A Pfaffian system of Appell's F4

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Abstract

In this note we will show that the Appell's system of partial differential equations (F_4) is equivalent to a Pfaffian system. By this Pfaffian system, we obtain an example of reducible cases of (F_4) which is equivalent to a "symmetric" (F_1) (see 4. 2).

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1. Notations.

z is an unknown function.

$$(F)\ (\alpha, \beta, \gamma; x)$$

$$x(1-x)z_{xx}+(\gamma-(\alpha+\beta+1)x)z_x-\alpha\beta z=0$$

$$(F_1)\ (\alpha, \beta, \beta', \gamma; x, y)$$

$$\begin{cases} x(1-x)z_{xx}+y(1-x)z_{xy}+(\gamma-(\alpha+\beta+1)x)z_x-\beta y z_y-\alpha\beta z=0 \\ y(1-y)z_{yy}+x(1-y)z_{xy}+(\gamma-(\alpha+\beta'+1)y)z_y-\beta' x z_x-\alpha\beta' z=0 \end{cases}$$

$$(F_4)\ (a, b, c, c'; X, Y)$$

$$\begin{cases} X(1-X)z_{XX}-Y^2z_{YY}-2XYz_{XY}+(c-(a+b+1)X)z_X-(a+b+1)Yz_Y-abz=0 \\ Y(1-Y)z_{YY}-X^2z_{XX}-2XYz_{XY}+(c'-(a+b+1)Y)z_Y-(a+b+1)Xz_X-abz=0 \end{cases}$$

$$(F_4')(a, b, c, c'; X, Y) \quad a=c+c'-1$$

(F_4) and $Xz_{XX}+Yz_{YY}+(X+Y-1)z_{XY}+(b+1)(z_X+z_Y)=0$

For generic $\alpha, \beta, \beta', \gamma,$

$$F(\alpha, \beta, \gamma; x) := \Sigma \frac{(\alpha, n)(\beta, n)}{(\gamma, n)(1, n)} x^n$$

is a solution of (F), and

$$F_1(\alpha, \beta, \beta', \gamma; x, y) := \Sigma \frac{(\alpha, m+n)(\beta, m)(\beta', n)}{(\gamma, m+n)(1, m)(1, n)} x^m y^n$$

is a solution of (F_1). (Kimura [1])

For generic a, b, c, c', (F_4) has the following 4 solutions (Kimura [1]):

(1) $F_4(a, b, c, c'; X, Y) = \Sigma \dfrac{(a, m+n)(b, m+n)}{(c, m)(c', n)(1, m)(1, n)} X^m Y^n$

(2) $X^{1-c}F_4(a-c+1, b-c+1, 2-c, c'; X, Y)$

(3) $Y^{1-c'}F_4(a-c'+1, b-c'+1, c, 2-c'; X, Y)$

(4) $X^{1-c}Y^{1-c'}F_4(a-c-c'+2, b-c-c'+2, 2-c, 2-c'; X, Y)$

If $a=c+c'-1$ then (1), (2) and (3) satisfy (F_4').

Let $\phi: C^2(x, y) \longrightarrow C^2(X, Y)$ $X=xy,\ Y=(x-1)(y-1)$ be a 2:1 mapping. The singularity of ϕ is $\{x=y\}$, and $\phi(\{x=y\})=\{D=0\}$, where $D=(X-Y)^2-2(X+Y)+1$. Then the pull back of (F_4) is

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$$\phi^*(F_4)\quad \begin{cases}(x-y)(x(1-x)z_{xx}+(c-(a+b+1)x)z_x-abz)+\varepsilon(y-1)(xz_x-yz_y)=0\\ (y-x)(y(1-y)z_{yy}+(c-(a+b+1)y)z_y-abz)+\varepsilon(x-1)(yz_y-xz_x)=0\end{cases}$$

where $\varepsilon=c+c'-a-b-1$.

2. A Pfaffian systems of $\phi^*(F_4)$

Let $v_0=z,\ v_1=xz_x,\ v_2=yz_y,\ v_3=xy(z_{xy}+\varepsilon\frac{z_x-z_y}{x-y})$

Then $\phi^*(F_4)$ is equivalent to the Pfaffian system

$$dv=Pv$$

where $P=P_{01}\frac{dx}{x}+P_{02}\frac{dy}{y}+P_{13}\frac{dx}{x-1}+P_{23}\frac{dy}{y-1}+P_{12}\frac{d(x-y)}{x-y}$

P_{ij} are the following constant (4, 4) matrices.

$$P_{01}=\begin{pmatrix}0&1&0&0\\0&1-c&0&0\\0&\varepsilon&0&1\\0&0&0&1-c\end{pmatrix}\qquad P_{02}=\begin{pmatrix}0&0&1&0\\0&0&\varepsilon&1\\0&0&1-c&0\\0&0&0&1-c\end{pmatrix}$$

$$P_{13}=\begin{pmatrix}0&0&0&0\\-ab&-(a+b+1-c+\varepsilon)&\varepsilon&0\\0&0&0&0\\0&0&-(a+\varepsilon)(b+\varepsilon)&-(a+b+1-c+\varepsilon)\end{pmatrix}$$

$$P_{23}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\-ab&\varepsilon&-(a+b+1-c+\varepsilon)&0\\0&-(a+\varepsilon)(b+\varepsilon)&0&-(a+b+1-c+\varepsilon)\end{pmatrix}$$

$$P_{12}=\begin{pmatrix}0&0&0&0\\0&\varepsilon&-\varepsilon&0\\0&-\varepsilon&\varepsilon&0\\0&0&0&0\end{pmatrix}$$

3. A Pfaffian System of (F_4)

Put $u_0=v_0,\ u_1=v_1+v_2,\ u_2=-v_1+v_2,\ u_3=v_3$. Then, since $x=(1+X-Y+\sqrt{D})/2$,

$y=(1+X-Y-\sqrt{D})/2$, we have $u_1=2Xz_X+(X+Y-1)z_Y$, $u_2=\sqrt{D}\,Z_Y$,

$u_3=X(Xz_{XX}+Yz_{YY}+(X+Y-1)z_{XY}+(2+a+b-c-c')(z_X+z_Y))$, and (F_4) is equivalent

to the Pfaffian system

$$du=\Omega u$$

where $\Omega=\Omega_1\frac{dx}{x}+\Omega_2\frac{dy}{y}+\Omega_3\frac{dD}{D}-\Omega_4\left(\frac{d(1+X-Y+\sqrt{D})}{1+X-Y+\sqrt{D}}-\frac{d(1+X-Y-\sqrt{D})}{1+X-Y-\sqrt{D}}\right)$

$$-\Omega_5\left(\frac{d(-1+X-Y+\sqrt{D})}{-1+X-Y+\sqrt{D}}-\frac{d(-1+X-Y-\sqrt{D})}{-1+X-Y-\sqrt{D}}\right).$$

Ω_i are the following constant (4, 4) matrices.

$$\Omega_1=\frac{1}{2}\begin{pmatrix}0 & 1 & 0 & 0\\ 0 & -(a+b-c') & 0 & 2\\ 0 & 0 & a+b-2c-c'+2 & 0\\ 0 & 0 & 0 & 2(1-c)\end{pmatrix}$$

$$\Omega_2=-\frac{1}{2}\begin{pmatrix}0 & 0 & 0 & 0\\ 2ab & a+b-c+1 & 0 & 0\\ 0 & 0 & -(a+b-c-2c'+1) & 0\\ 0 & (a-c-c'+1)(b-c-c'+1) & 0 & 2c'\end{pmatrix}$$

$$\Omega_3=-\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & a+b-c-c'+1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$$\Omega_4=\frac{1}{2}\begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & -(a+b-c') & 0\\ 0 & a+b-2c-c'+2 & 0 & -2\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$$\Omega_5=-\frac{1}{2}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & -(a+b-c-2c'+1) & 0\\ 2ab & a+b-c+1 & 0 & 0\\ 0 & 0 & -(a-c-c'+1)(b-c-c'+1) & 0\end{pmatrix}$$

4. Reducible Cases.

4. 1　$a+b-c-c'+1=0$

This means that $\varepsilon=0$ in the equation $\phi^*(F_4)$ in the section 1. Hence $\phi^*(F_4)$ is a "tensor product" of $(F)(a, b, c; x)$ and $(F)(a, b, c; y)$ (T. Sasaki, M. Yoshida [2] §§ 6. 5, 6. 6).

4. 2　$a-c-c'+1=0$

In this case $u_3=X(Xz_{XX}+Yz_{YY}+(X+Y-1)z_{XY}+(1+b)(z_X+z_Y))$, and from the Pfaffian system in the previours section, we have

$$du_3=((1-c)\frac{d_X}{X}-c'\frac{d_Y}{Y})u_3$$

Hence for any solution z of (F_4),

$$L(z):=X^c Y^{c'}(Xz_{XX}+Yz_{YY}+(X+Y-1)z_{XY}+(1+b)(z_X+z_Y))$$

is constant. If z is the solution (1) or (2) or (3) in the section 1, then $L(z)=0$.

If z is the solution (4), then $L(z)=-(1-c)(1-c')$.

The kernel of $L(z)$ is the solution space of (F_4') in (X, Y), and the solution space of $(F_1)(a, b, b, c+b)$ in (x, y).

Hence we have proved the following proposition.

PROPOSITION. *Let* $\phi:(x, y)\longrightarrow(X, Y)$ *be the 2 : 1 mapping defined by* $X=xy$, $Y=(x-1)(y-1)$. *Assume that* $a=c+c'-1$. *Then pull back* $\phi^*(F_4')$ *of* $(F_4')(a, b, c, c'; X, Y)$ *is* $(F_1)(a, b, b, b+c; x, y)$.

REFERENCES

[1] T. Kimura : Hypergeometric Functions of Two Variables, Tokyo Univ. (1973).

[2] T. Sasaki, M. Yoshida : Linear Differential Equations in Two Variables of Rank Four, MPI preprint (1986).