Parameter and Reliability Estimation for a Bivariate Exponential Distribution

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Zensho NAKAO* Zeng-Zhong LIU** and Masaya KINJO*

Abstract

Parameter and reliability estimators for a bivariate exponential distribution are derived by the moment and the maximum likelihood methods.

Key Words: Bivariate exponential distribution, Maximum likelihood estimation, Moment estimation, Parameters, Reliability function

1. Introduction

System elements are connected in series and/or in parallel. For example, in electric circuits, two resistors may be connected in series or in parallel; in electric power distribution, two generators may be used in parallel to supply energy; in multi-computer systems (space shuttles, for example) , three or more computers may be connected (redundantly) in parallel for reliability.

Suppose that such a system consists of two components in series or in paralled, and is shocked by three Poisson random interferences from outside: one shocking the first element, the second striking the second, and the third hitting the two components simultaneously. Then the survival probability function for the system can be shown to be given by a bivariate exponential distribution (BVE) with parameters λ_{01}, λ_{10}, and λ_{11} : $F(x, y) = P(X > x, Y > y) = exp[-\lambda_{01}x - \lambda_{10}y - \lambda_{11}max(x,y)]$, where X, Y denote the lifelength of the two components, $\lambda_{01} > 0, \lambda_{10} > 0, \lambda_{11} > 0, x, y \geq 0$[3].

We will present a set of estimators for the parameters and the reliability function of such systems by the method of moments and the maximum likelihood method. Bayesian and empirical Bayesian methods for estimating the parameters and the reliability function of a parallel BVE system where past data are available are given in [4,5] .

2. Derivation of a BVE

Let two independent Poisson processes $Z_1(t;\lambda_{01})$, $Z_2(t;\lambda_{10})$ shock the components, numbered 1,2, respectively, and let a third Poisson process $Z_{11}(t;\lambda_{11})$ shock the components 1,2 simultaneously. Let X, Y denote the lifelength of the 1,2 componets, and define the survival probability function by $F(x, y) = P(X > x, Y > y)$. Then $F(x, y) = P[Z_1(x;\lambda_{01}) = 0,\ Z_2(y;\lambda_{10}) = 0,\ Z_{11}(max(x, y);\ \lambda_{11}) = 0] = exp(-\lambda_{01}x) \bullet exp(-\lambda_{10}y) \bullet exp[-\lambda_{11}\,max(x, y)] = exp[-\lambda_{01}x - \lambda_{10}y - \lambda_{11}\,max(x, y)]$, where $\lambda_{01} > 0,\ \lambda_{10} > 0, \lambda_{11} > 0, x, y, \geq 0$[3].

Recall that a Poisson random process $Z(t;\lambda)$ is given by

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* Education of Common Course, Fac. of Eng.,

** Software Technology, Inc.

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$$P(Z(t;\lambda)=k)=\frac{(\lambda t)^k}{k!}\bullet exp(-\lambda t), k=0,1,2,\ldots$$

3. Parameter and reliability estimation for a BVE

a. Moment estimation for a BVE

It is known [3] that

$$E[X]=\frac{1}{\lambda_{01}+\lambda_{11}},\quad E[Y]=\frac{1}{\lambda_{10}+\lambda_{11}},$$
$$E[XY]=\frac{1}{\lambda}\bullet\left(\frac{1}{\lambda_{01}+\lambda_{11}}+\frac{1}{\lambda_{10}+\lambda_{11}}\right);$$

(where $\lambda=\lambda_{01}+\lambda_{10}+\lambda_{11}$, and $E[\ ]$ denotes the expected value).

Using the method of moments, we obtain a system of equations:

$$\frac{\sum_{i=1}^{n} x_i}{n}=\frac{1}{\lambda_{01}+\lambda_{11}},\quad \frac{\sum_{i=1}^{n} y_i}{n}=\frac{1}{\lambda_{10}+\lambda_{11}},$$
$$\frac{\sum_{i=1}^{n}(x_i, y_i)}{n}=\frac{1}{\lambda}\bullet\left(\frac{1}{\lambda_{01}+\lambda_{11}}+\frac{1}{\lambda_{10}+\lambda_{11}}\right); \text{(n:sample size)}.$$

Solving this system for the parameters, we get the estimators:

$$\hat{\lambda}_{01}=\frac{\sum_{i=1}^{n}(X_i+Y_i)}{\sum_{i=1}^{n}X_iY_i}-\frac{n}{\sum_{i=1}^{n}Y_i},\quad \hat{\lambda}_{10}=\frac{\sum_{i=1}^{n}(X_i+Y_i)}{\sum_{i=1}^{n}X_iY_i}-\frac{n}{\sum_{i=1}^{n}X_i},$$
$$\hat{\lambda}_{11}=\frac{n}{\sum_{i=1}^{n}X_i}-\frac{n}{\sum_{i=1}^{n}Y_i}-\frac{\sum_{i=1}^{n}(X_i+Y_i)}{\sum_{i=1}^{n}X_iY_i}.$$

Our method is slightly different from the moment method in [2] in that their third equation differs from ours given above, and that if $|\{i:x_i=y_i\}|=0$ ($|\ \ |$ denotes the cardinality of a set), then $\lambda_{11}=0$ in [2], which is a difficulty in that method. For a parallel system, the system is in an unfailed state at time t, if at least one components survive at the time t. Hence the reliability of the system is given by

$R(t)=P(X>t \text{ or } Y>t)=P(X>t)+P(Y>t)-P(X>t, Y>t)=F(t,0)+F(0,t)-F(t,t)=exp[-(\lambda_{01}+\lambda_{11})t]+exp[-(\lambda_{10}+\lambda_{11})t]-exp(-\lambda t).$

Thus the corresponding estimator $\hat{R}(t)$ for the reliability can be obtained by replacing λ's with their estimators $\hat{\lambda}$'s:

$\hat{R}(t)=exp[-(\hat{\lambda}_{01}+\hat{\lambda}_{11})t]+exp[-(\hat{\lambda}_{10}+\hat{\lambda}_{11})t]+exp(-\hat{\lambda}t)$, $(t\geq 0)$.

b. Maximum likelihood estimation for a BVE

For parallel systems, the maximum likelihood estimation for a BVE was made by F. Prochan and P. Sullo [6] ; here we will derive maximum likelihood estimators for a BVE in series, which turns out to be much simpler than for parallel systems. Let $(X,Y)\sim BVE(\lambda_{01},\lambda_{10}\lambda_{11})$, and $S=\{(1,0),(0,1),(1,1)\}$.

For $s\in S$, define a characteristic function V_s as follows [1] :

$$V_{(0,1)}=\begin{cases}1 & \text{if } X<Y,\\ 0 & \text{if } X\geq Y;\end{cases}\quad V_{(1,0)}=\begin{cases}1 & \text{if } X>Y,\\ 0 & \text{if } X\leq Y;\end{cases}\quad V_{(1,1)}=\begin{cases}1 & \text{if } X=Y,\\ 0 & \text{if } X\neq Y.\end{cases}$$

Thus, for every sample (X_i, Y_i), $i=1,\ldots,n$, we have $V_{s,i}$ defined, $i=1,\ldots,n$. Let $N_s=\sum_{i=1}^{n}V_{s,i}$, and let $U_i=min(X_i,Y_i)$, *i.e.*, the life time of the series system $i=1,\ldots,n$. Now, for a series system, the entire system fails when one unit fails. So the only information available is on the life time of the system, U_i, $i=1,\ldots,n$, and on which unit has the longer life time, $V_{s,i}, i=1,\ldots,n$. It is obvious that $V_{s,i}, i=1,\ldots,n$, are independent of $U_j, j=1,\ldots,n$ [1], and moreover, ($N_{(0,1)}, N_{(1,0)}, N_{(1,1)}$) has a trinomial distribution with probabilities $\frac{\lambda_{01}}{\lambda}, \frac{\lambda_{10}}{\lambda}, \frac{\lambda_{11}}{\lambda}$, respectively. So the joint probability is given (where $n=N_{(0,1)}+N_{(1,0)}+N_{(1,1)}$) by $P[N_{(0,1)}=n_{(0,1)}, N_{(1,0)}=n_{(1,0)}, N_{(1,1)}=n_{(1,1)}, U_i>t, i=1,\ldots,n]=P[N_{(0,1)}=n_{(0,1)}, N_{(1,0)}=n_{(1,0)}, N_{(1,1)}=n_{(1,1)}]\bullet P[U_i>t, i=1,\ldots,n]$

$$=\frac{n!}{n_{(0,1)}!\bullet n_{(1,0)}!\bullet n_{(1,1)}!}\bullet\lambda_{01}^{n_{(0,1)}}\bullet\lambda_{10}^{n_{(1,0)}}\bullet\lambda_{11}^{n_{(1,1)}}\bullet exp\left(-\lambda\sum_{i=1}^{n}U_i\right).$$

To derive maximum likelihood estimators, we take the logarithm of the likelihood function above, and set its gradient equal to zero:

$$ln\,P[N_{(0,1)}=n_{(0,1)}, N_{(1,0)}=n_{(1,0)}=N_{(1,1)}=n_{(1,1)}, U_i>t, i=1,\ldots,n]=ln\frac{n!}{n_{(0,1)}!\bullet n_{(1,0)}!\bullet n_{(1,1)}!}+n_{(0,1)}\bullet ln\lambda_{01}$$
$$+n_{(1,0)}\bullet ln\lambda_{10}+n_{(1,1)}\bullet ln\lambda_{11}-\lambda\sum_{i=1}^{n}U_i;\ \frac{\partial lnP}{\partial\lambda_{01}}=n_{(0,1)}\bullet\frac{1}{\lambda_{01}}-\sum_{i=1}^{n}U_i=0,\ \frac{\partial lnP}{\partial\lambda_{10}}=n_{(1,0)}\bullet\frac{1}{\lambda_{10}}-\sum_{i=1}^{n}U_i=0,$$
$$\frac{\partial lnP}{\partial\lambda_{11}}=n_{(1,1)}\bullet\frac{1}{\lambda_{11}}-\sum_{i=1}^{n}U_i=0.$$

Solving the resulting system of equations, we get the following estimators:

$$\hat{\lambda}_{01}=\frac{n_{(0,1)}}{\sum_{i=1}^{n}U_i},\quad \hat{\lambda}_{10}=\frac{n_{(1,0)}}{\sum_{i=1}^{n}U_i},\quad \hat{\lambda}_{11}=\frac{n_{(1,1)}}{\sum_{i=1}^{n}U_i},$$
$$\hat{\lambda}=\hat{\lambda}_{01}+\hat{\lambda}_{10}+\hat{\lambda}_{11}=\frac{n}{\sum_{i=1}^{n}U_i}.$$

For a series system, the system is in a failed state at time t if at least one components fail at the time t. Hence the reliability of the system is given by

$$R(t)=P(X>t,\ Y>t)=exp(-\lambda t).$$

Thus the corresponding estimator $\hat{R}(t)$ for the reliability becomes

$$\hat{R}(t)=exp(-\hat{\lambda}t)=exp\left(-\frac{n}{\sum_{n=1}^{n}U_i}t\right),\ (t\geq 0).$$

4. Simulation on parameter and reliability estimation

a. Methods

We give some detail for computer simulation: First, $RND(p)[p\in(0,1)]$ is used to produce random values in the unit interval (0,1). Secondly, we let $U\sim E(\lambda_{01})$, $V\sim E(\lambda_{10})$, $W\sim E(\lambda_{11})$, $(E(\lambda):\lambda)$: an exponential distrbution with parameter λ), and due to the fact that $1-exp(-\lambda_{01}U)\sim R(0,1)$, $1-exp(-\lambda_{11}V)\sim R(0,1)$, $1-exp(-\lambda_{11}W)\sim R(0,1)$ (where $R(0,1)$ is a uniform distribution in the interval $(0,1)$), we get samples (of size n) $U_1,\ldots,U_n$; $V_1,\cdots,V_n$; $W_1,\cdots,W_n$. Thirdly, we obtain independent random variables U, V, W, $X=\min(U,W)$, and $Y=\min(V,W)$, and get sample values of (X,Y) which is known to have a BVE distribution [3].

b. Results and analysis

Results of simulation are shown for moment estimation of parallel systems and for maximum likelihood estimation of series systems in Tables 1 and 2, respectively. We used the mean squared errors (MSE) for evaluation of the estimators, where

$$MSE=\frac{\sum_{i=1}^{n}(estimator_i-true\ value)^2}{n}.$$

Table 1

Parameter and reliability estimators of a BVE (Method of moments)

N	λ_{01} 0.1	λ_{10} 0.2	λ_{11} 0.3	R 0.728
50	0.129	0.204	0.308	0.718
100	0.159	0.305	0.242	0.755
200	0.097	0.171	0.285	0.741
300	0.133	0.247	0.281	0.738
MSE	0.0011	0.0035	0.0010	0.0003
N	λ_{01} 0.2	λ_{10} 0.1	λ_{11} 0.3	R 0.728
50	0.288	0.087	0.281	0.739
100	0.243	0.174	0.267	0.739
200	0.238	0.099	0.266	0.751
300	0.201	0.123	0.288	0.733
MSE	0.0028	0.0015	0.0007	0.0002
N	λ_{01} 0.3	λ_{10} 0.2	λ_{11} 0.1	R 0.682
50	0.361	0.242	0.104	0.842
100	0.195	0.118	0.210	0.794
200	0.390	0.239	0.050	0.886
300	0.292	0.239	0.089	0.865
MSE	0.0057	0.0029	0.0037	0.0014

(N: Sample sizes, MSE: Mean Squared errors, t = 1)

Table 2

Parameter and reliability estimators of a BVE

(Maximum likelihood method)

N	λ_{01} 0.1	λ_{10} 0.2	λ_{11} 0.3	R 0.549
50	0.106	0.225	0.331	0.516
100	0.114	0.234	0.284	0.532
200	0.086	0.208	0.260	0.575
300	0.103	0.217	0.288	0.544
MSE	0.0001	0.0005	0.0007	0.0005
N	λ_{01} 0.2	λ_{10} 0.1	λ_{11} 0.3	R 0.549
50	0.220	0.103	0.323	0.524
100	0.217	0.134	0.287	0.528
200	0.223	0.112	0.253	0.555
300	0.179	0.124	0.286	0.555
MSE	0.0004	0.0005	0.0008	0.0003
N	λ_{01} 0.3	λ_{10} 0.2	λ_{11} 0.1	R 0.549
50	0.369	0.171	0.119	0.517
100	0.306	0.222	0.072	0.549
200	0.333	0.215	0.075	0.536
300	0.293	0.240	0.082	0.541
MSE	0.0015	0.0008	0.0005	0.0003

(N: Sample sizes, MSE: Mean Squared errors, t = 1)

In Table 1, $0.0002 \leq MSE$ (reliability) ≤ 0.0014; $0.0011 \leq MSE(\lambda_{01}) \leq 0.057$; $0.0015 \leq MSE(\lambda_{10}) \leq 0.0035$; $0.0007 \leq MSE(\lambda_{11}) \leq 0.037$; while in Table 2, $0.003 \leq MSE$ (reliability) ≤ 0.0005; $0.0001 \leq MSE(\lambda_{01}) \leq 0.0015$; $0.0005 \leqq MSE(\lambda_{10}) \leq 0.008$; $0.0005 \leq MSE(\lambda_{11}) \leq 0.0008$

We see that, from Tables 1 and 2, the MSE's for both the moment and the maximum likelihood estimation are small enough as expected of the methods, and that, from Table 3, the MSE's for the maximum likelihood estimators are consistently smaller than those for the moment estimators.

Table 3

Comparison of MSE's

	Moment estimators			Maximum likelihood estimatorss		
	λ_{01} 0.1	λ_{10} 0.2	λ_{11} 0.3	λ_{01} 0.1	λ_{10} 0.2	λ_{11} 0.3
MSE	0.0011	0.0035	0.0010	0.0001	0.0005	0.0007
	λ_{01} 0.2	λ_{10} 0.1	λ_{11} 0.3	λ_{01} 0.2	λ_{10} 0.1	λ_{11} 0.3
MSE	0.0028	0.0015	0.0007	0.0004	0.0005	0.0008
	λ_{01} 0.3	λ_{10} 0.2	λ_{11} 0.1	λ_{01} 0.3	λ_{10} 0.2	λ_{11} 0.1
MSE	0.0057	0.0029	0.0037	0.0015	0.0008	0.0005

5. Conclusions

We obtained parameters and reliability estimators for a BVE where the system components are connected in series or in parallel; used the MSE's for evaluating the goodness of estimation and found that the MSE's are small enough as expected of the methods; in general, the maximum likelihood estimation suits better than the moment estimation for the parameters and the reliability function of a BVE; thus, for parallel systems, the estimators obtained in [6] and, for series systems, our estimators can be adopted.

There remain problems of hypothesis testing on the parameters of a BVE, which will be the subjects for our next investigation.

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