

On the Perimeter of a Triangle in a Minkowski Plane

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A *Minkowski plane* is a plane endowed with a norm that is not necessarily Euclidean. The Minkowski length of a curve in a Minkowski plane is defined in a manner analogous to the way it is defined in the Euclidean case. A circle of radius r in a Minkowski plane with norm $\|\cdot\|$ is the locus of a point X satisfying $\|X - P\| = r$ for a fixed “center” P . It is a convex curve that is symmetric with respect to its center. Golab’s theorem shows that the Minkowski length of a circle of unit radius lies between 6 and 8 (see [2]). Let L denote the Minkowski length. We establish the following:

Theorem 1. *Let ABC be a triangle in a Minkowski plane, and suppose that one is the minimum radius of a circle that encloses the triangle ABC . Then $4 \leq L(ABC) \leq 6$.*

Since there is an arbitrarily thin triangle with two sides close to a diameter of the circle, the lower bound 4 cannot be improved in *any* Minkowski plane. For the case of the Euclidean norm it is not difficult to see that the upper bound 6 can be reduced to $3\sqrt{3}$ [1]. However, in the max-norm $\|(x, y)\|_\infty = \max\{|x|, |y|\}$ any three corners of the unit circle (which is a Euclidean square) form a triangle with perimeter 6. Hence the upper bound of the theorem cannot be improved in general.

Proof. Let Γ be a circle circumscribed about ABC in the sense that it has smallest radius among all circles that contain A, B , and C in their convex hulls. It is easily seen that either the center of the circle is inside the triangle or on one of its sides, or there exists a congruent circumscribed circle with this property. So we may assume that the center of Γ is the origin O and lies

inside ABC or on one of its sides. The circle Γ is supposed to have radius 1. Since $\|A - B\| \leq 2$, $\|B - C\| \leq 2$, and $\|C - A\| \leq 2$, we have $L(ABC) \leq 6$.

We have to show that $L(ABC) \geq 4$. If $\|A - B\| = 2$, then

$$\|A - B\| + \|B - C\| + \|C - A\| \geq \|A - B\| + \|B - A\| = 4.$$

Hence we consider the case that $\|A - B\|$, $\|B - C\|$, and $\|C - A\|$ are all less than 2. Notice that in this case O lies inside ABC , while A , B , and C all lie on Γ .

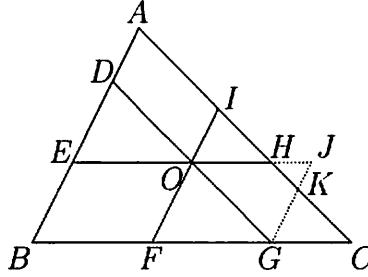


Figure 1.

Let D, E, F, G, H , and I be points on the sides of ABC such that $ADOI$, $BFOE$, and $CHOG$ are parallelograms (see Figure 1). Then

$$\begin{aligned} \|A - D\| + \|A - I\| &= \|A - D\| + \|D\| \geq \|A\| = 1, \\ \|B - E\| + \|B - F\| &= \|B - E\| + \|E\| \geq \|B\| = 1, \\ \|C - G\| + \|C - H\| &= \|C - G\| + \|G\| \geq \|C\| = 1. \end{aligned}$$

To prove that $L(ABC) \geq 4$, it is enough to show that

$$\|D - E\| + \|F - G\| + \|H - I\| > 1.$$

Among the similar triangles DEO , OFG , and IOH , we may suppose that DEO is the biggest and IOH the smallest. Then $\|D - E\| \geq \|F\| \geq \|I\|$. Let J be the point such that $OFGJ$ is a parallelogram, and let K be the intersection of GJ and AC . Then the triangle KGC is a translate of IOH , and

$$\begin{aligned} \|D - E\| &\geq \|F\| = \|J - G\| > \|J - K\|, \\ \|F - G\| &= \|J\|, \\ \|H - I\| &= \|C - K\|. \end{aligned}$$

Therefore,

$$\begin{aligned}\|D - E\| + \|F - G\| + \|H - I\| &> \|J - K\| + \|J\| + \|C - K\| \\ &\geq \|C\| = 1.\end{aligned}$$

■

Remark. The lower bound in the theorem is valid for any closed curve instead of a triangle. More precisely, let Ω be a simple closed curve in a Minkowski plane, and let one be the smallest radius of a circle that encloses Ω . Then $L(\Omega) \geq 4$.

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References

- [1] H. Maehara, On the total edge-length of a tetrahedron, *Amer. Math. Monthly* **108** (2001) 237-243.
- [2] A. C. Thompson, *Minkowski Geometry*, Cambridge University Press, New York 1996.

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