An Independence Test of a Multivariate Exponential Distribution

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Abstract

An independent test of a certain multivariate exponential distribution is done with respect to the equivalence of the marginal distributions, where simple restrictions are imposed on its parameters.

1. Introduction

Bemis et al [1] and Bhattacharyya and Johnson [2] did the independence test of a bivariate exponential distribution (BVED) with respect to the equivalence of the marginal distributions. Because the notion of a multivariate exponential (MVE) model is much more complicated than that of a BVE model, it is very difficult to do the independence test of a MVED. In this paper we perform only the independence test of a certain MVED.

Consider a model $\bar{F}(t_1,\cdots,t_k)=\exp\{-\lambda_1 t_1-\lambda_2 t_2\cdots-\lambda_k t_k-\lambda_0\max(t_1,t_2,\cdots,t_k)\}$, $t_i\geqq 0$, $i=1,\cdots,k$; $\underline{\lambda}\in\Lambda$, where $\underline{\lambda}=(\lambda_1,\cdots,\lambda_k,\lambda_0)$, $\Lambda=\{\underline{\lambda}: 0\leqq\lambda_i<\infty, i=0,\cdots,k, \lambda_0+\lambda_j>0,\ j=1,2\cdots,k\}$. We assume that $\lambda_1=\lambda_2=\lambda_3=\cdots=\lambda_k$. The notations below are the same as those in [3].

2. Notations and the model

In a parallel system of k elements, assume that $T_1,\cdots,T_k$ indicate the failure time of elements $1,2,\cdots,k$, respectively; $\{Z_i(t): t\geqq 0\}$, $i=1,2,\cdots,$ k is an independent Poison process with intensity λ_i; $Z_i(t)$ shocks the i-th element, $i=1,2,\cdots,k$; $Z_0(t)$ shocks simultaneously k elements. Obviously, $T_i=\inf\{t: Z_i(t)+Z_0(t)>0\}$. If $U_0,U_1,\cdots,U_k$ denote the time of the first appearing event from $Z_0(t),\cdots,Z_k(t)$, respectively, then we have $T_i=\min(U_0,U_i)$ according to a generalization of a theorem in [4].

Theorem: $\underline{T}\sim MVE(k,\ \underline{\lambda})$ if and only if there exist $k+1$ independent exponential random variables $\{U_i\}_{i=0}^{k}$ with intensity λ_i and $T_i=\min(U_0,U_i), i=1,\cdots,k$.

Assume E_k is a k-dimentional Euclidean space, $E_k^+=\{\underline{t}\in E_k: t_i\geqq 0, i=1,2,\cdots,k\}$ and the following notations will be used throughout the paper:

(i) $t_{(1)}=\min(t_1,t_2,\cdots,t_k)$.

(ii) $t_{(k)}=\max(t_1,t_2,\cdots,t_k)$.

(iii) $V_i(\underline{t})=\begin{cases}1, & t_i<t_{(k)},\\ 0, & \text{otherwise}.\end{cases}$

(iv) $S(\underline{t})=\begin{cases}1, & \text{there exist } i,j (i\neq j) \text{ with } t_i=t_j=t_{(k)},\\ 0, & \text{otherwise}.\end{cases}$

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We use an abbreviation $V_i(\underline{t}) = V_i$, and random variables are denoted by upper case letters.

Assume $\{\underline{T}_j = (T_{1j}, T_{2j}, \cdots, T_{kj}): j = 1, 2, \cdots, n\}$ are samples from MVE$(k, \underline{\lambda})$, $\{\underline{t}_j\}_{j=1}^{n}$ denote the coreesponding sample values, $V_{ij} = V_i(\underline{t}_j)$ and $S_j = S(\underline{t}_j)$.

Define (i) $n_0 = \sum_{j=1}^{n} S_j,\ n_i = \sum_{j=1}^{n} V_{ij}$

(ii) $n_i^{(c)} = \sum_{j=1}^{n} (1 - V_{ij})(1 - S_j)$

(iii) $n_0(i) = \sum_{j=1}^{n} (1 - V_{ij}) S_j = n - n_i - n_i^{(c)}$,

where $i = 1, \cdots, k$; random variables will be denoted by the upper case letters N_i, $N_i^{(c)}$, $N_0(i)$ below. The interpretation of the symbols n_i, n_0, $n_0(i)$ above are the following: n_i = the number of samples for which the failure time of the i-th element is strictly smaller than that of the system; n_0 = the number of samples for which at least two elements fail at the same time; $n_0(i)$ = the number of samples for which the i-th element fails with some other element at the same time; $n_i^{(c)}$ = the number of samples for which the i-th element fails last and fails with no other element at the same time. With the notations above we can obtain the following result.

Lemma: $P(S = 0) = \sum_{i=1}^{k} \theta_i$, where $\theta_i = \int_0^{\infty} \nu_i \exp(-\nu_i t) \prod_{j \neq 0, i} F_j(t) dt$, $i = 1, \cdots, k$;
$\nu_i = \lambda_i + \lambda_0$, $F_j(t) = 1 - \exp(-\lambda_j t)$.

Proof: $P(S = 0) = \sum_{i=1}^{k} P(V_i = 0, S = 0) = \sum_{i=1}^{k} P(T_i > U_j, \text{ for all } j \neq i, j \neq 0)$

$$= \sum_{i=1}^{k} \int_0^{\infty} \nu_i e^{-\nu_i t} \prod_{j \neq 0, i} F_j(t) dt = \sum_{i=1}^{k} \theta_i.$$

3. Likelyhood ratio test

As mentioned in [4], an MVED includes a singular part under the Lebesgue measure. However, it is possible that the MVE is absolutely continuous under a new measure. In [3] is defined a new measure under which the MVED is absolutely continuous and has a density. The measure is defined as follows:

$\mu(A) = \mu_k(A) + \sum \mu_{k-r+1}(A \cap \{\underline{t} \in E_k^+ : t_{i_1} = t_{i_2} = \cdots = t_{i_r} = t_{(k)}\})$, $A \in \beta_k^+$(Borel sets),
$\{i_1, \cdots, i_r\}$ takes all subsets of $\{1, 2, \cdots, k\}$, $r = 2, 3, \cdots, k$.

The MVED has a density function given by:

$$f(\underline{t}, \underline{\lambda}) = \lambda_0^s \left[\prod_{i=1}^{k} \lambda_i^{v_i} \nu_i^{(1-s)(1-v_i)} \right] \bar{F}(\underline{t}).$$

The sample likelyhood function of the MVE$(k, \underline{\lambda})$ is defined by:

$$L(\underline{\lambda}) = \prod_{j=1}^{n} f(\underline{t}_j, \underline{\lambda}) = \lambda_0^{\Sigma S_j} \prod_{i=1}^{k} \lambda_i^{\Sigma V_{ij}} \nu_i^{\Sigma(1-S_j)(1-V_{ij})} \exp\left(-\sum_{j=1}^{n}\sum_{i=1}^{k} \lambda_i \cdot t_{ij} - \lambda_0 \sum_{j=1}^{n} t_{(k)j}\right)$$

$$= \lambda_0^{n_0} \prod_{i=1}^{k} \lambda_i^{n_i} \nu_i^{n_i(c)} \exp\left(-\sum_{j=1}^{n}\sum_{i=1}^{k} \lambda_i t_{ij} - \lambda_0 \sum_{j=1}^{n} t_{(k)j}\right).$$

When $\lambda_0 = 0$, we obtain:

$$P(N_0 > 0) = 1 - P(N_0 = 0) = 1 - \prod_{j=1}^{n} P(S_j = 0) = 1 - [P(S = 0)]^n = 1 - \left(\sum_{i=1}^{k} \theta_i\right)^n$$

$$=1-\left\{k\int_0^{\infty}\lambda_1\exp(-\lambda_1 t)[1-\exp(-\lambda_1 t)]^{k-1}\,dt\right\}^n=1-\left\{[1-\exp(-\lambda_1 t)]^k\Big|_0^{\infty}\right\}^n$$
$$=1-(1-0)^n=0.$$

Thus, $N_0=0$(a.s.), and hence, $S_j=0$ for all j and $n_1+n_1^{(c)}=n$. Therefore, the likelyhood function above becomes:

$$L(\lambda_1,\lambda_2,\cdots,\lambda_k,0)=\prod_{l=1}^{k}\lambda_l^{n}\exp(-\sum_{j=1}^{n}\sum_{l=1}^{k}\lambda_l t_{lj}).$$

The hypothesis we want to test is, then:

$$H_0:\lambda_0=0 \longleftrightarrow K_0:\lambda_0>0$$

Assume $\delta>0$, we first test the hypothesis:

$$H_1:\lambda_0=0 \longleftrightarrow K_1:\lambda_0=\delta.$$

Obviously, when $N_0>0$, we must reject the hypothesis $\lambda_0=0$. When $N_0=0$, we can use the conditional lykelyhood ratio test: in accordance with the theorem of Neyman-Pearson, the MP test of the hypothesis is

$$\phi=\begin{cases}1, & \dfrac{L(\underline{\lambda})}{L(\lambda_1,\cdots,\lambda_k,0)}\geqq C_\alpha(C_\alpha>0),\\ 0, & \text{otherwise,}\end{cases}$$

where $$\frac{L(\underline{\lambda})}{L(\lambda_1,\cdots,\lambda_k,0)}=\frac{\prod_{j=1}^{n}f(\lambda_1,\cdots,\lambda_k,\lambda_0)}{\prod_{j=1}^{n}f(\lambda_1,\cdots,\lambda_k,0)}=\nu^{\sum_{j=1}^{n}n_j^{(c)}}\exp(-\delta\sum_{j=1}^{n}t_{(k)j})=\nu^{n}\exp(-\delta\sum_{j=1}^{n}t_{(k)j}).$$

$$(\text{Note}: n-n_0=\sum_{j=1}^{n}n_j^{(c)})$$

So when $\sum_{j=1}^{n}t_{(k)j}\leqq c(\alpha,\lambda_1)$, we reject hypothesis H_0. ($c(\alpha,\lambda_1)$ is a constant which depends on α and λ_1.) Obviously, the rejection region is

$$\left\{N_0>0\right\}\cup\left\{\sum_{j=1}^{n}t_{(k)j}\leqq c(\alpha,\lambda_1)\right\}.$$

The value of $c(\alpha,\lambda_1)$ is computed in the following: Assume the significance level is α. Then we have

$$P_{\lambda_0=0}\left\{N_0>0\right\}+P_{\lambda_0=0}\left\{N_0=0,\ \sum_{j=1}^{n}t_{(k)j}\leqq c(\alpha,\lambda_1)\right\}=\alpha.$$

Due to the result above, $P_{\lambda_0=0}\{N_0>0\}=0$, and hence,

$$\alpha=P_{\lambda_0=0}\left\{\sum_{j=1}^{n}t_{(k)j}\leqq c(\alpha,\lambda_1)\right\}.$$

Now, $$P_{\lambda_0=0}(\lambda_1 t_{(k)j}\leqq t)=\prod_{l=1}^{k}P(t_l<t/\lambda_1)=[1-\exp(-t)]^k.$$

Below we find the distribution of $\sum_{j=1}^{n}\lambda_1 t_{(k)j}$. Assume $y_j=\lambda_1 t_{(k)j}$. Then $\{y_j\}$ is independent and the same distribution as well, $j=1,\cdots,n$. Performing the transformation:

$$\begin{cases}Z_1=\sum_{j=1}^{n}y_j,\\ Z_2=y_2,\\ \quad\vdots\\ Z_n=y_n,\end{cases}$$

we have $Z_1 \geqq Z_2+Z_3+\cdots+Z_n$. Thus the joint density of $(Z_1, \cdots, Z_n)$ is:

$$f(Z_1,\cdots,Z_n)=\prod_{j=1}^{k} k[1-\exp(-y_j)]^{k-1}\exp(-y_j)\left|\frac{\partial(y_1,\cdots,y_n)}{\partial(Z_1,\cdots,Z_n)}\right|$$

$$=k^n\prod_{j=2}^{n}[1-\exp(-Z_j)]^{k-1}\exp(-Z_j)\cdot\left\{1-\exp[-(Z_1-\sum_{j=2}^{n}Z_j)]\right\}^{k-1}\exp[-(Z_1-\sum_{j=2}^{n}Z_j)].$$

So the marginal density function of Z_1 is:

$$g(Z_1)=\underset{Z_1\geqq Z_2+\cdots+Z_n}{\int\cdots\int} k^n\prod_{j=2}^{n}[1-\exp(-Z_j)]^{k-1}\left\{1-\exp[-(Z_1-\sum_{j=2}^{n}Z_j)]\right\}^{k-1}\cdot\exp(-Z_1)dZ_2\cdots dZ_n$$

$$=\underset{\substack{Z_1\geqq Z_2+\cdots+Z_n\\ Z_i\geqq 0,\ i=2,\cdots,n}}{\int\cdots\int} k^n\prod_{j=2}^{n}[1-\exp(-Z_j)]^{k-1}\sum_{i=0}^{k-1}C_{k-1}^{i}(-1)^i\cdot$$

$$\exp[-(1+i)Z_1]\exp(i\sum_{j=2}^{n}Z_j)dZ_2\cdots dZ_n=k^n\sum_{i=0}^{k-1}C_{k-1}^{i}(-1)^i\exp[-(1+i)Z_1]\cdot$$

$$\underset{\substack{Z_1\geqq Z_2+\cdots Z_n\\ Z_i\geqq 0,\ i=2,\cdots,n}}{\int\cdots\int}\prod_{j=2}^{n}[1-\exp(-Z_j)]^{k-1}\cdot\exp(i\sum_{j=2}^{n}Z_j)dZ_2\cdots dZ_n.$$

Therefore, $C(\alpha, \lambda_1)$ is a constant such that $\alpha=\int_0^{\lambda_1 C(\alpha,\lambda_1)}g(Z_1)dZ_1$. In other words, if $c^*(\alpha)$ satisfies $\alpha=\int_0^{C^*(\alpha)}g(Z_1)dZ_1$, then $c(\alpha, \lambda_1)=c^*(\alpha)/\lambda_1$ because the critical region given above does not depend on the constant δ, so the test

$$\phi(\underline{t})=\begin{cases}1, & N_0>0 \text{ or } \sum_{j=1}^{n}t_{(k)j}\leqq c(\alpha,\lambda_1)\\ 0, & \text{otherwise}\end{cases}$$

is a UMP test of the hypothesis H_0: $\lambda_0=0 \longleftrightarrow K_0$: $\lambda_0>0$.

Now we calculate its power function $\beta_\phi(\lambda_0)$:

$$\beta_\phi(\lambda_0)=P_{\lambda=\lambda_0}\left\{N_0>0\right\}+P_{\lambda=\lambda_0}\left\{N_0=0,\ \sum_{j=1}^{n}t_{(k)j}\leqq c(\alpha,\lambda_1)\right\}.$$

$P_{\lambda_0}(N_0>0)=1-P_{\lambda=\lambda_0}(N_0=0)=1-[P_{\lambda=\lambda_0}(S_j=0)]^n=1-[P_{\lambda=\lambda_0}(S=0)]^n$,

where $P_{\lambda=\lambda_0}(S=0)=\sum_{i=1}^{k}\theta_i=\sum_{i=1}^{k}\int_0^\infty \nu_i\exp(-\nu_i t)[1-\exp(-\lambda_1 t)]^{k-1}dt$

$$=\sum_{i=1}^{k}\int_0^\infty(\lambda_1+\lambda_0)\exp(-(\lambda_1+\lambda_0)t\cdot[1-\exp(-\lambda_1 t)]^{k-1}dt$$

$$=\sum_{i=1}^{k}\frac{\lambda_1+\lambda_0}{k\lambda_1}\left\{\exp(-\lambda_0 t)[1-\exp(-\lambda_1 t)]^k\Big|_0^\infty\right.$$

$$\left.+\int_0^\infty\lambda_0\exp(-\lambda_0 t)[1-\exp(-\lambda_1 t)]^k dt\right\}$$

$$=\frac{k(\lambda_1+\lambda_0)}{k\lambda_1}\int_0^\infty \lambda_0\exp(-\lambda_0 t)\sum_{l=0}^{k}C_k^l(-1)^l\exp(-i\lambda_1 t)dt$$

$$=\frac{\lambda_1+\lambda_0}{\lambda_1}\sum_{l=0}^{k}C_k^l(-1)^l\int_0^\infty \lambda_0\exp[-(\lambda_0+i\lambda_1)t]dt$$

$$=\frac{\lambda_1+\lambda_0}{\lambda_1}\sum_{l=0}^{k}C_k^l(-1)^l\lambda_0\frac{1}{\lambda_0+i\lambda_1}=\frac{\lambda_0(\lambda_1+\lambda_0)}{\lambda_1}\sum_{l=0}^{k}C_k^l(-1)^l\frac{1}{\lambda_0+i\lambda_1}.$$

To calculate the density function of $t_{(k)}$, we first find its distribution function:

$$F(t_1,\cdots,t_k)=\bar{F}(0,\cdots,0)-\bar{F}(t_1,0,\cdots,0)-\bar{F}(0,t_2,\cdots,0)-\cdots-\bar{F}(0,0,\cdots,t_k)$$
$$+\bar{F}(t_1,t_2,0,\cdots,0)+\cdots+F(0,\cdots,t_{k-1},t_k)-\cdots(-1)^k\bar{F}(t_1,\cdots,t_k).$$

Thus,

$$P_{\lambda=\lambda_0}(t_{(k)}<t)=P_{\lambda=\lambda_0}(t_1<t,\cdots,t_k<t)=1-k\exp[-(\lambda_1+\lambda_0)t]+C_k^2\exp(-2\lambda_1 t-$$
$$\lambda_0 t)-\cdots+(-1)^k\exp(-k\lambda_1 t-\lambda_0 t)=1+\sum_{l=1}^{n}(-1)^lC_k^l\exp(-i\lambda_1 t-\lambda_0 t).$$

So the density function of $t_{(k)}$ becomes:

$$f_{tk}(t)=\sum_{l=1}^{k}(-1)^lC_k^l(-i\lambda_1-\lambda_0)\exp(-i\lambda_1 t-\lambda_0 t).$$

Assume $y_j=t_{(k)j}$, $j=1,\cdots,n$. By transforming

$$Z_1=\sum_{j=1}^{n}y_j$$
$$Z_2=y_2,$$
$$\vdots$$
$$Z_n=y_n,$$

we get $Z_1\geqq Z_2+\cdots+Z_n$. Therefore, the joint density function of $(Z_1,\cdots,Z_k)$ is:

$$f(Z_1,\cdots,Z_n)=\prod_{j=1}^{n}\left[\sum_{l=1}^{k}(-1)^lC_k^l(-i\lambda_1-\lambda_0)\exp(-i\lambda_1 y_j-\lambda_0 y_j)\right]$$
$$=\prod_{j=2}^{n}\left[\sum_{l=1}^{k}(-1)^lC_k^l(-i\lambda_1-\lambda_0)\exp(-i\lambda_1 Z_j-\lambda_0 Z_j)\right]\cdot$$
$$\left[\sum_{l=1}^{k}(-1)^lC_k^l(-i\lambda_1-\lambda_0)\exp((-i\lambda_1-\lambda_0)(Z_1-\sum_{j=2}^{n}Z_j))\right]$$
$$=\sum_{l=1}^{k}(-1)^lC_k^l(-i\lambda_1-\lambda_0)\exp[(-i\lambda_1-\lambda_0)Z_1]\cdot\exp\left[(i\lambda_1+\lambda_0)\sum_{j=2}^{n}Z_j\right]\cdot$$
$$\prod_{j=2}^{n}\left\{\sum_{l=1}^{k}(-1)^lC_k^l(-i\lambda_1-\lambda_0)\cdot\exp[-i\lambda_1-\lambda_0)Z_j]\right\}.$$

So the marginal density function of Z_1 is:

$$h(Z_1)=\sum_{l=1}^{k}(-1)^lC_k^l(-i\lambda_1-\lambda_0)\exp(-(i\lambda_1+\lambda_0)Z_1)\cdot$$
$$\int\cdots\int\exp\left[(i\lambda_1+\lambda_0)\sum_{j=2}^{n}Z_j\right]\prod_{j=2}^{n}\left[\sum_{l=1}^{k}(-1)^lC_k^l(-i\lambda_1-\lambda_0)\right]\cdot$$
$$Z_1\geqq Z_2+\cdots Z_n$$
$$Z_1\geqq 0,\ i=2,\cdots,n$$
$$\exp[(-i\lambda_1-\lambda_0)Z_j]dZ_2\cdots dZ_n.$$

Finally, the power function $\beta_\phi(\lambda_0)$ is given by:

$$\beta_\phi(\lambda_0)=P_{\lambda=\lambda_0}\{N_0>0\}+P_{\lambda=\lambda_0}\{N_0=0\}P_{\lambda=\lambda_0}\left\{\sum_{j=1}^{n}t_{(k)j}\leq C(\alpha,\lambda_1)\right\}$$

$$=1-\frac{\lambda_0^n(\lambda_1+\lambda_0)^n}{\lambda_1^n}\left[\sum_{i=0}^{k}C_k^i(-1)^i\frac{1}{\lambda_0+i\lambda_1}\right]^n+\frac{\lambda_0^n(\lambda_1+\lambda_0)^n}{\lambda_1^n}\left[\sum_{i=0}^{k}C_k^i(-1)^i\frac{1}{\lambda_0+i\lambda_1}\right]^n\int_0^{C(\alpha,\lambda_1)}h(Z_1)dZ_1.$$

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