Observing an angle from various viewpoints

Yoichi Maeda[1] and Hiroshi Maehara[2]

[1] Tokai University, Hiratsuka, Kanagawa, 259-1292 Japan
[2] Ryukyu University, Nishihara, Okinawa, 903-0213 Japan

Abstract. Let AOB be a triangle in R^3. When we look at this triangle from various viewpoints, the angle $\angle AOB$ changes its appearance, and its 'visual size' is not constant. We prove, nevertheless, that the average visual size of $\angle AOB$ is equal to the true size of the angle when viewpoints are chosen at random on the surface of a sphere centered at O. We also present a formula to compute the variance of the visual size.

1 Introduction

Let $\angle AOB$ be a fixed angle determined by three points O, A, B in the three dimensional Euclidean space R^3. When we look at this angle, its appearance changes according to our viewpoint. Let us denote by

$$\angle_P AOB$$

the dihedral angle of the two faces OAP, OBP of the (possibly degenerate) tetrahedron $POAB$, see Figure 1. This angle $\angle_P AOB$ is called the *visual angle* of $\angle AOB$ from the viewpoint P, and its size (measure) is called the *visual size* of $\angle AOB$ from P. For an angle with fixed size ω $(0 < \omega < \pi)$, its visual size can

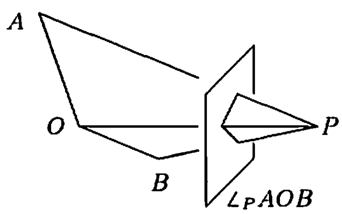

Fig. 1. A visual angle

vary from 0 to π depending on the viewpoint.

For a given angle $\angle AOB$ in R^3, take a random point P distributed uniformly on the unit sphere centered at O. Then the size of $\angle_P AOB$ is a random variable, which is called the *random visual size* of $\angle AOB$. It will be clear that this random visual size depends only on the size of $\angle AOB$. So, for an angle $\angle AOB$ of size $\omega (0 < \omega < \pi)$, we may denote its random visual size by Θ_ω.

Theorem 1. *For any angle of size $\omega\,(0 < \omega < \pi)$, the expected value of the random visual size Θ_ω is equal to ω, that is,* $\mathbf{E}(\Theta_\omega) = \omega$.

Thus, when we observe an angle from several viewpoints, each chosen at random, the average visual size is approximately equal to the true size.

For potential applications of Theorem 1, let us present a formula to compute the variance $\mathbf{V}(\Theta_\omega)$ of Θ_ω.

Theorem 2. $\mathbf{V}(\Theta_\omega) = \mathbf{V}(\Theta_{\pi-\omega})$ *and*

$$\mathbf{V}(\Theta_\omega) = \int_0^\pi \int_0^\pi \left(\frac{\pi}{2} - \tan^{-1}\left(\frac{\cot\omega \sin y + \cos x \cos y}{\sin x}\right)\right)^2 \frac{\sin y}{2\pi} dx dy - \omega^2.$$

Though this looks complicated, we can easily compute the variance from this double integral by computer. The following table shows the values of $\mathbf{V}(\Theta_\omega)$ for $\omega = k\pi/12$, $k = 1, 2, 3, 4, 5, 6$.

ω	$\pi/12$	$2\pi/12$	$3\pi/12$	$4\pi/12$	$5\pi/12$	$6\pi/12$
$\mathbf{V}(\Theta_\omega)$	0.0699	0.1874	0.3042	0.3988	0.4595	0.4804

2 Proof of Theorem 1

Let $\angle AOB$ be an angle of size ω, and let P be a random point on the unit sphere S^2 centered at O in R^3. We may suppose that A, B lie on S^2. Then the spherical distance $\widehat{AB}$ of A and B is equal to ω. (We denote the shortest geodesic connecting A, B and its length by the same notation $\widehat{AB}$.) Notice that $\angle_P AOB$ is equal to the interior angle $\angle P$ of the spherical triangle ΔAPB. Since P is a *random* point on S^2, we have $\Theta_\omega = \angle P$.

Now, consider the (polar) dual triangle $\Delta A^*B^*P^*$ of the spherical triangle ΔABP:

$$\Delta A^*B^*P^* = H(A) \cap H(B) \cap H(P),$$

where $H(A)$ denotes the hemisphere with pole A. Let $\angle P^*$ $(=: \tau)$ denote the interior angle at P^* of this spherical triangle $\Delta A^*B^*P^*$, see Figure 2. Then, by the duality (see, e.g. [1] Chapter 2), we have

$$\widehat{AB} + \angle P^* = \pi, \quad \widehat{A^*B^*} + \angle P = \pi.$$

Hence $\angle P^* = \pi - \omega$ and $\widehat{A^*B^*} = \pi - \Theta_\omega$. Let $\Lambda = H(A) \cap H(B)$. Then the angle of the lune Λ is equal to $\pi - \omega$, and its area is equal to $2(\pi - \omega)$. Note that since P is a random point on S^2, the boundary $\partial H(P)$ of $H(P)$ is a random great circle, and $\widehat{A^*B^*} = \partial H(P) \cap \Lambda$.

Here we recall Santaló's chord theorem:

Theorem[2] *Let $\Omega \subset S^2$ be a subset obtained as the intersection of a number of hemispheres. Let G be a random great circle, and let φ be the length of the arc $G \cap \Omega$. ($G \cap \Omega = \emptyset$ implies $\varphi = 0$.) Then* $\mathbf{E}(\varphi) =$*area*$(\Omega)/2$.

Applying this theorem, we have

$$\mathbf{E}(\widehat{A^*B^*}) = \frac{\text{area}(\Lambda)}{2} = \pi - \omega.$$

Therefore $\mathbf{E}(\pi - \Theta_\omega) = \pi - \omega$, and $\mathbf{E}(\Theta_\omega) = \omega$. This proves Theorem 1.

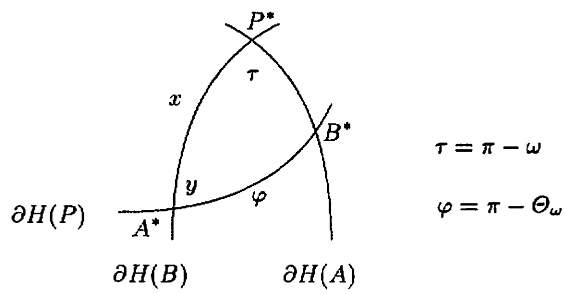

Fig. 2. The polar dual $\triangle P^*A^*B^*$

3 Proof of Theorem 2

Since $\pi - \Theta_\omega$ can be regarded as $\Theta_{\pi-\omega}$, Θ_ω and $\Theta_{\pi-\omega}$ have the same variance.

Now, let $\angle AOB$ be an angle of size ω with $\overline{OA} = \overline{OB} = 1$, and let P be a random point on the unit sphere S^2 centered at O in R^3 as in the proof of Theorem 1. Let $\triangle A^*B^*P^*$ be the polar dual of $\triangle ABP$, and let $\Lambda = H(A) \cap H(B)$, $\tau = \pi - \omega$. Let φ denote the length of the arc $\widehat{A^*B^*} = \partial H(P) \cap \Lambda$. Then by Santalo's chord theorem, $\mathbf{E}(\varphi) = \tau$. Since $\varphi = \pi - \Theta_\omega$ by duality, we have $\mathbf{V}(\varphi) = \mathbf{V}(\pi - \Theta_\omega) = \mathbf{V}(\Theta_\omega)$. So, we consider the variance of φ. Let $y = \angle A^*$ and $x = \widehat{P^*A^*}$, see Figure 2. By applying the spherical cosine law for angles (see e.g. [1]) to $\triangle P^*A^*B^*$,

$$\cos \angle B^* = -\cos\tau\cos y + \sin\tau\sin y\cos x,$$
$$\cos\tau = -\cos y\cos\angle B^* + \sin y\sin\angle B^*\cos\varphi.$$

Hence

$$\begin{aligned}
\cos\varphi &= \frac{\cos\tau + \cos y\cos\angle B^*}{\sin y\sin\angle B^*} \\
&= \frac{\cos\tau + \cos y(-\cos\tau\cos y + \sin\tau\sin y\cos x)}{\sin y\sin\angle B^*} \\
&= \frac{\cos\tau - \cos\tau\cos^2 y + \cos y\sin\tau\sin y\cos x}{\sin y\sin\angle B^*} \\
&= \frac{\cos\tau\sin y + \cos y\sin\tau\cos x}{\sin\angle B^*}
\end{aligned}$$

On the other hand, by the spherical sine law (see [1]), we have

$$\frac{\sin\varphi}{\sin\tau} = \frac{\sin x}{\sin\angle B^*},$$

and hence

$$\sin\varphi = \frac{\sin\tau\sin x}{\sin\angle B^*}.$$

Therefore,

$$\cot\varphi = \frac{\cos\tau\sin y + \cos y\sin\tau\cos x}{\sin\tau\sin x} = \frac{\cot\tau\sin y + \cos x\cos y}{\sin x}.$$

Since $0 \le \varphi \le \pi$, we have

$$\varphi = \frac{\pi}{2} - \tan^{-1}\left(\frac{\cot\tau\sin y + \cos x\cos y}{\sin x}\right).$$

Notice that since $\partial H(P)$ is a random great circle, x and y are mutually *independent.* (Indeed, relative to the position of the random great circle $\partial H(P)$, the point P^* on the fixed great circle $\partial H(B)$ can be regarded as a *uniform random point* on this great circle.) Since $\Pr(y \le y_0) = \Pr(\widehat{PB} \ge \pi - y_0) = (1-\cos y_0)/2$, the angle y is distributed on the interval $[0,\pi]$ with probability density $\frac{1}{2}\sin y$; and x is distributed uniformly on $[0,\pi]$. Therefore,

$$\mathbf{V}(\varphi) = \int_0^\pi\int_0^\pi \varphi^2\frac{\sin y}{2\pi}dxdy - \mathbf{E}(\varphi)^2$$
$$= \int_0^\pi\int_0^\pi \left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\cot\tau\sin y + \cos x\cos y}{\sin x}\right)\right]^2\frac{\sin y}{2\pi}dxdy - \tau^2.$$

Since $\mathbf{V}(\Theta_{\pi-\omega}) = \mathbf{V}(\Theta_\omega)$, $\mathbf{V}(\varphi)$ is equal to $\mathbf{V}(\Theta_\tau)$. Now, by replacing τ with ω, we have the second formula of Theorem 2.

References

1. G. A. Jennings, *Modern Geometry with Applications*, Springer-Verlag, New York 1994.
2. L. A. Santaló, Integral formulas in Crofton's style on the sphere and some inequalities referring to spherical curves. *Duke Math. J.* 9(1942) 707–722.

This article was processed using the LaTeX macro package with LLNCS style