The Primitive Form of E^<12>

The Primitive Form of E_{12}

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Abstract

In this paper we will formally determine the primitive form of the singularity of E_{12} type which is given by $\frac{1}{7}x^7+\frac{1}{3}y^3+\frac{1}{2}z^2$. We can calculate the coefficients of the power series expansion of the primitive form inductively. By the theory of K. Saito and M. Saito, thus determined power series has a non empty range of convergence, but a experimental calculation strongly suggests that the range of convergence is bordered by the set defined by $F_{1x}=F_{1y}=0$.

1. Notations.

Let F_1 be a universal unfolding of $f=\frac{1}{7}x^7+\frac{1}{3}y^3+\frac{1}{2}z^2$ given by

$F_1=f-t_{-2}x^5y-t_4x^4y-t_{10}x^2y-t_{12}x^5-t_{16}x^2y-t_{18}x^4-t_{24}x^3-t_{28}y-t_{30}x^2-t_{36}x.$

Then the primitive form $\zeta^{(0)}$ is uniquely expressible by the form

$$\zeta^{(0)}=v(x,\ y,\ F_1,\ t)\ dxdydz,$$

$v=v_0+v_3t_{-2}^{\frac{3}{}}g_3+v_6t_{-2}^{\frac{6}{}}g_6+v_7t_{-2}^{\frac{7}{}}g_7+v_9t_{-2}^{\frac{9}{}}g_9+v_{10}t_{-2}^{\frac{10}{2}}g_{10}+v_{12}t_{-2}^{\frac{12}{2}}g_{12}+v_{13}t_{-2}^{\frac{13}{2}}g_{13}+v_{15}t_{-2}^{\frac{15}{2}}g_{15}+$

$v_{16}t_{-2}^{\frac{16}{2}}g_{16}+v_{19}t_{-2}^{\frac{19}{2}}g_{19}+v_{22}t_{-2}^{\frac{22}{2}}g_{22},$

where $g_3=x$, $g_6=x^2$, $g_7=y$, $g_9=x^3$, $g_{10}=xy$, $g_{12}=x^4$, $g_{13}=x^2y$, $g_{15}=x^5$, $g_{16}=x^3y$, $g_{19}=x^4y$, $g_{22}=x^5y$, and 1 are a basis of $C\{x,\ y,\ z\}/(f_x,\ f_y,\ f_z)$.

$v_i=v_i(t_{-2}^{\frac{21}{2}}F_1,\ t_{-2}^{\frac{2}{}}t_4,\ t_{-2}^{\frac{5}{}}t_{10},\cdots\cdots,t_{-2}^{\frac{18}{2}}t_{36})=\Sigma v_{ijL}t_{-2}^{\frac{i+21j}{2}}F_1^j g_i t^L,$

where each L is a multi-index such that t^L is of weight 0 (the weight of t_i is i).

In the section 2, we determine v_i formally, and in the section 3 we will deal with the special case when $t_4=t_{10}=\cdots\cdots=t_{36}=0$.

2. General Case.

We set $v_{000}=1$, and v_{ijL} are determined inductively on the $i+21j+|L|$ (=degree with respect to t) by the equalities

$K^{(i+22j+|L|}((\partial/\partial t_{-2})^{i+21j}(\partial/\partial t)^L\zeta^{-i-21j-|L|)},g_{22-i})=0.$

These are equivalent to the following ones:

$$(\partial/\partial t_{-2})^{i+21j}(\partial/\partial t)^L\zeta^{-i-20j-|L|}\ |_{t=0}=0$$

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in a suitable sense.

In this way coefficients v_{ijL} of v are uniquely determined, and conversely it can be shown that thus determined $\xi^{(0)}$ actually is a primitive form.

3. The Case when $t_{+}=0$

In this section we restrict the case that all the parameter t with positive weights are equal to zero, that is $t_4=t_{10}=\cdots\cdots=t_{36}=0$. Then vi are the functions of $t_{-2}^{21} F_1$, and we denote

$$n=i_0+21j_0,\ m=i+21j$$

$$v_i=\Sigma v_{ij}(t_{-2}^{21} F_1)^j$$

$$\xi^{(0)}=\Sigma v_{ij} t_{-2}^{i+21j} F_1^j g_i dxdydz.$$

From now on we omit the factor dxdydz, and ∂ denotes $\partial/\partial t_{-2}$.

Remember that $\partial \nabla^{-1} g=\nabla^{-1}\partial g+x^5yg$, hense we have the following equalities:

$$\partial^n \nabla^{-n} \xi^{(0)}=\Sigma v_{ij} \partial^n \nabla^{-n} t_{-2}^{m} F_1^j g_i$$

$$\partial^n \nabla^{-n} t_{-2}^{m} F_1^j g_i=\Sigma\binom{n}{m+k}\nabla^{-(m+k)}\partial^{m+k}(t_{-2}^{m} F_1^j g_i)(x^5y)^{n-(m+k)}$$

$$=\Sigma\binom{n}{m+k}\binom{m+k}{k}m!\nabla^{-(m+k)}\partial^k F_1^j g_i(x^5y)^{n-(m+k)}$$

$$=\Sigma\frac{n!\ j(j-1)\cdots\cdots(j-k+1)}{(n-m-k)!\ k!}(-1)^k\nabla^{-(m+k)}F_1^{j-k} g_i(x^5y)^{n-m}$$

all modulo t_{-2}.

Since $F_1 g=(\frac{41}{42}+weight\ of\ g)\ \nabla^{-1}g,$

$F_1^{j-k} g_i(x^5y)^{n-m}=w(w+1)\cdots\cdots(w+j-k-1)\nabla^{-j+k}g_i(x^5y)^{n-m}$

mod t_{-2}, where $\frac{2i+44(n-m)+41}{42}=n-m-j+\frac{n}{21}+\frac{41}{42}.$

By the lemma1,

$$\Sigma(-1)^k\frac{j(j-1)\cdots\cdots(j-k+1)}{(n-m-k)!\ k!}w(w+1)\cdots\cdots(w+j-k-1)=\frac{1}{(n-m)!}(\frac{2n+41}{42}-j,\ j),$$

where $(a,\ j)=a(a+1)\cdots\cdots(a+j-1).$

Hence

$$\partial^n \nabla^{-n} t_{-2}^{m} F_1^j g_i=\frac{n!}{(n-m)!}(\frac{2n+41}{42}-j,\ j)\nabla^{-m-j}g_i(x^5y)^{n-m}.$$

When $g_i(x^5y)^{n-m}=x^p y^q$ and $g_{i_0}=x^{p_0} y^{q_0}$,

$$g_i(x^5y)^{n-m}=((p-6)(p-13)\cdots\cdots)((q-2)(q-5)\cdots\cdots)\nabla^{-(n-m-j+j_0)}g_{i_0}$$

$$=7^{[p]}((p_0+1)/7,\ [p])3^{[q]}((q_0+1)/3,\ [q])\nabla^{-(n-m+j_0-j)}g_{i_0}.$$

By the lemma 3, $[p] = [\frac{5}{7}n] - [\frac{5}{7}m]$, $[q] = [\frac{1}{3}n] - [\frac{1}{3}m]$.

Put $w_0 = (2i_0 + 41)/42$, *then* $(2n+41)/42 = w_0 + j_0$.

To sum up,

$$\partial^n \nabla^{-n} t_{-2}^{m} F_1^j g_i = \frac{n!}{(n-m)!}(w_0 + j_0 - j,\ j) 7^{[5n/7]-[5m/7]} 3^{[n/3]-[m/3]} ((p_0+1)/7,\ [\frac{5}{7}n] - [\frac{5}{7}m])((q_0+1)/3,\ [\frac{1}{3}n] - [\frac{1}{3}m]) \nabla_{-(n+j_0)} g_{i_0},$$

$$\partial^n \nabla^{-n} t_{-2}^{n} F_1^{j_0} g_{i_0} = n! w_0 (w_0+1) \cdots\cdots (w_0 + j_0 - 1).$$

where $n = i_0 + 21 j_0$, $m = i + 21j$, $g_{i_0} = x^{p_0} y^{q_0}$, $w_0 = (2i_0 + 41)/42$.

4. Lemmata.

Lemma 1. *Let* $w = n - i - 22j + a$, $s = n - i - 21j$, *then*

$$\sum_{k=0}^{\min(s,\ j)} (-1)^k \frac{j(j-1)\cdots\cdots(j-k+1)}{(s-k)!\ k!} w(w+1)\cdots\cdots(w+j-k-1) = \frac{1}{s!}(a-j,\ j).$$

Proof. When $s \leqq j$, differentiate $x^j x^{-w-j+s}$ s times by the Leibniz rule and put $x=1$. When $s \geqq j$, differentiate $x^j x^{-w-j+s}$ j times and put $x=1$.

Lemma 2. *Let* g_i *be as in the section 1, and put* $g_i = x^{a_i} y^{b_i}$. *Then* $a_i \equiv 5i \pmod 7$, $b_i \equiv i \pmod 3$.

Proof. The weight of g_i is $i/21$, and that of $x^{a_i} y^{b_i}$ is $a_i/7 + b_i/3$. Hense $i \equiv 3a_i$, $5i \equiv 15a_i \equiv a_i \pmod 7$, $i \equiv 7b_i \equiv b_i \pmod 3$.

Lemma 3. *Let* $m = i + 21j$ *and* g_i *be as in the section 1. Put* $g_i(x^5 y)^{n-m} = x^p y^q$. *Then* $[p/7] = [5n/7] - [5m/7]$, $[q/3] = [n/3] = [m/3]$.

Proof. Let a_i, b_i be as in the lemma 2. Then $p = 5n - 5i - 105j + a_i$, and by the lemma 2, $[5r/7] = 15j + [5i/7] = 15j + (15i - a_i)/7$, hense $[p/7] = [5n/7] - (5i - a_i/7 - 15j = [5n/7] - [5r/7]$. By the same way, $[q/3] = [n/3] - [m/3]$.

References

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