) \end{cases}$$

Theorem 7.3 [46] *Let C_τ be centered at O, and x, y be two points on C_τ such that $\angle xOy$ has a rational degree measure. Then there is no path in $C_\tau(A_+)$ connecting x, y.*

Proof. Suppose there is a path P connecting x and y. Let ρ be the rotation around O through the angle $\angle xOy$. Since $\angle xOy$ has a rational degree, there is an integer n such that $\rho^n(x) = x$. Then by connecting the paths

$$P, \rho(P), \dots, \rho^{n-1}(P)$$

end by end, we get a cycle K of $C_\tau(A_+)$. Its winding number around O is clearly not zero, a contradiction to Theorem 7.1. ∎

Since each connected component of $C_\alpha(A_+)$ is complete, $C_\alpha(A_+)$ has infinitely many components. (Take a $\theta > 0$ such that $\sin\theta \notin A_+$, and let $p_1, p_2, \dots$ be points on C_α such that $\angle p_i O p_{i+1} = \theta$. Then these points belong to distinct components.) All components of $C_\alpha(A_+)$ are clearly isomorphic. By the above theorem, it follows that $C_\tau(A_+)$ has also infinitely many components. Each component of $C_\tau(A_+)$ is bipartite as already seen. It is never a complete bipartite graph. To see this, consider a path $wxyz$ of $C_\tau(A_+)$ such that $d(w,x) = d(x,y) = d(y,z) < \tau$. Then $d(w,z) \notin A_+$, for otherwise, in the 4-cycle $wxyz$, the edge wz has no partner-edge, contradicting Theorem 7.2. Hence no component of $C_\tau(A_+)$ is complete bipartite.

Problem 7.1 *Is $C_\tau(A_+)$ always isomorphic to $C_\pi(A_+)$? In other words, is there a bijection $f : C_\tau \to C_\pi$ that satisfies the condition $d(x,y) \in A_+ \Leftrightarrow d(f(x), f(y)) \in A_+$?*

8 Integral- and rational-distance graphs

If G is a finite graph represented by a rational-distance graph in the plane, then by blowing up the plane suitably, we can get a representation by an integral-distance graph in the plane. Hence, any finite graph represented by a rational-distance graph can be also represented by an integral-distance graph in the plane. This is no longer true for graphs with infinitely many vertices as seen from the following result.

Let $\infty = |Z_+|$, and $K(\infty, \infty)$ be the complete bipartite graph with both partite-sets of size ∞.

Theorem 8.1 $\dim_{Q_+} K(\infty,\infty) = 2$, $\dim_{Z_+} K(\infty,\infty) = 3$.

Proof. (i) It is clear that $\dim_{Q_+} K(\infty,\infty) > 1$. Let C_r be a circle with radius $r = \sqrt{8}$, and let v_i, $i \in Z_+$ be the sequence of points on C_r

such that $d(v_i, v_{i+1}) = 2$, $i \in Z_+$. Then by elementary geometry, it follows that $d(v_1, v_3) = \sqrt{14}$ which is an *irrational number whose square is a rational.* Applying Ptolemy's theorem to the quadrilateral $v_1v_2v_3v_4$, we have $d(v_1, v_4) = d(v_1, v_3)d(v_2, v_4)/2 - 2 = 5$, a rational. Similarly, it can be proved by induction that (1) if $n - 1$ is odd, then $d(v_1, v_n)$ is a rational, and (2) if $n - 1$ is even then $d(v_1, v_n)$ is an irrational whose square is a rational. Hence the rational-distance graph on the set $\{v_1, v_2, v_3, \ldots\}$ is isomorphic to $K(\infty, \infty)$.

(*ii*) First we show that $K(3, \infty)$ is not an integral-distance graph in the plane. Suppose that $K(3, \infty)$ is represented by the integral-distance graph on $\{u, v, w\} \cup X$ in the plane, where $\{u, v, w\}$ is one partite-set and X is the other infinite partite-set. For a point $x \in X$, $d(u, x), d(v, x)$ are integers, but $d(u, v)$ is not an integer. Hence x, u, v are not collinear. Therefore, no point of X lies on the lines uv, vw, wu. Now, for any $x \in X$,

$$m = |d(u, x) - d(v, x)| \tag{3}$$
$$n = |d(v, x) - d(w, x)| \tag{4}$$

are integers satisfying

$$0 \le m < d(u, v),\ 0 \le n < d(v, w). \tag{5}$$

Let us interpret these in the following way: Every points $x \in X$ can be obtained as an intersection point of two *hyperbolas* (3), (4) with foci u, v and v, w for some m, n satisfying (5). Now, two hyperbolas (3), (4) intersect in at most four points, and there are only finitely many such hyperbolas by (5). Therefore the total number of intersection points must be finite. This contradicts that X is infinite.

Now, let
$A = \{(\cos \frac{n\pi}{\sqrt{3}}, \sin \frac{n\pi}{\sqrt{3}}, 0) \mid n = 1, 2, 3, \ldots\}$,
$B = \{(0, 0, \sqrt{n^2 - 1}) \mid n = 2, 3, 4, \ldots\}$.
Then the integral-distance graph on $A \cup B \subset R^3$ is isomorphic to $K(\infty, \infty)$. ∎

Can every finite graph be represented by an integral-distance graph in the plane [58]?

Theorem 8.2 (Maehara, Ota, Tokushige [53]) *For every finite graph G, there is a point set X on a circle such that $X(Z_+)$ is isomorphic to G.* ∎

This was proved by, first choosing a complete rational-distance graph on a point-set on a circle, and then blowing up the plane suitably so that only the prescribed edges come to integral lengths.

This theorem implies that for a finite connected graph G with more than two vertices,

$$\dim_{Z_+} G = \begin{cases} 1 & \textit{if } G \textit{ is complete} \\ 2 & \textit{otherwise.} \end{cases}$$

Since $dim_1 G$, $sph(G) := \dim_I G$, $\dim_{A_+} G$ are all unbounded for finite graphs G, the fact $\dim_{Z_+} G \leq 2$ is rather curious.

Kemnitz and Harborth [33] conjecture that every planar graph can be embedded in the plane in such a way that each edge is a straight line segment of integer length.

Let K_∞ denote the complete graph with countably infinite vertices.

Theorem 8.3 (Klee, Wagon [35]) *Let C_r be a circle of radius r. Then the following three are equivalent.*

(1) $r^2 \in Q_+$.

(2) $C_r(Q_+)$ *contains a subgraph isomorphic to* K_∞.

(3) $C_r(Q_+)$ *contains a* 3*-cycle.* ▮

There is a graph with countably many vertices that is not an integral-distance graph in any R^n. Let $K_\infty - e$ denote the graph ontained from K_∞ by removing an edge.

Theorem 8.4 *The graph $K_\infty - e$ is not an integral-distance graph in any dimension, while $K_\infty - e$ is a rational-distance graph in the plane.*

Proof. (i) Suppose that $K_\infty - e$ is isomorphic to $X(Z_+)$ for some $X \subset R^n$. Then there are two points $p, q \in X$ such that $d(p,q)$ is not an integer. Since the integral-distance graph on $X - \{p\}$ is complete, $X - \{p\}$ lies on a line by Theorem 1.2. Similarly, $X - \{q\}$ lies on a line. Therefore X lies on a line. In this case $d(p,q)$ must be an integer, a contradiction.

(ii) By Theorem 8.3, there is a set X of size ∞ on the circle $\{(x,y,0) \mid x^2 + y^2 = 2\}$ in R^3 such that $X(Q_+)$ is complete. Let $p = (0,0,\sqrt{2})$, $q = (0,0,-\sqrt{2})$. Then $Y := \{p,q\} \cup X$ lies on the sphere of radius $\sqrt{2}$ in R^3, and $Y(Q_+)$ is isomorphic to $K_\infty - e$. Take a point $u \in X$, and let f denote the inversion of R^3 with respect to the unit sphere centered at u. Notice that since Δuvw is similar to $\Delta uf(w)f(v)$ for $v, w \in R^3 - \{u\}$, the distance $d(f(v), f(w))$ is calculated as

$$d(f(v), f(w)) = \frac{d(v, w)}{d(u, v) \cdot d(u, w)}.$$

Hence, it follows that the rational-distance graph on $f(Y - \{u\})$ is isomorphic to $K_\infty - e$. Since $f(Y - \{u\})$ lies on a plane, $K_\infty - e$ is a rational-distance graph on the plane. ▮

Erdős conjectured that $R^n(Q_+)$ is countably chromatic, and this conjecture was proved by Komjáth [36]. From this the next follows.

Theorem 8.5 *Let $X \subset R^n$ be an infinite set such that $X(Q_+)$ is complete. Then X is a countable set.*

Let us give a direct proof of this fact. In 1-dimensional case ($n = 1$), the assertion is clearly true. Assuming that the assertion is true up to $(n-1)$-dimensional case, let us consider n-dimensional case. We may suppose that X contains n points $p_1, p_2, \ldots, p_n$ that span an $(n-1)$-dimensional simplex. Then, for any point $x \in X$, $d(x, p_i)$, $i = 1, \ldots n$, are rationals. Thus each points $x \in X$ can be obtained as an intersection point of n spheres in R^n with centers $p_i, i = 1, \ldots, n$ and rational radii. Since the n points $p_1, \ldots, p_n$ span an $(n-1)$-simplex, such n spheres intersect in at most two points. Since the number of combination of n rational radii is countable, the set of intersection points is also countable. Hence X is countable. ▮

Thus, a complete graph with uncountably many vertices is not a rational-distance graph in any dimension.

Problem 8.1 *Is there a countably infinite graph that is not a rational-distance graph in any dimension?*

Problem 8.2 *Is there a countably infinite graph that is not a rational-distance graph in the plane?*

Problem 8.3 *Is there a finite graph that is not a rational-distance graph in the plane?*

Example 8.1 $\dim_{Q_+} K(3,3,3) = 2$.

Let ρ denote the counter-clockwise rotation around the origin in R^2 through the angle 120°. Take 9 points in R^2 in the following way:

$$\begin{array}{lll} u_1 = (2\sqrt{3},0), & u_2 = (11\sqrt{3},0), & u_3 = (-13\sqrt{3},0) \\ v_1 = \rho(u_1), & v_2 = \rho(u_2), & v_3 = \rho(u_3) \\ w_1 = \rho(v_1), & w_2 = \rho(v_2), & w_3 = \rho(v_3) \end{array}$$

Then the Q_+-distance graph of these 9 points is isomorphic to $K(3,3,3)$.

Example 8.2 $\dim_{Z_+}(P_\infty \times K_2) = \dim_{Q_+}(P_\infty \times K_2) = 2$, *where* $P_\infty \times K_2$ *is the Cartesian product of the two-way-infinite path* P_∞ *and* K_2. *See* Figure 3.

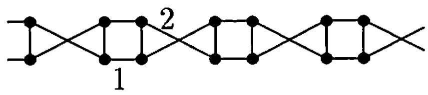


Figure 3: $P_\infty \times K_2$

Example 8.3 $\dim_{Q_+} K(\infty,\infty,\infty) \leq 3$, $\dim_{Q_+} K(\infty,\infty,\infty,\infty) \leq 4$.

Proof. (i) Represent $K(\infty,\infty)$ as a rational-distance graph on a point-set on the circle of radius $\sqrt{8}$ centered at the origin in the xy-plane as in the proof of Theorem 8.1, and then take infinite points $(0,0,\sqrt{n^2-8})$, $n = 9,10,11,\ldots$ on the z-axis.

(ii) Let X, Y be point-sets on the circles

$$\{(x,y,0,0) \in R^4 \mid x^2+y^2=8\}, \ \{(0,0,z,w) \in R^4 \mid z^2+w^2=8\},$$

respectively, each representing $K(\infty,\infty)$ as in Theorem 8.1. Then the rational-distance graph on $X \cup Y$ is isomorphic to $K(\infty,\infty,\infty,\infty)$. ∎

Following Stewart [65], let us call an n-point-set X in the plane an *n-pack* if

(1) no three points of X lie on a line,
(2) no four points of X lie on a circle, and
(3) $X(Z_+)$ is complete.

It is known that there is a 6-pack, see Harborth [25], Harborth and Kemnitz [26]. The following is an open problem.

Problem 8.4 [22,35,65] *Is there a 7-pack?*

The next theorem relates the existence of an n-pack to the Q_+-dimension of a complete n-partite graph. For $n \geq 1$, let

$$a_n = \binom{n-1}{2} + \binom{n-1}{3} + 1.$$

Theorem 8.6 *If* $\dim_{Q_+} K(a_1, a_2, \ldots, a_n) \leq 2$ *then there is an n-pack.*

For example, if $K(1,1,2,5,11,21,36)$ can be represented by a rational-distance graph in the plane, then there is a 7-pack.

Proof. Suppose that there are n disjoint sets $Y_i \subset R^2$ of size a_i, $i = 1,2,\ldots,n$ such that the rational-distance graph on $Y = Y_1 \cup Y_2 \cup \ldots \cup Y_n$ is isomorphic to $K(a_1, a_2, \ldots, a_n)$. Then, for each $i = 1,2,\ldots,n$, we can choose a point $x_i \in Y_i$ so that $\{x_1, x_2, \ldots, x_n\}$ satisfies the conditions (1),(2) of n-pack. To see this, suppose that we could choose $\{x_1, \ldots, x_k\}$, $k < n$, satisfying (1) and (2). Then $\{x_1, \ldots, x_k\}$ determines $\binom{k}{2}$ lines and $\binom{k}{3}$ circles. None of these $\binom{k}{2}$ lines can contain more than one point of Y_{k+1}. (For otherwise, some two of Y_{k+1} become adjacent in $Y(Q_+)$.) Similarly, none of these $\binom{k}{3}$ circles can contain more than one point of Y_{k+1}. (For otherwise, applying Ptolemy's theorem, Y_{k+1} contains an adjacent pair.) Since Y_{k+1} contains $a_{k+1} = \binom{k}{2} + \binom{k}{3} + 1$ points, there is at least one point in Y_{k+1} that lies on none of $\binom{k}{2}$ lines and $\binom{k}{3}$ circles. Hence we can choose $x_i \in Y_i$ so that $X = \{x_1, \ldots, x_n\}$ satisfies (1) and (2). Since $d(x_i, x_j) \in Q_+$, by dilating the set X suitably, we get an n-pack. ∎

9 $\sqrt{Q}$-distance graphs

Let $\sqrt{Q}$ denote the set of positive real numbers whose squares are rational numbers. Since the distance between a pair of rational points in R^n belongs to $\sqrt{Q}$, this distance set is of some interest. A set $X \subset R^n$ is called a *$\sqrt{Q}$-set* if $X(\sqrt{Q})$ is a complete graph. Thus a subset of $Q^n \subset R^n$ is a $\sqrt{Q}$-set.

Answering to a problem in [51], Kumada proved the following result by applying the theory of p-adic number field.

Theorem 9.1 (Kumada [37]) *Every $\sqrt{Q}$-set in R^n is isometrically embeddable into Q^{n+3}.* ∎

Since the integer 7 cannot be written as the sum of three squares, the $\sqrt{Q}$-set $\{0, \sqrt{7}\}$ in R^1 is not isometrically embeddable in Q^3. Hence, the above theorem is best possible in general.

Example 9.1 *Let $\{x, p_0, p_1, p_2\}$ and $\{y, p_0, p_1, p_2\}$ be two $\sqrt{Q}$-sets in the plane. If p_0, p_1, p_2 are not collinear, then the 5-point-set $\{x, y, p_0, p_1, p_2\}$ is also a $\sqrt{Q}$-set, see* Figure 4(a).

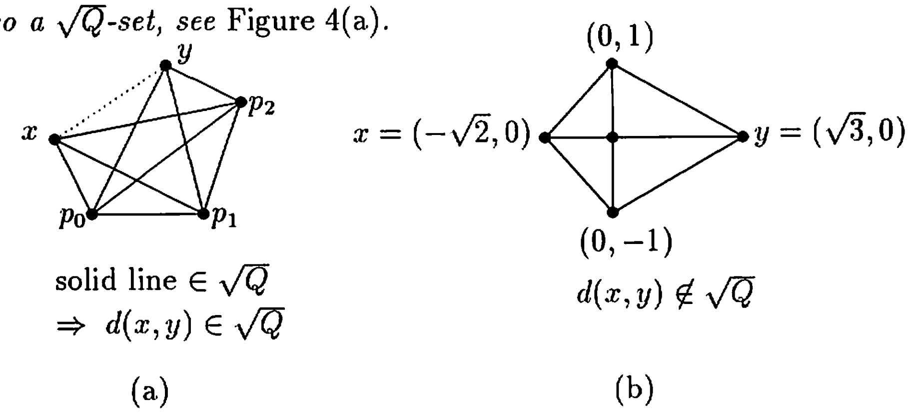


Figure 4: Example

Proof. We may suppose that $p_0 = O$, the origin. Then we can put $x = \lambda_1 p_1 + \lambda_2 p_2$. We show that $\lambda_1, \lambda_2 \in Q$. First, notice that $\|u\|^2, \|v\|^2, d(u,v)^2 \in Q$ implies $u \cdot v \in Q$, where $\cdot$ denotes the inner product. Now, in the simultaneous equation

$$\begin{aligned} p_1 \cdot x &= \lambda_1 p_1 \cdot p_1 + \lambda_2 p_1 \cdot p_2 \\ p_2 \cdot x &= \lambda_2 p_2 \cdot p_1 + \lambda_2 p_2 \cdot p_2 \end{aligned}$$

on λ_1, λ_2, the coefficients $p_1 \cdot p_1, p_1 \cdot p_2, p_2 \cdot p_1, p_2 \cdot p_2$ are all rationals. Since O, p_1, p_2 are not collinear, we have

$$\begin{vmatrix} p_1 \cdot p_1 & p_1 \cdot p_2 \\ p_2 \cdot p_1 & p_2 \cdot p_2 \end{vmatrix} \neq 0,$$

and hence, by Cramer's rule, we have $\lambda_1, \lambda_2 \in Q$. Similarly, we can put $y = \mu_1 p_1 + \mu_2 p_2$ $(\mu_1, \mu_2 \in Q)$. Then $d(x,y)^2 = \|(\lambda_1 - \mu_2)p_1 + (\lambda_2 - \mu_2)p_2\|^2 \in Q$. ∎

Note that the condition p_0, p_1, p_2 are not collinear is necessary in this example as seen from Figure 4(b).

A point-set $X \subset R^n$ is said to be in *general position* if every $k+1$ points of X span a k-dimensional simplex for $k \leq n$. Similarly to the above example, the following holds.

Theorem 9.2 [48] *Let X, Y be two $\sqrt{Q}$-sets in R^n. If they share at least $n+1$ points in general position, then $X \cup Y$ is a $\sqrt{Q}$-set.* ∎

An abstract graph G with N vertices is said to be *n-valid* if for any set $X \in R^n$ of N points *in general position*, the condition

(∗) $X(\sqrt{Q})$ contains a subgraph isomorphic to G

implies that X is a $\sqrt{Q}$-set.

For example, the complete graph K_N is n-valid for any n. It follows from Theorem 9.2 that $K_{n+3} - \{\text{an edge}\}$ is n-valid.

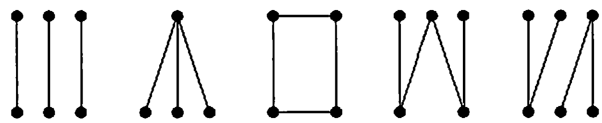
Let us denote by G_m a graph with m vertices.

Theorem 9.3 [48]

(1) *For any $m \leq n+2$, G_m is m-valid if and only if $G_m = K_m$.*

(2) *G_{n+3} is n-valid if and only if $G_{n+3} = K_{n+3}$ or $G_{n+3} = K_{n+3} - \{an\ edge\}$.*

(3) *G_{n+4} is n-valid if and only if the complement of G contains none of the graphs in* Figure 5. ∎



$3K_2$ $K(1,3)$ C_4 P_5 $2P_2$

Figure 5: Theorem 9.3

Similarly to Theorem 8.3, the next holds.

Theorem 9.4 *Let C_r be a circle of radius r. Then the following three are equivalent.*

(1) $r \in \sqrt{Q}$.

(2) *$C_r(\sqrt{Q})$ contains a subgraph isomorphic to K_∞.*

(3) *$C_r(\sqrt{Q})$ contains a 3-cycle.*

Proof. Since $C_r(Q_+)$ is a subgraph of $C_r(\sqrt{Q})$, (1) implies (2) by Theorem 8.3. So, we show that (3) implies (1). Suppose that C_r contains three points x, y, z such that all $a = d(x,y), b = d(y,z), c = d(z,x)$ belong to $\sqrt{Q}$. Let $\theta = \angle xyz$. Then by the cosine theorem, $\cos\theta = (a^2 + b^2 - c^2)/(2ab)$. Hence $\cos^2\theta \in Q_+$. By the sine theorem, $2r = c/\sin\theta$. Since $\sin^2\theta = 1 - \cos^2\theta \in Q_+$, (3) implies (1). ∎

The next theorem shows that $\sqrt{Q}$-dimension $\dim_{\sqrt{Q}} G$ for finite graphs G is also unbounded.

Theorem 9.5 $\dim_{\sqrt{Q}} K(1, \underbrace{2, 2, \ldots, 2}_{n}) = n.$

Proof. (*i*) First we show that $\dim_{\sqrt{Q}} K(1, \underbrace{2, 2, \ldots, 2}_{n}) \geq n$. To this end, we prove the assertion that if a point-set X in some R^N satisfies

$$X(\sqrt{Q}) \cong K(1, \underbrace{2, 2, \ldots, 2}_{n}),$$

then X contains a subset Y that spans an n-dimensional flat and satisfies $Y(\sqrt{Q}) \cong K_{n+1}$.

If $n = 1$, then this is clearly true. Suppose that the assertion is true for $n = k$. Let $X \cup \{p, q\}$ be a point-set in some R^N such that

$$(X \cup \{p,q\})(\sqrt{Q}) \cong K(1, \underbrace{2, 2, \ldots, 2}_{k+1}) \text{ and } X(\sqrt{Q}) \cong K(1, \underbrace{2, 2, \ldots, 2}_{k}).$$

Then, by the inductive hypothesis, X contains a Y that spans a k-dimensional flat and $Y(\sqrt{Q}) \cong K_{k+1}$. Then $Y \cup \{p\}$ and $Y \cup \{q\}$ are both $\sqrt{Q}$-sets. If both p, q lie on the flat spanned by Y, then by Theorem 9.2, $Y \cup \{p, q\}$ is a $\sqrt{Q}$-set, which implies that $d(p,q) \in \sqrt{Q}$, a contradiction. Hence, one of p, q, say, p does not lie on the flat spanned by Y. Then $Y \cup \{p\}$ spans a $(k+1)$-dimensional flat, and $(Y \cup \{p\})(\sqrt{Q}) \cong K_{k+2}$.

(*ii*) Now, the $\sqrt{Q}$-distance graph on the $2n+1$ points

$$\begin{array}{l}
(0,0,0,0,\ldots,0),\\
(\sqrt{2},0,0,\ldots,0), (\sqrt{3},0,0,\ldots,0),\\
(0,\sqrt{2},0,\ldots,0), (0,\sqrt{3},0,\ldots,0),\\
\quad\ldots\\
(0,0,0,\ldots,\sqrt{2}), (0,0,0,\ldots,\sqrt{3})
\end{array}$$

in R^n is isomorphic to $K(1,\underbrace{2,2,\ldots,2}_{n})$. This completes the proof. ∎

References

[1] M. Aigner and M. Ziegler, *Proofs from THE BOOK, second edition* (Springer-Verlag 2001 New York).

[2] N. H. Anning and P. Erdős, Intgeral distances, *Bull. Amer. Math. Soc.* 51(1945) pp.598–600.

[3] L. Asimov and B. Roth, The rigidity of graphs, *Trans. Amer. Math. Soc.* 245(1978) pp.279–289.

[4] E. D. Bolker and B. Roth, When is a bipartite graph a rigid framework?, *Pacific J. Math.* 90(1980) pp.27–44.

[5] H. Breu and D. G. Kirkpatrick, On the complexity of of recognizing intersection and touching graphs of discs, in: F. J. Brandenburg (Ed.), Graph drawing, Proceedings Graph Drawing '95, Passau, September 1995, *Lecture Notes in Computer Science 1027*, pp. 88-98 (Springer-Verlag 1996 Berlin)

[6] H. Breu and D. G. Kirkpatrick, Unit disk graph recognition is NP-hard, *Comput. Geom.* 9(1998) pp.3-24.

[7] K. B. Chilakamarri, Unit distance graph in rational n-space, *Discrete Math.* 69(1978) pp.213-218.

[8] K. B. Chilakamarri, Unit-distance graphs in Minkowski metric spaces *Geom. Dedicata* 37(1991) pp.345-356.

[9] K. B. Chilakamarri, The unit-distance graph problem: A brief survey and some new results, *Bull. ICM* 8(1993) pp.39–60.

[10] R. Connelly, The rigidity of suspensions, *J. Differential Geom.* 13(1978) pp.399-408.

[11] R. Connelly, A counterexample to the rigidity conjecture for polyhedra, *Inst. Hautes Études Sci. Publ. Math.* 47(1978) pp.333-338.

[12] R. Connelly, I. Ku. Sabitov and A. Walz, The bellow conjecture, *Beiträge zur Algebra und Geometrie* 38(1997) pp.1-10.

[13] H. T. Croft, K. J. Falconer and R. K. Guy, *Unsolved Problems in Geometry* (Springer-Verlag 1991 New York).

[14] W. A. Deuber and X. Zhu, The chromatic numbers of distance graphs, *Discrete Math.* 165/166 (1997) 195-204.

[15] R. B. Eggleton, P. Erdős and D. K. Skilton, Coloring the real line, *J. Combin. Theory B* 39(1985) pp.86–100.

[16] R. B. Eggleton, P. Erdős and D. K. Skilton, Reserch problems 77, *Discrete Math.* 58(1986) p.323.

[17] R. B. Eggleton, P. Erdős and D. K. Skilton, Coloring prime distance graphs, *Graphs and Combinatorics* 6(1990) pp.17–32.

[18] P. Erdős, F. Harary and W. T. Tutte, On the dimension of a graph, *Mathematica* 12(1965) pp.118-122.

[19] S. V. Gervacio and H. Maehara, Subdividing a graph toward a unit-distance graph in the plane, *European J. Combin.* 21(2000) pp.223-229.

[20] H. Gluck, Almost all simply connected closed surface are rigid, *Lecture Notes in Math. 438 Geometric Topology*, pp.225-239 (Springer-Verlag 1975 Berlin).

[21] I. Gutman, A definition of dimensionality and distance for graphs, *Geometric Representation of Relational Data* (J. C. Lingoes, ed.) pp.713-723, (Mathesis Press 1977 Michigan).

[22] R. K. Guy, *Unsolved Problems in Number Theory*, Springer-Verlag, New York 1981.

[23] H. Hadwiger and H. Debrunner, *Combinatorial Geometry in the Plane* (Holt, Reinhalt and Winston 1964 New York).

[24] P. Hanlon, Counting interval graphs, *Trans. Amer. Math. Soc.* 271(1982) pp.383-426.

[25] H. Harborth, Regular points with unit distances, *Intuitive Geometry* (Ed. by K. Böröczky and G. Fejes Tóth) pp.239–253 (North-Holland 1987 Budapest).

[26] H. Harborth and A. Kemnitz, Diameters of integral point sets, *Intuitive Geometry* (Ed. by K. Böröczky and G. Fejes Tóth) pp.255–266 (North-Holland 1987 Budapest).

[27] T. H. Havel, The combinatorial distance geometry approach to the calculation of molecular conformation, Ph D Thesis, Group in Biophysics, University of California, Berkeley, 1982.

[28] Petr Hliněný and Jan Kratochvíl, Representing graphs by disks and balls (a survey of recognition-complexity results), *Discrete Math.* 229(2001) pp.101-124.

[29] M. Homma, M. Kato and H. Maehara, On extremal points of a real algebraic function, *Mathematica Japonica* 35(1990) pp.57–64.

[30] M. Homma and H. Maehara, Algebraic distance graphs and rigidity, *Trans. Amer. Math. Soc.* 319(1990) pp.561–572.

[31] A. Kemnitz and M. Marangio, Chromatic numbers of integer distance graphs, *Discrete Math.* 233 (2001) 239-246.

[32] A. Kemnitz and M. Marangio, Colorings and list colorings of integer distance graphs, *Proceedings of the Thirty-second Southeastern International Conference on Combinatorics, Graph Theory and Computing* (Baton Rouge, LA, 2001) Congr. Numer. 151(2001) 75–84.

[33] A. Kemnitz and H. Harborth, Plane integral drawings of planar graphs, *Discrete Math.* 236(2001) 191-195.

[34] A. B. Kempe, On a general method of describing plane curve of the nth degree by linkwork, *Proc. London Math. Soc.* 7(1876) pp.213-216.

[35] V. Klee and S. Wagon, *Old and New Unsolved Problems in Plane Geometry and Number Theory* (MAA 1991).

[36] P. Komjáth, A decomposition theorem for R^n, *Proc. Amer. Math. Soc.* 120(1994) pp.921-927.

[37] T. Kumada, Isometric embeddings of metric Q-vector space into Q^N, *European J. Combin.* 19(1998) pp.701-706.

[38] H. Maehara, On time graphs, *Discrete Math.* 32(1980) pp.281–289.

[39] H. Maehara, Space graphs and sphericity, *Discrete Appl. Math.* 7(1984) pp.55-64.

[40] H. Maehara, On the Euclidean dimension of a complete multipartite graph, *Discrete Math.* 72(1988) pp.285-289.

[41] H. Maehara, Dispersed points and geometric embedding of complete bipartite graphs, *Discrete Comput. Geom.* 6(1991) pp.57-67.

[42] H. Maehara, Distances in a unit distance graph in the plane, *Discrete Appl. Math.* 31(1991) pp.193-200.

[43] H. Maehara, A rigid unit-bar-framework without triangle, *Math. Japonica* 36(1991) pp.681-683.

[44] H. Maehara, On a special case of Connelly's suspension theorem, *Ryukyu Math. J.* 4(1991) pp.35-45.

[45] H. Maehara, Extending a flexible unit-bar-framework to a rigid one, *Discrete Math.* 108(1992) pp.167-174.

[46] H. Maehara, Distance graphs in Euclidean space, *Ryukyu Math. J.* 5(1992) pp.33-51.

[47] H. Maehara, A note on rigid graphs in the plane, Proceedings of JSPS Workshop on Graph Theory and Combinatorics, 55-59 (Ateneo de Manila Univ. Press 1993 Philippines).

[48] H. Maehara, On $\sqrt{Q}$-distances, *European J. Combinatorics* 17(1996) pp.271–277.

[49] H. Maehara, A minimal rigid graph in the plane that is not constructible by ruler and compass, *Yokohama J. Math.* 10(1998) pp.109-111.

[50] H. Maehara, Vector fields and quadratic surfaces, *Ryukyu Math. J.* 11(1998) pp.53-63.

[51] H. Maehara, Embedding a set of rational points in lower dimensions, *Discrete Math.* 192(1998) pp.273-279.

[52] H. Maehara and K. Chinen, An infinitesimally rigid unit-bar-framework in the plane which contains no triangle, *Ryukyu Math. J.* 8(1995) pp.37–41.

[53] H. Maehara, K. Ota and N. Tokushige, Every graph is an integral distance graph in the plane, *J. Combin. Th. A*, 80(1997) pp.290-294.

[54] H. Maehara, J. Reiterman, V. Rödl and E. Šiňajova, Embedding trees in Euclidean spaces, *Graphs and Combinatorics* 4(1988) pp.43–47.

[55] H. Maehara and N. Tokushige, A spatial Unit-bar-framework which is rigid and triangle-free, *Graphs and Combinatorics* 12(1996) pp.341–344.

[56] H. Maehara and N. Tokushige, When does a planar bipartite graph admit a continuous deformation? *Theoretical Computer Science* 263(2001) pp.345-354.

[57] L. Moser and W. Moser, Problem 10, *Canad. Math. Bull.* 4(1961) pp.187–189.

[58] W. Moser, Problems, problems, problems, *Discrete Appl. Math.* 31 (1991) pp.201–225.

[59] W. Moser and J. Pach, Recent developments in combinatorial geometry, in *New Trends in Discrete and Computational Geometry* (ed. J. Pach) (Springer-Verlag 1993 New York).

[60] B. Roth, Rigid and flexible frameworks, *Amer. Math. Monthly* 88(1981) pp.6-21.

[61] I. Z. Ruzsa, Zs. Tuza and M. Voigt, Distance graphs with finite chromatic number, *J. Combin. Theory Ser. B* 85(2002) 181–187.

[62] I. Ku. Sabitov, On the problem of volume conservation for flexible polyhedra (Russian), *Uspekhi matem. nauk* 50(1995) pp.223–224.

[63] I. Ku. Sabitov, The volume as a metric invariant of polyhedra, *Discrte Comput. Geom.* 20(1998) pp.405-425.

[64] I. Stewart, *Galois Theory* (Chapman and Hall 1973 London).

[65] I. Stewart, *Another Fine Math You've Got Me into ...* (Freeman and Company 1992 New York).

[66] M. Voigt and H. Walther, On the chromatic number of special distance graphs, *Discrete Math.* 97(1991) 395-397.

[67] W. Whiteley, Infinitesimal motions of a bipartite framework, *Pacific J. Math.* 110(1984) pp.233-255.

[68] W. Wunderlich, Projective invariance of shaky structures, *Acta Mech.* 42(1982) pp.171-181.