Response to "Comment on 'Uncertainty principle for proper time and mass' "

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Response to "Comment on 'Uncertainty principle for proper time and mass'" [J. Math. Phys. 42, 3975 (2001)]

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Krikorian asserts the following.

(1) The Lagrangian

$$\mathrm{L}=M(\dot{\tau}-c^{-1}\sqrt{-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}})+eA_{\mu}(x)\dot{x}^{\mu} \tag{1}$$

which we used for the description of a clock[1] is singular in a sense.

(2) When one considers the variational problem for such a singular Lagrangian, "the transversality condition"[2] must be taken into account in addition to the Euler condition.

(3) If both of these two conditions are taken into account for the Lagrangian L, it follows that the value of the variable M must be equal to zero along extremal curves. Therefore our conclusion[1] that the variable M can be identified with the rest mass of the particle is not correct. The objective of this response is to question the second of his assertions.

Let us consider a simple Lagrangian

$$\mathrm{L}'=-mc\sqrt{(\dot{x}^{0})^{2}-(\dot{x}^{1})^{2}} \tag{2}$$

for the description of a particle with rest mass m, where the dynamical variables are the space–time coordinates (x^0,x^1) (assuming one-dimensional space for simplicity) and the dot denotes the differential with respect to the proper time τ. The Euler–Lagrange equations are

$$\ddot{x}^{0}=\ddot{x}^{1}=0. \tag{3}$$

The Lagrangian L' is singular in the sense that

$$\det(\partial^{2}\mathrm{L}'/\partial\dot{x}^{\mu}\partial\dot{x}^{\nu})=0. \tag{4}$$

The Lagrangian L is also singular in this sense, and, in our judgment, this is the reason Krikorian asserts that the transversality condition must be taken into account when we consider the variational problem for the Lagrangian L.

Following his assertion, let us consider an extremal $x^{\mu}(\tau)(\tau_1\leqslant\tau\leqslant\tau_2)$ for the Lagrangian L' which, besides the Euler condition, satisfies the transversality condition. Without loss of generality we may assume that the first end point is fixed. In this case the transversality condition reduces to

$$\dot{x}^{0}(\tau_2)=0 \quad \text{or} \quad \dot{x}^{1}(\tau_2)=0. \tag{5}$$

Combining the condition (5) with the Euler condition (3), we are led to the conclusion that

$$\dot{x}^{1}(\tau)=0 \quad (\text{for all } \tau). \tag{6}$$

That is to say, only a very limited class of motions is admissible if we consider the variational problem for the Lagrangian L' under the transversality condition.

On the other hand, there is a general method[3] by which we can formulate Hamilton's principle for such a singular Lagrangian. When we apply the method to the Lagrangian L', we can actually describe all free motions of the relativistic particle. That is to say, at least for the Lagrangian L′, there is no need to introduce the transversality condition and to limit the extent of the admissible motions. We dealt with the variational problem for the Lagrangian L by this method and got a set of motion equations which seems to be very reasonable.

Our question is then why we have to introduce an extra condition which limits the extent of the admissible motions when there is a method by which we can adequately represent Hamilton's principle and derive a set of reasonable motion equations.

[1] S. Kudaka and S. Matsumoto, J. Math. Phys. **40**, 1237 (1999).
[2] M. Morse, *The Calculus of Variations in the Large* (American Mathematical Society, New York, 1934).
[3] P. A. M. Dirac, Can. J. Math. **2**, 129 (1950); Proc. R. Soc. London, Ser. A **246**, 326 (1958).