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A Non-parametric Test for Equivalence of Marginal Distributions of a Bivariate Exponential Distribution

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Abstract

An asymptotically non-parametric method was used to test the equivalence of the marginal distributions of a bivariate exponential distribution and to compute Pitman's relative efficiency.

1. Introduction

We will use an asymptotically non-parametric method due to Raviv [5] to test the equivalence of the marginal distributions of a bivariate exponential distribution (BVED) [3, 4]. Raviv applied the method to a general continuous distribution and compared the method to the paired-t-test as well; the result was found to be satisfactory. We will use the method (called the R-test) to test the equivalence of the marginal distributions of a BVED and to calculate Pitman's relative efficiency. We will see that the minimum Pitman's relative efficiency is 3. By simulation, we will also find that the difference between the power of R-test and that of the t-test is small.

2. R-test and its applications

Assume $(\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}), \dots, (\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}) \sim \text{BVED}(\lambda_1, \lambda_2, \lambda_0)$ and note that the marginal distributions are given by

$$F(x) = 1 - e^{-\nu_1 x}, G(y) = 1 - e^{-\nu_2 y}, x, y \geq 0 \quad \text{where} \quad \nu_1 = \lambda_1 + \lambda_0, \nu_2 = \lambda_2 + \lambda_0.$$

We are going to test the following:

$$H_0: \nu_1 = \nu_2 \leftrightarrow H_1: \nu_1 < \nu_2.$$

$$\text{Define } Z_{ij} = \begin{cases} 1 & x_i > y_j, \\ 0 & \text{otherwise,} \end{cases} \quad i, j = 1, 2, \dots, n;$$

$$R = \sum_{i,j} Z_{ij}, \quad U = \sum_i \sum_j Z_{ij} - \sum_i Z_{ii} = U - S,$$

$$T_i = \sum_{j=1}^n (Z_{ij} + Z_{ji}) / (n-1), \quad i = 1, 2, \dots, n;$$

$$\bar{T} = \sum_{i=1}^n T_i / n; \quad \hat{\nu} = \sum_{i=1}^n (T_i - \bar{T})^2 / n.$$

It is known that $n^{\frac{1}{2}} \left[\frac{R}{n(n-1)} - p \right] / \hat{\nu}^{\frac{1}{2}}$ approximates to the standard normal distribution, where $p = EZ_{ij} = P(X_i > Y_j)$, $i \neq j$ [5]. When $n^{\frac{1}{2}} \left[\frac{R}{n(n-1)} - \frac{1}{2} \right] / \hat{\nu}^{\frac{1}{2}} > \Phi^{-1}(1 - \alpha)$,

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we reject hypothesis H_0 with probability α , and the test is called the R-test [5]. When a bivariate continuous distribution is a bivariate normal distribution, the paired-t-test is a UMPU [1].

To see if the power of the R-test has some loss, Raviv introduced Pitman's relative efficiency and calculated it:

$$\text{ARE}(R, t) = 12 \text{Var}_0(x) \left\{ \int f^2(x) dx \right\}^2 \left[\frac{1 - \rho_0(X, Y)}{1 - \rho_0\{F(x), F(y)\}} \right],$$

where $f(x)$ is the density function of X , ρ_0 is the correlation coefficient under H_0 . So when we use the R-test to test the equivalence of marginal distributions of a BVED, we are interested in calculating Pitman's relative efficiency of the R-test against the t-test, i. e., $\text{ARE}(R, t)$.

Change the BVED to the following:

$$\begin{aligned} F(x, y) &= \exp \{ -\lambda_1 x - (\lambda_1 + \theta) y - \lambda_0 \max(x, y) \} \\ &= \exp \{ -\lambda_1 x - (1 + \theta/\lambda_1) y - \lambda_0/\lambda_1 \cdot \max(\lambda_1 x, \lambda_1 y) \} \quad (\#) \end{aligned}$$

From the above, we can see that the R-test and the t-test do not depend on parameter λ_1 , so we can let $\lambda_1 = 1$ (where $\lambda_2 = \lambda_1 + \theta$), and the test

$$H_0: \lambda_1 = \lambda_2 \leftrightarrow H_1: \lambda_1 < \lambda_2$$

can be changed to the test:

$$H_0: \theta = 0 \leftrightarrow H_1: \theta > 0.$$

Furthermore, $f(x) = \nu_1 e^{-\nu_1 x}$ (the density function of x), $x > 0$, and

$$\int_0^\infty f^2(x) dx = \int_0^\infty \nu_1^2 \exp(-2\lambda_1 x) dx = \frac{\nu_1}{2},$$

$$\rho_0(X, Y) = \frac{\lambda_0}{\lambda}, \quad \text{where } \begin{cases} \nu_1 = \lambda_1 + \lambda_0 = 1 + \lambda_0, \\ \lambda = \lambda_1 + \lambda_1 + \lambda_0 = 2 + \lambda_0. \end{cases}$$

Assume that $F(X)$ is the distribution function of X . Then

$$F(X) = 1 - e^{-\nu_1 X}, \quad F_{11}(Y) = 1 - e^{-\nu_1 Y};$$

$$\begin{aligned} E_{11}F(X)F(Y) &= \int_0^\infty \int_0^\infty (1 - e^{-\nu_1 x})(1 - e^{-\nu_1 y}) dF(x, y) \\ &= \frac{1 + \lambda_0}{4 + 3\lambda_0} \quad [2], \end{aligned}$$

$$E_{11}F(X) = E_{11}F(Y) = 1/2,$$

$$D_{11}F(X) = D_{11}F(Y) = 1/12.$$

Thus,

$$\begin{aligned} \rho_0\{F(X), F(Y)\} &= \frac{\frac{1 + \lambda_0}{4 + 3\lambda_0} - \frac{1}{4}}{\frac{1}{12}} \\ &= \frac{12(1 + \lambda_0)}{4 + 3\lambda_0} - 3. \end{aligned}$$

And finally,

$$\begin{aligned} \text{ARE}(R, t) &= 12 \cdot \frac{1}{(1 + \lambda_0)^2} \left(\frac{1 + \lambda_0}{2} \right)^2 \cdot \frac{1 - \frac{\lambda_0}{\lambda_0 + 2}}{1 + 3 - \frac{12(1 + \lambda_0)}{4 + 3\lambda_0}} \\ &= \frac{9}{2} \left(1 - \frac{2}{3\lambda_0 + 6} \right). \end{aligned}$$

Therefore, $\min_{\lambda_0 \geq 0} \text{ARE}(R, t) = 3$.

3. Simulation

We drew 50 groups of samples of size 40 from a BVED model. The table indicates the rejected proportion number by the t-test and the R-test, respectively. When $L = 0$, this is the test for equivalence of marginal distributions. The R-test is somewhat more conservative than the t-test, i. e., type I error probability of the t-test is larger than that of the R-test, but the difference is small.

Table

The rejected proportion number of 50 groups of samples by the t-test and R-test

$$p1 = t_{78}(0.05) = 1.668$$

$$p2 = \Phi^{-1}(1 - 0.05) = \Phi^{-1}(0.95) = 1.64$$

λ_0	L^*	t	R
0.1	0	0.08	0
	0.2	0.2	0.14
	0.4	0.34	0.28
	0.6	0.54	0.3
0.2	0	0.1	0.02
	0.2	0.16	0.04
	0.4	0.26	0.18
	0.6	0.56	0.38
0.8	0	0.02	0
	0.2	0.1	0
	0.4	0.1	0
	0.6	0.2	0.1

* L indicates θ in (//).

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