Solution of Differential Equation by Means of Finite Element Method

Bulletin of the College of Science
University of the Ryukyus No. 48 :15−34 (1989)

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Daigoro HAYASHI*

*Department of Marine Sciences, University of the Ryukyus, Nishihara, Okinawa, 903−01, Japan.

(Received 1 June 1989)

Abstract

The finite element method may be defined as the modern "Method of Weighted Residuals" (MWR). This paper describes how to solve the differential equations which are essential in order to explain quantiatively a number of valuable geological and geodynamic problems.

The methods to solve linear differential equation, non-linear equation, non-linear non-steady equation, Laplace equation and incompreeible Newtonian flow problem are explained by means of the Galerkin finite element method.

Introduction

The writer intends to explain how any differential equation is solved by means of the "Finite Element Method" (FEM). As well known, the FEM is a powerful technique for any differential equation if one wants to solve them numerically. Also we know there are a number of cases where we need to treat them numerically because of the difficulty to handle them analytically, in which case the FEM gives us most useful methods.

The other technique called as the "Finite Difference Method" (FDM) is also the popular method to solve any differential equation, but there are some difficulties with treating boundary conditions.[1) 2) 3)]

If the differential equation concerned has its correspondent functional, we can solve it by means of the variational principle.[4) 5) 6) 7) 8) 9) 10)] We are better, however, to recognize the FEM as follows because it is not always the case that any differential equation has its correspondent functional.

The FEM is a method which belongs to a broad regime called the "Method of Weighted Residuals" (MWR), where several methods are recognized as the Galerkin method, least-squares method, collocation method and subdomain method.[11) 12) 13) 14) 15) 16)]

In this paper, the writer prefers the Galerkin method to introduce the final form of differential equations because the method is most basic and most extensively applicable to any differential equation.

The method of weighted residuals (MWR) will be explained as follows.

Let consider the next differential equation

$$-Au-f=0 \qquad \text{in } \Lambda \tag{1}$$

with boundary condition

$$Bu=g \qquad \text{on } \Gamma \tag{2}$$

where u is the unknown function, and f and g are the known functions. A and B are linear differential operators in the domain Λ and on the domain boundary Γ.

In order to solve the differential equation (1), we introduce the next function $\hat{u}$ to approximate the unknown function u.

$$\hat{u}=a_i \, \Phi_i \tag{3}$$

where Φ_i is called "basis function" which is the linearly independent functions to satisfy the boundary condition (2), and a_i are constants.

If we substitute the approximate function (3) into (1), the equation (1) doesn't keep zero because it is impossible to take infinite number of independent functions. Therefore there is a difference from zero which is called "residual" ε as follows.

$$\varepsilon=-A\hat{u}-f \tag{4}$$

Here we take the inner product of ε and w_i and set it zero, where the function w_i is called "weighting function".

$$(\varepsilon, w_i)=0 \tag{5}$$

From the set of equations (5) it is possible to calculate the constants a_i, therefore we can determine the approximate solution $\hat{u}$.

There are several methods among the MWR, and if we choose the weighting function same as the basis function, the method is called "Galerkin method" where the weighting function and basis function are specially called test function and trial function, respectively.

The main difference of the traditional MWR and the finite element method is attributed how to choose the basis function (or interpolation function). If we take the interpolation function to satisfy the whole domain, the method is equal to the traditional MWR. If the interpolation function will be satisfied only within a part of the domain (finite element), the method is called "finite element method".

1. Ordinary Differential Equation (1)

$$d^2u/dx^2 + \alpha u + \beta = 0 \quad (0 < x < 1) \tag{1.1}$$

$$u=u(x) \tag{1.2}$$

We will solve this equation as a simplest case of differential equation, in order to show how to handle with this by means of the Galerkin finite element method.

1.1 Galerkin finite element equation

The Galerkin finite element equation of the equation (1.1) is obtained through the following processes.

Let set the function

$$\hat{u}(x) = \Phi_i(x) u_i \qquad (1.1.1)$$

as the approximate function of u(x), we get a residual ε as

$$\varepsilon = d^2\hat{u}/dx^2 + \alpha\hat{u} + \beta \qquad (1.1.2)$$

Therefore, the inner product is

$$(\varepsilon, \Phi_i) = \int_0^1 (d^2u/dx^2 + \alpha u + \beta)\, \Phi_i\, dx = 0 \qquad (1.1.3)$$

where the symbol ^ is removed for the sake of simplicity.

Using partial integration, we obtain

$$(du/dx)\, \Phi_i |_0^1 - \int_0^1 \{ (du/dx)(d\Phi_i/dx) - \alpha u \Phi_i - \beta\Phi_i \} dx = 0 \qquad (1.1.4)$$

Although the approximate function $\hat{u}(x)$ is defined within the domain, it is not defined at the boundary points, and the first term is simply rewritten as

$$(du/dx)\, \Phi_i |_0^1 = (du/dx)|_0^1 \qquad (1.1.5)$$

Therefore the equation (1.1.4) is changed as

$$\int_0^1 \{ (du/dx)(d\Phi_i/dx) - \alpha u \Phi_i - \beta\Phi_i \} dx = (du/dx)|_0^1 \qquad (1.1.6)$$

Since the boundary condition is the Dirichlet type, we get

$$\int_0^1 \{ (du/dx)(d\Phi_i/dx) - \alpha u \Phi_i - \beta\Phi_i \} dx = 0 \qquad (1.1.7)$$

As the next relation holds from (1.1.1)

$$du/dx = (d/dx)(\Phi_i u_i) = (d\Phi_i/dx) u_i \qquad (1.1.8)$$

we obtain

$$\left[\int_0^1 \{ (d\Phi_i/dx)(d\Phi_j/dx) - \alpha\Phi_i\Phi_j \} dx\right] u_j = \int_0^1 \beta\Phi_i\, dx \qquad (1.1.9)$$

Let divide the domain $0 < x < 1$ to equal n subdomains and set length of each subdomain h ($h = 1/n$), we get

$$\left[\sum_{e=1}^{n} \int_0^h \{ (d\Phi_N^{(e)}/dx)(d\Phi_M^{(e)}/dx) - \alpha\Phi_N^{(e)}\Phi_M^{(e)} \} dx\, \Delta_{Ni}^{(e)}\, \Delta_{Mj}^{(e)}\right] u_j = \sum_{e=1}^{n} \int_0^h \beta\Phi_N^{(e)}\, dx\, \Delta_{Ni}^{(e)} \qquad (1.1.10)$$

where $\Delta_{Ni}^{(e)}$ and $\Delta_{Mj}^{(e)}$ are called "Boolean matrix" having the property

$\Delta_{Ni}^{(e)} = 1$, if the local node N coincide with the global node i.

$= 0$, otherwise.

Replacing the lengthy integrated terms by simpler notations as

$$A_{NM}^{(e)} \equiv \int_0^h \{ (d\Phi_N^{(e)}/dx)(d\Phi_M^{(e)}/dx) \} dx \qquad (1.1.11)$$

$$B_{NM}^{(e)} \equiv \int_0^h (-\alpha\Phi_N^{(e)}\Phi_M^{(e)})\, dx \qquad (1.1.12)$$

$$F_N^{(e)} \equiv \int_0^h \beta\Phi_N^{(e)}\, dx\, \Delta_{Ni}^{(e)} \qquad (1.1.13)$$

we have the Galerkin finite element equation which is called global form.

$$\left\{ \sum_{e=1}^{n} (A_{NM}^{(e)} + B_{NM}^{(e)})\, \Delta_{Ni}^{(e)}\, \Delta_{Mj}^{(e)} \right\} u_j = \sum_{e=1}^{n} F_N^{(e)}\, \Delta_{Ni}^{(e)} \qquad (1.1.14)$$

1.2 Linear interpolation function

Next linear function is specified as the approximate function of unknown function u(x) within the subdomain.

$$u^{(e)}(x) = a_1 + a_2 x \tag{1.2.1}$$

where $u^{(e)}(x)$ is the function $u(x)$ defined within the subdomain, and

$$u^{(e)}(0) = a_1 = u_1 \tag{1.2.2}$$

$$u^{(e)}(h) = a_1 + a_2 h = u_2 \tag{1.2.3}$$

Therefore we obtain

$$a_1 = u_1 \tag{1.2.4}$$

$$a_2 = (u_2 - u_1)/h \tag{1.2.5}$$

Substituting them to the equation (1.2.1), we have

$$u^{(e)} = u_1 + (u_2 - u_1)\, x/h = \Phi_i^{(e)} u_i \tag{1.2.6}$$

where

$$\Phi_1^{(e)} = 1 - (x/h) \tag{1.2.7}$$

$$\Phi_2^{(e)} = x/h \tag{1.2.8}$$

Differentiating each interpolation function $\Phi_1^{(e)}$ and $\Phi_2^{(e)}$ by x, we get

$$d\Phi_1^{(e)}/dx = -1/h \tag{1.2.9}$$

$$d\Phi_2^{(e)}/dx = 1/h \tag{1.2.10}$$

As we have the explicit form of interpolation function, we can calculate each component of $A_{NM}^{(e)}$, $B_{NM}^{(e)}$ and $F_N^{(e)}$ as follows.

$$A_{11}^{(e)} = 1/h \tag{1.2.11}$$

$$A_{12}^{(e)} = -1/h \tag{1.2.12}$$

$$A_{21}^{(e)} = A_{12}^{(e)} = -1/h \tag{1.2.13}$$

$$A_{22}^{(e)} = 1/h \tag{1.2.14}$$

$$B_{11}^{(e)} = -\alpha h/3 \tag{1.2.15}$$

$$B_{12}^{(e)} = -\alpha h/6 \tag{1.2.16}$$

$$B_{21}^{(e)} = B_{12}^{(e)} = -\alpha h/6 \tag{1.2.17}$$

$$B_{22}^{(e)} = -\alpha h/3 \tag{1.2.18}$$

$$F_1^{(e)} = \beta h/2 \tag{1.2.19}$$

$$F_2^{(e)} = \beta h/2 \tag{1.2.20}$$

2. Ordinary Differential Equation (2)

$$d^2u/dx^2 + \alpha u + \beta x = 0 \quad (0 < x < 1) \tag{2.1}$$

$$u = u(x) \tag{2.2}$$

Let solve the equation (2.1) as the other simple case of differential equations but including the term of x. The way to induce its Galerkin finite element equation is similar to the former case. Therefore, we will omit the detail description and write directly the final finite element equation.

$$\left[\sum_{e=1}^{n} \int_0^h \{ (d\Phi_N^{(e)}/dx)(d\Phi_M^{(e)}/dx) - \alpha \Phi_N^{(e)} \Phi_M^{(e)} \}\, dx\, \Delta_{Ni}^{(e)}\, \Delta_{Mj}^{(e)} \right] u_j = \sum_{e=1}^{n} \int_0^h \beta x\, \Phi_N^{(e)}\, dx\, \Delta_{Ni}^{(e)} \tag{2.3}$$

Replacing the integral terms by simpler terms as

$$A_{NM}^{(e)} \equiv \int_0^h \{ (d\Phi_N^{(e)}/dx)(d\Phi_M^{(e)}/dx) \}\, dx \tag{2.4}$$

$$B_{NM}^{(e)} \equiv \int_0^h (-\alpha \Phi_N^{(e)} \Phi_M^{(e)})\, dx \tag{2.5}$$

$$F_i \equiv \sum_{e=1}^{n} \int_0^h \beta x \Phi_N^{(e)} dx \Delta_{Ni}^{(e)} \tag{2.6}$$

we have

$$\{ \sum_{e=1}^{n} (A_{NM}^{(e)} + B_{NM}^{(e)}) \Delta_{Ni}^{(e)} \Delta_{Mj}^{(e)} \} u_j = F_i \tag{2.7}$$

2.1 Linear interpolation function

Since both terms of $A_{NM}^{(e)}$ and $B_{NM}^{(e)}$ is exactly equal to the former case, we need to seek the explicit form only for F_i . Considering the linear function for the approximate function of coordinate x within each subdomain as

$$x^{(e)} = \Phi_1^{(e)} x_1^{(e)} + \Phi_2^{(e)} x_2^{(e)} = \Phi_N^{(e)} x_N^{(e)} \tag{2.1.1}$$

and substituting (2.1.1) into (2.6), we get

$$F_i = \sum_{e=1}^{n} (\int_0^h \beta \Phi_N^{(e)} \Phi_M^{(e)} dx \Delta_{Ni}^{(e)}) x_M^{(e)} \tag{2.1.2}$$

Replacing the integrated term by $C_{NM}^{(e)}$ as

$$C_{NM}^{(e)} \equiv \beta \int_0^h \Phi_N^{(e)} \Phi_M^{(e)} dx \tag{2.1.3}$$

we have

$$F_i = \sum_{e=1}^{n} (C_{NM}^{(e)} \Delta_{Ni}^{(e)}) x_M^{(e)} \tag{2.1.4}$$

The term $C_{NM}^{(e)}$ is same as $B_{NM}^{(e)}$ except for the sign and coefficient β.

2.2 Quadratic interpolation function

One of the quadratic approximate functions for $u^{(e)}(x)$ and $x^{(e)}$ may be realized as follows.

(1) If we take

$$u^{(e)}(x) = a_1 + a_2 x + a_3 x^2 \tag{2.2.1}$$

this form is changed to

$$u^{(e)}(x) = \Phi_N^{(e)} u^{(e)} \tag{2.2.2}$$

where

$$\Phi_1^{(e)} = 1 - (3x/h) + 2(x/h)^2 \tag{2.2.3}$$

$$\Phi_2^{(e)} = 4 \{ (x/h) - (x/h)^2 \} \tag{2.2.4}$$

$$\Phi_3^{(e)} = \{ -(x/h) + 2(x/h)^2 \} \tag{2.2.5}$$

$$d\Phi_1^{(e)}/dx = \{ -3 + (4x/h) \} / h \tag{2.2.6}$$

$$d\Phi_2^{(e)}/dx = 4 \{ 1 - 2(x/h) \} / h \tag{2.2.7}$$

$$d\Phi_3^{(e)}/dx = \{ -1 + 4(x/h) \} / h \tag{2.2.8}$$

(2) Similarly the form of approximate function of $x^{(e)}$ is shown as

$$x^{(e)} = a_1 + a_2 x + a_3 x^2 \tag{2.2.9}$$

and is rewritten to

$$x^{(e)} = \Phi_N^{(e)} x_N^{(e)} \tag{2.2.10}$$

The explicit forms of each $\Phi_N^{(e)}$ are same as those of $u^{(e)}$.

Now, we can obtain each component of $A_{NM}^{(e)}$, $B_{NM}^{(e)}$ and $C_{NM}^{(e)}$ as follows.

(3) Explicit form of $A_{NM}^{(e)}$

$$A_{11}^{(e)} = 7/(3h) \tag{2.2.11}$$

$A_{12}{}^{(e)} = -8/(3h)$ (2.2.12)

$A_{13}{}^{(e)} = 1/(3h)$ (2.2.13)

$A_{22}{}^{(e)} = 16/(3h)$ (2.2.14)

$A_{23}{}^{(e)} = -8/(3h)$ (2.2.15)

$A_{33}{}^{(e)} = 7/(3h)$ (2.2.16)

(4) Explicit form of $B_{NM}{}^{(e)}$

$B_{11}{}^{(e)} = 2\alpha h/15$ (2.2.17)

$B_{12}{}^{(e)} = \alpha h/15$ (2.2.18)

$B_{13}{}^{(e)} = -\alpha h/30$ (2.2.19)

$B_{22}{}^{(e)} = 16\alpha h/30$ (2.2.20)

$B_{23}{}^{(e)} = \alpha h/15$ (2.2.21)

$B_{33}{}^{(e)} = 2\alpha h/15$ (2.2.22)

(5) Explicit form of $C_{NM}{}^{(e)}$

As the form of $C_{NM}{}^{(e)}$ is same as of $B_{NM}{}^{(e)}$ except for the sign and coefficient β, we need not obtain its components.

3. Linear Non-steady Differential Equation

$$\partial u/\partial t - \alpha\partial^2 u/\partial x^2 = 0 \quad (0 < x < 1) \tag{3.1}$$

$$u = u(x, t) \tag{3.2}$$

This is one of the simplest equation among non-steady differential equations.

3.1 Finite element equation

Supposing the next function

$$u(x) = \Phi_N(x)\, u_N \tag{3.1.1}$$

as one of the local approximate functions, residual ε is represented as

$$\varepsilon = \partial u/\partial t - \alpha\partial^2 u/\partial x^2 \tag{3.1.2}$$

Dividing equally the domain $0 < x < 1$ into n subdomains and taking the length of each subdomain as h (h=1/n), its local inner product is written as

$$(\varepsilon, \Phi_N) = \int_0^h (\partial u/\partial t - \alpha\partial^2 u/\partial x^2)\,\Phi_N\, dx = 0 \tag{3.1.3}$$

It should be noted that we will consider the local form hereafter. Applying partial integration to the equation (3.1.3) and using

$$\partial u/\partial t = \Phi_N\, \partial u_N/\partial t = \Phi_N\, \dot{u}_N \tag{3.1.4}$$

we obtain

$$\left(\int_0^h \Phi_N\, \Phi_M\, dx\right)\dot{u}_M + \left\{\int_0^h \alpha(d\Phi_N/dx)(d\Phi_M/dx)\,dx\right\}u_M = \alpha(\partial u/\partial x)\overset{*}{\Phi}_N \Big|_0^h \tag{3.1.5}$$

From the Dirichlet boundary condition, the equation takes the next form.

$$\left(\int_0^h \Phi_N\, \Phi_M\, dx\right)\dot{u}_M + \left\{\int_0^h \alpha(d\Phi_N/dx)(d\Phi_M/dx)\,dx\right\}u_M = 0 \tag{3.1.6}$$

Replacing the integrated terms by simpler terms as

$$A_{NM} \equiv \int_0^h \Phi_N\, \Phi_M\, dx \tag{3.1.7}$$

$$B_{NM} \equiv \int_0^h \alpha(d\Phi_N/dx)(d\Phi_M/dx)\,dx \tag{3.1.8}$$

we have

$$A_{NM}\dot{u}_M + B_{NM}u_M = 0 \quad (3.1.9)$$

3.2 Linear interpolation function

Supposing a linear function as the approximate function, we will obtain each component of A_{NM} and B_{NM} as follows.

$$A_{11} = h/3 \quad (3.2.1)$$

$$A_{12} = h/6 \quad (3.2.2)$$

$$A_{21} = A_{12} = h/6 \quad (3.2.3)$$

$$A_{22} = h/3 \quad (3.2.4)$$

$$B_{11} = \alpha/h \quad (3.2.5)$$

$$B_{12} = -\alpha/h \quad (3.2.6)$$

$$B_{21} = B_{12} = -\alpha/h \quad (3.2.7)$$

$$B_{22} = \alpha/h \quad (3.2.8)$$

3.3 Iteration scheme

In order to solve (3.1.9), we have to consider the equation as global form as follows.

$$A_{ij}\dot{u}_j + B_{ij}u_j = 0 \quad (3.3.1)$$

We approximate u_j and $\dot{u}_j$ as

$$u_j = (u_j^{(n+1)} + u_j^{(n)})/2 \quad (3.3.2)$$

$$\dot{u}_j = (u_j^{(n+1)} - u_j^{(n)})/\delta t \quad (3.3.3)$$

which is called "Crank - Nicolson" scheme. Substituting (3.3.2) and (3.3.3) to (3.3.1), we get

$$(A_{ij} + \delta t B_{ij}/2)u_j^{(n+1)} = (A_{ij} - \delta t B_{ij}/2)u_j^{(n)} \quad (3.3.4)$$

Replacing complex terms by simpler terms as

$$\hat{A}_{ij} \equiv A_{ij} + \delta t B_{ij}/2 \quad (3.3.5)$$

$$F_i^{(n)} \equiv (A_{ij} - \delta t B_{ij}/2)u_j^{(n)} \quad (3.3.6)$$

We have

$$\hat{A}_{ij}u_j^{(n+1)} = F_i^{(n)} \quad (3.3.7)$$

4. Non-linear Differential Equation

$$\alpha(d^2u/dx^2) - u(du/dx) = 0 \quad \text{in } \Lambda \quad (4.1)$$

$$u = u(x) \quad (4.2)$$

This is a simple non-linear differential equation with which we can demonstrate how to treat with the non-linear case.

4.1 Finite element equation

Let consider the next approximate function of $u(x)$

$$u(x) = \Phi_N(x)u_N = \Phi_N u_N \quad (4.1.1)$$

The inner product of the equation (4.1) is given as

$$(\varepsilon, \Phi_N) = \int_\Lambda \{ \alpha (d^2 u / dx^2) - u(du/dx) \} \Phi_N \, d\Lambda = 0 \quad (4.1.2)$$

Using partial integration, we change the equation as follows.

$$\{ \int_\Lambda \alpha (d\Phi_N / dx)(d\Phi_M / dx) d\Lambda \} u_M - \{ (1/2) \int (d\Phi_N / d\Lambda) \Phi_M \Phi_Q \, d\Lambda \} u_M u_Q = \int_\Gamma \{ \alpha (du/dx) - u^2/2 \} \overset{*}{\Phi}_N \, d\Gamma \quad (4.1.3)$$

Since the right hand side integral vanishes, we obtain

$$\{ \int_\Lambda \alpha (d\Phi_N / dx)(d\Phi_M / dx) d\Lambda \} u_M - \{ (1/2) \int_\Lambda (d\Phi_N / dx) \Phi_M \Phi_Q \, d\Lambda \} u_M u_Q = 0 \quad (4.1.4)$$

Replacing the integrated terms by simpler terms as

$$B_{NM} \equiv \int_\Lambda \alpha (d\Phi_N / dx)(d\Phi_M / dx) d\Lambda \quad (4.1.5)$$

$$A_{NMQ} \equiv -(1/2) \int_\Lambda (d\Phi_N / dx) \Phi_M \Phi_Q \, d\Lambda \quad (4.1.6)$$

We have the local finite element equation.

$$B_{NM} u_M + A_{NMQ} u_M u_Q = 0 \quad (4.1.7)$$

4.2 Linear interpolation function

Supposing a linear function as the approximate function of u(x), we obtain each component of A_{NMQ} and B_{NM} as follows.

(1) $A_{111} = 1/6$ (4.2.1)

$A_{112} = 1/12$ (4.2.2)

$A_{121} = A_{112} = 1/12$ (4.2.3)

$A_{122} = 1/6$ (4.2.4)

$A_{211} = -1/6$ (4.2.5)

$A_{212} = -A_{112} = -1/12$ (4.2.6)

$A_{221} = -A_{112} = -1/12$ (4.2.7)

$A_{222} = -A_{122} = -1/6$ (4.2.8)

(2) $B_{11} = \alpha/h$ (4.2.9)

$B_{12} = -\alpha/h$ (4.2.10)

$B_{21} = B_{12} = -\alpha/h$ (4.2.11)

$B_{22} = B_{11} = \alpha/h$ (4.2.12)

4.3 Iteration scheme

As (4.1.7) is expressed as local form, we will write this as global form.

$$B_{ij} u_j + A_{ijk} u_j u_k = 0 \quad (4.3.1)$$

In order to solve this non-linear equation by "Newton-Raphson method", considering the function $R_i(u)$ as

$$R_i(u) = A_{ijk} u_j u_k + B_{ij} u_j = 0 \quad (4.3.2)$$

and appling the Taylor expansion to (4.3.2), we have

$$R_i(u) = R_i(u^{(0)} + \delta u) = R_i(u^{(0)}) + (\partial R_i(u^{(0)}) / \partial u_j)(u_j - u_j^{(0)}) = 0 \quad (4.3.3)$$

Therefore, we have a next formula at the initial stage.

$$J_{ij}^{(0)} u_j = J_{ij}^{(0)} u_j^{(0)} - R_i(u^{(0)}) \quad (4.3.4)$$

where

$$J_{ij}^{(0)} = \partial R_i(u^{(0)}) / \partial u_j = (A_{ijk} + A_{ikj}) u_k^{(0)} + B_{ij} \quad (4.3.5)$$

Similarly, we obtain the next expression for the step r+1.

$$J_{ij}^{(r)} u_j^{(r+1)} = J_{ij}^{(r)} u_j^{(r)} - R_i(u^{(r)}) \quad (4.3.6)$$

where

$$J_{ij}^{(r)} = (A_{ijk} + A_{ikj}) u_k^{(r)} + B_{ij} \quad (4.3.7)$$

$$R_i(u^{(r)}) = A_{ijk} u_j^{(r)} u_k^{(r)} + B_{ij} u_j^{(r)} \quad (4.3.8)$$

5. Non-linear Non-steady Differential Equation

$$\partial u/\partial t + u(\partial u/\partial x) - \nu(\partial^2 u/\partial x^2) = 0 \quad (0 < x < 1) \quad (5.1)$$

$$u = u(x, t) \quad (5.2)$$

This equation is a famous equation known as " Burger's equation" which is the simplest non–linear non–steady differential equation, and it is the remarkable point that there are analytical solutions for this equation.

5.1 Finite element equation

Supposing the next function as the approximate function of $u(x,t)$,

$$u(x) = \Phi_N(x) u_N = \Phi_N u_N \quad (5.1.1)$$

we realize the semi-discrete approximation of the equation (5.1). The inner product of the equation (5.1) is represented as follows.

$$(\varepsilon, \Phi_N) = \int_0^h \{ \partial u/\partial t + u(\partial u/\partial x) - \nu(\partial^2 u/\partial x^2) \} \Phi_N \, dx = 0 \quad (5.1.2)$$

Applying the next relation to (5.1.2)

$$\partial u/\partial t = \Phi_N (\partial u_N/\partial t) = \Phi_N \dot{u}_N \quad (5.1.3)$$

we have

$$\int_0^h (\partial u/\partial t) \Phi_N \, dx + \int_0^h \{ \partial(u^2/2)/\partial x \} \Phi_N \, dx - \nu \int_0^h (\partial^2 u/\partial x^2) \Phi_N \, dx = 0 \quad (5.1.4)$$

Using the partial integration, we obtain

$$(\int_0^h \Phi_N \Phi_M \, dx) \dot{u}_M - \{ (1/2) \int_0^h (d\Phi_N/dx) \Phi_M \Phi_Q \, dx \} u_M u_Q + \{ \nu \int_0^h (d\Phi_N/dx)(d\Phi_M/dx) dx \} u_M = \{ -u^2/2 + \nu(\partial n^u/\partial x) \} \overset{*}{\Phi}_N |_0^h \quad (5.1.5)$$

As the integral of right hand side varnishes, we get

$$(\int_0^h \Phi_N \Phi_M \, dx) \dot{u}_M - \{ (1/2) \int_0^h (d\Phi_N/dx) \Phi_M \Phi_Q \, dx \} u_M u_Q + \{ \nu \int_0^h (d\Phi_N/dx)(d\Phi_M/dx) dx \} u_M = 0 \quad (5.1.6)$$

Replacing integral terms by simpler terms as

$$B_{NM} \equiv \int_0^h \Phi_N \Phi_M \, dx \quad (5.1.7)$$

$$A_{NMQ} \equiv -(1/2) \int_0^h (d\Phi_N/dx) \Phi_M \Phi_Q \, dx \quad (5.1.8)$$

$$C_{NM} \equiv \nu \int_0^h (d\Phi_N/dx)(d\Phi_M/dx) dx \quad (5.1.9)$$

We obtain the local finite element equation.

$$B_{NM} \dot{u}_M + A_{NMQ} u_M u_Q + C_{NM} u_M = 0 \quad (5.1.10)$$

5.2 Linear interpolation function

Supposing a linear function as the approximate function of $u(x, t)$ and executing the calculation to seek the explicit representation of matrices B_{NM}, A_{NMQ} and C_{NM}, we obtain them as follows.

(1) $B_{11} = h/3$ (5.2.1)

$B_{12} = h/6$ (5.2.2)

$B_{21} = B_{12} = h/6$ (5.2.3)

$B_{22} = h/3$ (5.2.4)

(2) A_{NMQ}

This is exactly same as the matrix A_{NMQ} of chapter 4.

(3) C_{NM}

This is also same as the matrix B_{NM} of chapter 4 except for the coefficient ν.

5.3 Iteration scheme

As the equation (5.1.10) is a local form, we need to change it to the global form as follows.

$$B_{ij}\,\dot{u}_j + A_{ijk}\,u_j\,u_k + C_{ij}\,u_j = 0 \tag{5.3.1}$$

In order to solve this non-steady differetial equation, we will use the Crank-Nicolson scheme as follows. Applying the next approximation for u_j and $\dot{u}_j$

$$u_j = (u_{j(n+1)} + u_j^{(n)})/2 \tag{5.3.2}$$

$$\dot{u}_j = (u_{j(n+1)} - u_j^{(n)})/\delta t \tag{5.3.3}$$

we have

$$(1/2)\,\delta t A_{ijk}\,u_k^{(n+1)}\,u_{j(n+1)} + \{B_{ij} + (1/2)\,\delta t C_{ij}\}\,u_j^{(n+1)} = \{B_{ij} - (1/2)\,\delta t\,(A_{ijk}\,u_k^{(n)} + C_{ij})\}\,u_j^{(n)} \tag{5.3.4}$$

Replacing complicated terms by simpler terms as

$$\hat{A}_{ijk} \equiv (1/2)\,\delta t\,A_{ijk} \tag{5.3.5}$$

$$\hat{B}_{ij} \equiv B_{ij} + (1/2)\,\delta t C_{ij} \tag{5.3.6}$$

$$\hat{F}_i \equiv \{B_{ij} - (1/2)\,\delta t\,(A_{ijk}\,u_k^{(n)} + C_{ij})\}\,u_j^{(n)} \tag{5.3.7}$$

we obtain

$$\hat{A}_{ijk}\,u_j^{(n+1)}\,u_{k(n+1)} + \hat{B}_{ij}\,u_j^{(n+1)} = \hat{F}_i \tag{5.3.8}$$

Replacing $u_j^{(n+1)}$ by u_j for simplicity, we have

$$\hat{A}_{ijk}\,u_j\,u_k + \hat{B}_{ij}\,u_j = \hat{F}_i \tag{5.3.9}$$

In order to apply the Newton-Raphson method to this equation, supposing the next function $R_i(u)$ as

$$R_i(u) = \hat{A}_{ijk}\,u_j\,u_k - (\hat{F}_i - \hat{B}_{ij}\,u_j) = 0 \tag{5.3.10}$$

we obtain the next relation for the initial stage.

$$J_{ij}^{(0)}\,u_j = J_{ij}^{(0)}\,u_j^{(0)} - R_i(u^{(0)}) \tag{5.3.11}$$

where

$$J_{ij}^{(0)} = \partial R_i(u^{(0)})/\partial u_j = (\hat{A}_{ijk} + \hat{A}_{ikj})\,u_k^{(0)} + \hat{B}_{ij} \tag{5.3.12}$$

Similarly we have the next formula for the step $r+1$.

$$J_{ij}^{(r)} u_j^{(r+1)} = J_{ij}^{(r)} u_j^{(r)} - R_i(u^{(r)}) \quad (5.3.13)$$

where

$$J_{ij}^{(r)} = (\hat{A}_{ijk} + \hat{A}_{ikj}) u_k^{(r)} + \hat{B}_{ij} \quad (5.3.14)$$

and

$$R_i(u^{(r)}) = \hat{A}_{ijk} u_j^{(r)} u_k^{(r)} - (\hat{F}_i - \hat{B}_{ij} u_j^{(r)}) \quad (5.3.15)$$

6. Non-linear steady Thermal Conduction

This is one of the examples for the non-linear differential equation described at chapter 4. The problem is written as

$$\alpha(\partial^2 T/\partial x^2) + (\beta/T)\exp(\gamma/T) = 0 \quad \text{in } \Lambda \quad (6.1)$$

$$T = T(x) \quad (6.2)$$

The local form of the Galerkin inner product of this equation is expressed as

$$\{-\alpha \int_0^h (\partial\Phi_N/\partial x)(\partial\Phi_M/\partial x)\,dx\} T_M + \int_0^h f(T)\Phi_N\,dx = 0 \quad (6.3)$$

The global form of this is shown as

$$B_{ij} T_j + \sum_{e=1}^{n} \Delta_{Ni} \Delta_{Mj} \Delta_{Qk} \{\int_0^h (\beta/\Phi_M T_M)\exp(\gamma/\Phi_Q T_Q)\Phi_N\,dx\} = 0 \quad (6.4)$$

Introducing $R_i(T)$ as

$$R_i(T) = B_{ij} T_j + \sum_{e=1}^{n} \Delta_{Qk}^{Ni}{}_{Mj} \{\int_0^h (\beta/\Phi_M T_M)\exp(\gamma/\Phi_Q T_Q)\Phi_N\,dx\} = 0 \quad (6.5)$$

and applying the Taylor expansion to the equation (6.5), we obtain the next relation.

$$J_{ij}^{(0)} T_j = J_{ij}^{(0)} T_j^{(0)} - R_i(T^{(0)}) \quad (6.6)$$

where

$$J_{ij}^{(0)} = \partial R_i(T^{(0)})/\partial T_j = B_{ij} + \sum_{e=1}^{n} \Delta_{Qk}^{Ni}{}_{Mj} [\int_0^h \{-\beta\Phi_N \Phi_j (\Phi_Q T_Q + \gamma) / (\Phi_Q T_Q)^3\}\exp(\gamma/\Phi_Q T_Q)\,dx] \quad (6.7)$$

7. Non-linear Non-steady Thermal Conduction

This is one of the case for chapter 5. The governing equation of this problem is expressed as

$$\partial T/\partial t + \alpha(\partial^2 T/\partial x^2) + (\beta/T)\exp(\gamma/T) = 0 \quad \text{in } \Delta \quad (7.1)$$

$$T = T(x, t) \quad (7.2)$$

7.1 Finite element equation

Supposing the next function

$$T(x,t) = \Phi_N(x) T_N \quad (7.1.1)$$

as the approximate function of $T(x,t)$, the residual ε is written as

$$\varepsilon = \partial T/\partial t + \alpha(\partial^2 T/\partial x^2) + f(T) \quad (7.1.2)$$

where

$$f(T) = (\beta/T)\exp(\gamma/T) \quad (7.1.3)$$

Therefore the inner product of (7.1) is expressed in the local form as

$$(\epsilon, \Phi_N) = \int_0^h \{ (\partial T/\partial t) + \alpha(\partial^2 T/\partial x^2) + f(T) \} \Phi_N \, dx = 0 \qquad (7.1.4)$$

Applying the next relation to (7.1.4)

$$\partial T/\partial t = \Phi_N (\partial T_N/\partial t) = \Phi_N \dot{T}_N \qquad (7.1.5)$$

we have

$$(\int_0^h \Phi_N \Phi_M \, dx)\dot{T}_M + \{ -\alpha \int_0^h (\partial \Phi_N/\partial x)(\partial \Phi_M/\partial x) dx \} T_M$$
$$+ \beta \int_0^h (1/\Phi_M T_M) \exp(\gamma/\Phi_Q T_Q) \Phi_N \, dx = 0 \qquad (7.1.6)$$

Replacing complex terms by simpler terms as

$$A_{NM} \equiv \int_0^h \Phi_N \Phi_M \, dx \qquad (7.1.7)$$

$$B_{NM} \equiv -\alpha \int_0^h (\partial \Phi_N/\partial x)(\partial \Phi_M/\partial x) \, dx \qquad (7.1.8)$$

$$C_N \equiv \beta \int_0^h (1/\Phi_M T_M) \exp(\gamma/\Phi_Q T_Q) \Phi_N \, dx \qquad (7.1.9)$$

we obtain the local finite element equation.

$$A_{NM}\dot{T}_M + B_{NM}T_M + C_N = 0 \qquad (7.1.10)$$

7.2 Linear interpolation function

Supposing a linear function for the approximate function of $T(x,t)$, we can obtain each component of A_{NM} and B_{NM} by direct integration.

It should be noted that it needs the numerical integration to obtain each component of C_N which is expanding as

$$C_N = \beta \int_0^h \{ 1/(\Phi_1 T_1 + \Phi_2 T_2) \} \exp\{ \gamma/(\Phi_1 T_1 + \Phi_2 T_2) \} \Phi_N \, dx \qquad (7.2.1)$$

Differentiating C_N in terms of T_1 and T_2, we get

$$\partial C_N/\partial T_1 = \beta \int_0^h \{ -\Phi_1 (\Phi_1 T_1 + \Phi_2 T_2 + \gamma)/(\Phi_1 T_1 + \Phi_2 T_2)^3 \}$$
$$\exp\{ \gamma/(\Phi_1 T_1 + \Phi_2 T_2) \} \Phi_N \, dx \qquad (7.2.2)$$

$$\partial C_N/\partial T_2 = \beta \int_0^h \{ -\Phi_2 (\Phi_1 T_1 + \Phi_2 T_2 + \gamma)/(\Phi_1 T_1 + \Phi_2 T_2)^3 \}$$
$$\exp\{ \gamma/(\Phi_1 T_1 + \Phi_2 T_2) \} \Phi_N \, dx \qquad (7.2.3)$$

7.3 Iteration scheme

Writing (7.1.10) as global form, we have

$$A_{ij}\dot{T}_j + B_{ij}T_j + \sum_{e=1}^{n} \Delta_{Ni} \Delta_{Mj} \Delta_{Qk} \{ \beta \int_0^h (1/\Phi_M T_M) \exp(\gamma/\Phi_Q T_Q) \Phi_N \, dy \} = 0 \qquad (7.3.1)$$

Applying the Crank-Nicolson scheme to the equation (7.3.1)

$$T_j = (T_j^{(n+1)} + T_j^{(n)})/2 \qquad (7.3.2)$$

$$\dot{T}_j = (T_j^{(n+1)} - T_j^{(n)})/\delta t \qquad (7.3.3)$$

we have

$$\{ \hat{A}_{ij} + (\delta t/2) B_{ij} \} T_j^{(n+1)} + \delta t C_i (T^{(n+1)}) = \{ A_{ij} - (\delta t/2) B_{ij} \} T_j^{(n)} \qquad (7.3.4)$$

Replacing complicated terms by simpler terms as

$$\hat{A}_{ij} \equiv A_{ij} + (1/2)\, \delta t B_{ij} \qquad (7.3.5)$$

$$\hat{C}_i \equiv \delta t C_i \qquad (7.3.6)$$

$$\hat{F}_i^{(n)} \equiv \{ A_{ij} - (1/2)\, \delta t B_{ij} \} T_j^{(n)} \tag{7.3.7}$$

we obtain

$$\hat{A}_{ij}\ T_j^{(n+1)} + \hat{C}_i(T_i^{(n+1)}) = \hat{F}_i^{(n)} \tag{7.3.8}$$

Abbreviating the superscripts(n+1) and (n) for the sake of simplicity, we get

$$\hat{A}_{ij}\ T_j + \hat{C}_i\ (T_i) = \hat{F}_i \tag{7.3.9}$$

Setting the function $R_i(T)$ as

$$R_i(T) = \hat{A}_{ij}\ T_j + \hat{C}_i(T_i) - \hat{F}_i = 0 \tag{7.3.10}$$

and applying the Taylor expansion to (7.3.10), we have

$$J_{ij}^{(0)}\ T_j = J_{ij}^{(0)}\ T_j^{(0)} - R_i(T^{(0)}) \tag{7.3.11}$$

where

$$J_{ij}^{(0)} = \partial R_i\ (T^{(0)}) / \partial T_j = \hat{A}_{ij} + \delta t\ \partial C_i / \partial T_j \tag{7.3.12}$$

8. Laplace Equation

$$\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 = 0 \qquad \text{in } \Lambda \tag{8.1}$$

$$u = u(x, y) \tag{8.2}$$

Rewriting this as tensor form, we get

$$u,_{ii} = 0 \qquad (i = 1, 2) \tag{8.3}$$

This is one of the simplest partial differential equations which has analytical solutions under special boundary conditions, however this equation can frequently describe some interesting geological phenomena.

8.1 Finite element equation

Taking the next function

$$u\ (x_i) = \Phi_N\ (x_i)\ u_N \tag{8.1.1}$$

as the approximate function of $u(x_i)$, the residual ϵ is represented as

$$\epsilon = u,_{ii}\ (x_i) = u,_{ii} \tag{8.1.2}$$

Therefore the inner product is obtained as

$$(\epsilon, \Phi_N) = \int_\Lambda u,_{ii}\ \Phi_N\ d\Lambda = 0 \tag{8.1.3}$$

Using the Green-Gauss' theorem, we get

$$\int_\Lambda u,_i\ \Phi_{N,i}\ d\Lambda = \int_\Gamma u,_i\ u_i\ \overset{*}{\Phi}_N\ d\Gamma \tag{8.1.4}$$

Considering the next relation

$$u,_i = \Phi_{N,i}\ u_N \tag{8.1.5}$$

we obtain

$$\left(\int_\Lambda \Phi_{M,i}\ \Phi_{N,i}\ d\Lambda\right) u_M = F_N \tag{8.1.6}$$

Replacing the left hand integral as

$$A_{MN} \equiv \int_\Lambda \Phi_{M,i}\ \Phi_{N,i}\ d\Lambda \tag{8.1.7}$$

we obtain the local finite element equation.

$$A_{MN}\, u_M = F_N \tag{8.1.8}$$

8.2 Linear triangular element

Supposing a linear function as the approximate function of $u(x_i)$, we have

$$u(x_i) = \Phi_N(x_i)\, u_N \tag{8.2.1}$$

where

$$\Phi_N = a_N + b_N x + c_N y \quad (N=1,3) \tag{8.2.2}$$

$$\Phi_{N,1} = b_N \tag{8.2.3}$$

$$\Phi_{N,2} = c_N \tag{8.2.4}$$

and

$$a_1 = (x_2 y_3 - x_3 y_2)/(2A) \tag{8.2.5}$$

$$b_1 = (y_2 - y_3)/(2A) \tag{8.2.6}$$

$$c_1 = (x_3 - x_2)/(2A) \tag{8.2.7}$$

$$a_2 = (x_3 y_1 - x_1 y_3)/(2A) \tag{8.2.8}$$

$$b_2 = (y_3 - y_1)/(2A) \tag{8.2.9}$$

$$c_2 = (x_1 - x_3)/(2A) \tag{8.2.10}$$

$$a_3 = (x_1 y_2 - x_2 y_1)/(2A) \tag{8.2.11}$$

$$b_3 = (y_1 - y_2)/(2A) \tag{8.2.12}$$

$$c_3 = (x_2 - x_1)/(2A) \tag{8.2.13}$$

and A is the area of the triangle.
The matrix A_{MN} is easily obtained by using the above coefficients.

8.3 Quadratic triangular element

Supposing a quadratic function as the approximate function of $u(x_i)$, we have

$$u(x_i) = \Phi_N(x_i)\, u_N \tag{8.3.1}$$

where

$$\Phi_1 = (2L_1 - 1)L_1 \tag{8.3.2}$$

$$\Phi_2 = (2L_2 - 1)L_2 \tag{8.3.3}$$

$$\Phi_3 = (2L_3 - 1)L_3 \tag{8.3.4}$$

$$\Phi_4 = 4L_1 L_2 \tag{8.3.5}$$

$$\Phi_5 = 4L_2 L_3 \tag{8.3.6}$$

$$\Phi_6 = 4L_3 L_1 \tag{8.3.7}$$

and L_N is defined as

$$L_N = a_N + b_N x + c_N y \quad (N=1,6) \tag{8.3.8}$$

Therefore their derivatives are denoted as

$$\Phi_{1,1} = (4L_1 - 1)\, b_1 \tag{8.3.9}$$

$$\Phi_{2,1} = (4L_2 - 1)\, b_2 \tag{8.3.10}$$

$$\Phi_{3,1} = (4L_3 - 1)\, b_3 \tag{8.3.11}$$

$$\Phi_{4,1} = 4(b_1 L_2 + b_2 L_1) \tag{8.3.12}$$

$$\Phi_{5,1} = 4(b_2 L_3 + b_3 L_2) \tag{8.3.13}$$

$$\Phi_{6,1} = 4(b_3 L_1 + b_1 L_3) \tag{8.3.14}$$

$\Phi_{1,2} = (4L_1 - 1)c_1$
$\Phi_{2,2} = (4L_2 - 1)c_2$
$\Phi_{3,2} = (4L_3 - 1)c_3$
$\Phi_{4,2} = 4(c_1 L_2 + c_2 L_1)$
$\Phi_{5,2} = 4(c_2 L_3 + c_3 L_2)$
$\Phi_{6,2} = 4(c_3 L_1 + c_1 L_3)$

The form of matrix A_{MN} is easily obtained by using the above coefficients.

8.4. Linear rectangular element

Supposing a linear function for the approximate function of $u(x_i)$ and using the rectangular element as

$$u(x_i) = \Phi_N(x_i)u_N \qquad (N=1,4) \tag{8.4.1}$$

the explicit form of Φ_N is written as follows.

$$\Phi_1 = (1-\xi)(1-\eta) \tag{8.4.2}$$
$$\Phi_2 = \xi(1-\eta) \tag{8.4.3}$$
$$\Phi_3 = \xi\eta \tag{8.4.4}$$
$$\Phi_4 = (1-\xi)\eta \tag{8.4.5}$$

where each ξ and η are defind as

$$\xi = x/(2a) \tag{8.4.6}$$
$$\eta = y/(2b) \tag{8.4.7}$$

Therefore the derivatives of Φ_N is represented as follows.

$$\Phi_{1,1} = -(1-\eta)/(2a) \tag{8.4.8}$$
$$\Phi_{2,1} = (1-\eta)/(2a) \tag{8.4.9}$$
$$\Phi_{3,1} = \eta/(2a) \tag{8.4.10}$$
$$\Phi_{4,1} = -\eta/(2a) \tag{8.4.11}$$
$$\Phi_{1,2} = -(1-\xi)/(2b) \tag{8.4.12}$$
$$\Phi_{2,2} = -\xi/(2b) \tag{8.4.13}$$
$$\Phi_{3,2} = \xi/(2b) \tag{8.4.14}$$
$$\Phi_{4,2} = (1-\xi)/(2b) \tag{8.4.15}$$

We can easily obtain the explicit form of matrix A_{MN}.

9. Incompressible Newtonian Flow

$$\sigma_{ij,j} + \rho f_i = 0 \quad \text{in } A \quad (i,j = 1, 2) \tag{9.1}$$
$$v_{i,i} = 0 \tag{9.2}$$

These are useful equations which can describe many geological phenomena, for example, diapirism, folding, magma migration and others.

9.1 Finite element equation

Taking $\hat{\sigma}_{ij}(x_i)$ and $\hat{v}_i(x_i)$ as approximate functions of $\sigma_{ij}(x_i)$ and $v_i(x_i)$, one of the residuals ϵ_1 is represented as

$$\varepsilon_1 = \hat{\sigma}_{ij,j} + \rho f_i \quad (9.1.1)$$

Removing the ^ for the sake of simplicity, we get

$$\varepsilon_1 = \sigma_{ij,j} + \rho f_i \quad (9.1.2)$$

The other residual ε_2 is written as

$$\varepsilon_2 = v_{i,i} \quad (9.1.3)$$

Therefore the inner products of the equations (9.1) and (9.2) are

$$(\varepsilon_1, \Phi_N) = \int_\Lambda (\sigma_{ij,j} + \rho f_i)\Phi_N \, d\Lambda = 0 \quad (9.1.4)$$

$$(\varepsilon_2, \psi_Q) = \int_\Lambda v_{i,i} \psi_Q \, d\Lambda = 0 \quad (9.1.5)$$

Applying the Green - Gauss' theorem for (9.1.4), we have

$$\int_\Lambda \sigma_{ij} \Phi_{N,j} \, d\Lambda = \int_\Gamma S_i \overset{*}{\Phi}_N \, d\Gamma + \int_\Lambda \rho f_i \Phi_N \, d\Lambda \quad (9.1.6)$$

According to the constitutive law of Newtonian fluid

$$\sigma_{ij} = -p\delta_{ij} + \mu(v_{i,j} + v_{j,i}) \cdot \quad (9.1.7)$$

the equation (9.1.6) is written in terms of $p(x_i)$ and $v_i(x_i)$ as,

$$-\int_\Lambda p\Phi_{N,i} \, d\Lambda + \int_\Lambda \mu(v_{i,j} + v_{j,i})\Phi_{N,j} \, d\Lambda = \int_\Gamma S_i \overset{*}{\Phi}_N \, d\Gamma + \int_\Lambda \rho f_i \Phi_N \, d\Lambda \quad (9.1.8)$$

Supposing the approximate functions of $v_i(x_i)$ and $p(x_i)$ as

$$\hat{v}_i(x_i) = \Phi_N(x_i) v_{Ni} \quad (9.1.9)$$

$$\hat{p}(x_i) = \psi_Q(x_i) P_Q \quad (9.1.10)$$

we have a next formula from (9.1.9) as

$$v_{i,j}(x_i) = \Phi_{N,j} v_{Ni} \quad (9.1.11)$$

and the finite element equation of (9.1.8) is expressed as

$$\left(-\int_\Lambda \psi_Q \Phi_{N,i} \, d\Lambda\right) P_Q + \left\{\int_\Lambda \mu (\Phi_{M,j} \Phi_{N,j} \delta_i{}^k + \Phi_{M,i} \Phi_{N,j} \delta_j{}^k) d\Lambda\right\} v_{Mk} = \int_\Gamma S_i \overset{*}{\Phi}_N \, d\Gamma + \int_\Lambda \rho f_i \Phi_N \, d\Lambda \quad (9.1.12)$$

While the other inner product (9.1.5) is written similarly using the Green-Gauss' theorem as

$$\left(\int_\Lambda \Phi_N \psi_{Q,i} \, d\Lambda\right) v_{Ni} = \int_\Gamma v_i n_i \overset{*}{\psi}_Q \, d\Gamma \quad (9.1.13)$$

Since the Dirichlet boundary condition is considered, we obtain next equations from (9.1.12) and (9.1.13).

$$\left(-\int_\Lambda \psi_Q \Phi_{N,i} \, d\Lambda\right) P_Q + \left\{\int_\Lambda \mu (\Phi_{M,j} \Phi_{N,j} \delta_i{}^k + \Phi_{M,i} \Phi_{N,j} \delta_j{}^k) d\Lambda\right\} v_{Mk} = \int_\Lambda \rho f_i \Phi_N \, d\Lambda \quad (9.1.14)$$

$$\left(\int_\Lambda \Phi_N \psi_{Q,i} \, d\Lambda\right) v_{Ni} = 0 \quad (9.1.15)$$

Applying the next replacements to (9.1.14) and (9.1.15)

$$C_{NiQ} \equiv -\int_\Lambda \psi_Q \Phi_{N,i} \, d\Lambda \quad (9.1.16)$$

$$D_{NMik} \equiv \int_\Lambda \mu (\Phi_{M,j} \Phi_{N,j} \delta_i{}^k + \Phi_{M,i} \Phi_{N,j} \delta_j{}^k) \, d\Lambda \quad (9.1.17)$$

$$E_{Ni}{}^{(b)} \equiv \int_\Lambda \rho f_i \Phi_N \, d\Lambda \quad (9.1.18)$$

$$R_{QiN} \equiv \int_\Lambda \Phi_N \psi_{Q,i} \, d\Lambda \quad (9.1.19)$$

We obtain the local finite element equation.

$$C_{NiQ} P_Q + D_{NMik} v_{Mk} = E_{Ni}{}^{(b)} \quad (9.1.20)$$

$$R_{QiN} v_{Ni} = 0 \quad (9.1.21)$$

9.2 Linear triangular element

Supposing the next linear functions as the approximate function of $v_i(x_i)$ and $p(x_i)$

$$v_i(x_i) = \Phi_N(x_i)\, v_{Ni} \qquad (9.2.1)$$

$$p(x_i) = \Phi_N(x_i)\, P_N \quad (N=1,3) \qquad (9.2.2)$$

the interpolation function Φ_N is explicitly written as follows.

$$\Phi_N = a_N + b_N x + c_N y \qquad (9.2.3)$$

where coefficients are defined same as the former case.

The form of composite finite element matrix is easily represented by the above coefficients.

9.3 Quadratic and linear triangular element

We consider here also the triangular element, but the approximate function of $v_i(x_i)$ is taken as quadratic function and that of $p(x_i)$ is still linear as follows.

$$v_i(x_i) = \Phi_N(x_i)\, v_{Ni} \quad (N=1,6) \qquad (9.3.1)$$

$$p(x_i) = \psi_Q(x_i)\, P_Q \quad (Q=1,3) \qquad (9.3.2)$$

The explicit form of interplolation function Φ_N is taken as

$$\Phi_1 = (2L_1 - 1)L_1 \qquad (9.3.3)$$

$$\Phi_2 = (2L_2 - 1)L_2 \qquad (9.3.4)$$

$$\Phi_3 = (2L_3 - 1)L_3 \qquad (9.3.5)$$

$$\Phi_4 = 4L_1L_2 \qquad (9.3.6)$$

$$\Phi_5 = 4L_2L_3 \qquad (9.3.7)$$

$$\Phi_6 = 4L_3L_1 \qquad (9.3.8)$$

therefore their derivatives are written as

$$\Phi_{1,1} = b_1(4L_1 - 1) \qquad (9.3.9)$$

$$\Phi_{2,1} = b_2(4L_2 - 1) \qquad (9.3.10)$$

$$\Phi_{3,1} = b_3(4L_3 - 1) \qquad (9.3.11)$$

$$\Phi_{4,1} = 4(b_2L_1 + b_1L_2) \qquad (9.3.12)$$

$$\Phi_{5,1} = 4(b_3L_2 + b_2L_3) \qquad (9.3.13)$$

$$\Phi_{6,1} = 4(b_3L_1 + b_1L_3) \qquad (9.3.14)$$

$$\Phi_{1,2} = c_1(4L_1 - 1) \qquad (9.3.15)$$

$$\Phi_{2,2} = c_2(4L_2 - 1) \qquad (9.3.16)$$

$$\Phi_{3,2} = c_3(4L_3 - 1) \qquad (9.3.17)$$

$$\Phi_{4,2} = 4(c_2L_1 + c_1L_2) \qquad (9.3.18)$$

$$\Phi_{5,2} = 4(c_3L_2 + c_2L_3) \qquad (9.3.19)$$

$$\Phi_{6,2} = 4(c_3L_1 + c_1L_3) \qquad (9.3.20)$$

where L_N is defined as

$$L_N = a_N + b_N x + c_N y \qquad (9.3.21)$$

Coefficients a_N , b_N and c_N are same as the former case. The final form of composite finite element matrix is easily obtained by using these coefficients.

9.4 Linear rectangular element

Supposing linear functions for the approximate function of $v_i\ (x_i)$ and $p(x_i)$ in the rectangular element, we have

$$v_i(x_i) = \Phi_N\ (x_i)\, v_{Ni} \qquad (9.4.1)$$

$$p(x_i) = \Phi_N(x_i)\, P_N \qquad (N=1,4) \qquad (9.4.2)$$

where the form of the interpolation function Φ_N is described as follows.

$$\Phi_1 = (1-\xi)(1-\eta) \qquad (9.4.3)$$

$$\Phi_2 = \xi(1-\eta) \qquad (9.4.4)$$

$$\Phi_3 = \xi\eta \qquad (9.4.5)$$

$$\Phi_4 = (1-\xi)\,\eta \qquad (9.4.6)$$

and

$$\xi = x/(2a) \qquad (9.4.7)$$

$$\eta = y/(2b) \qquad (9.4.8)$$

We have their derivatives from the above formulae as

$$\Phi_{1,1} = -(1-\eta)/(2a) \qquad (9.4.9)$$

$$\Phi_{2,1} = (1-\eta)/(2a) \qquad (9.4.10)$$

$$\Phi_{3,1} = \eta/(2a) \qquad (9.4.11)$$

$$\Phi_{4,1} = -\eta/(2a) \qquad (9.4.12)$$

$$\Phi_{1,2} = -(1-\xi)/(2b) \qquad (9.4.13)$$

$$\Phi_{2,2} = -\xi/(2b) \qquad (9.4.14)$$

$$\Phi_{3,2} = \xi/(2b) \qquad (9.4.15)$$

$$\Phi_{4,2} = (1-\xi)/(2b) \qquad (9.4.16)$$

The final form of composite finite element matrix is easily obtained by using the above coefficients.

9.5 Linear isoparametric element

Supposing isoparametric linear functions as the approximate functions of $v_i\ (x_i)$ and $p(x_i)$, we have

$$v_i\ (x_i) = \Phi_N(x_i)\, v_{Ni} \qquad (9.5.1)$$

$$p(x_i) = \Phi_N(x_i)\, P_N \quad (N=1,4) \qquad (9.5.2)$$

where the form of interpolation function Φ_N is

$$\Phi_1 = (1-\xi)(1-\eta)/4 \qquad (9.5.3)$$

$$\Phi_2 = (1+\xi)(1-\eta)/4 \qquad (9.5.4)$$

$$\Phi_3 = (1+\xi)(1+\eta)/4 \qquad (9.5.5)$$

$$\Phi_4 = (1-\xi)(1+\eta)/4 \qquad (9.5.6)$$

These derivatives in terms of ξ and η are

$$\Phi_{1,1} = \partial\Phi_1/\partial\xi = -(1-\eta)/4 \qquad (9.5.7)$$

$$\Phi_{2,1} = (1-\eta)/4 \tag{9.5.8}$$
$$\Phi_{3,1} = (1+\eta)/4 \tag{9.5.9}$$
$$\Phi_{4,1} = -(1+\eta)/4 \tag{9.5.10}$$
$$\Phi_{1,2} = \partial\Phi_1/\partial\eta = -(1-\xi)/4 \tag{9.5.11}$$
$$\Phi_{2,2} = -(1+\xi)/4 \tag{9.5.12}$$
$$\Phi_{3,2} = (1+\xi)/4 \tag{9.5.13}$$
$$\Phi_{4,2} = (1-\xi)/4 \tag{9.5.14}$$

Region integral is expressed in terms of $|J|$ as follows.

$$\int_\Lambda dA = \int_{-1}^{1}\int_{-1}^{1} |J|\, d\xi\, d\eta \tag{9.5.15}$$

where

$$|J| = \{(b_1c_2 - c_1b_2) + (b_1d_2 - d_1b_2)\xi + (d_1c_2 - c_1d_2)\eta\}/16 \tag{9.5.16}$$

Finite element matrices are represented as follows.

(1) $$C_{NiQ} = -(1/32)\int_{-1}^{1}\int_{-1}^{1}(A_{Ni} + B_{Ni}{}^1\xi_1 + B_{Ni}{}^2\xi_2)(1+\xi_{Q1}\xi_1)(1+\xi_{Q2}\xi_2)\,d\xi\, d\eta \tag{9.5.17}$$

(2) $$R_{QiN} = -C_{QiN} \tag{9.5.18}$$

(3) $$D_{NMik} = (\mu/8)\int_{-1}^{1}\int_{-1}^{1}[\{(A_{Nj} + B_{Nj}{}^1\xi_1 + B_{Nj}{}^2\xi_2)(A_{Mj} + B_{Mj}{}^1\xi_1 + B_{Mj}{}^2\xi_2)/(\alpha_0 + \alpha_1\xi_1 + \alpha_2\xi_2)\}\delta i^k + \{(A_{Nj} + B_{Nj}{}^1\xi_1 + B_{Nj}{}^2\xi_2)(A_{Mi} + B_{Mi}{}^1\xi_1 + B_{Mi}{}^2\xi_2)/(\alpha_0 + \alpha_1\xi_1 + \alpha_2\xi_2)\}\delta j^k]\,d\xi\, d\eta \tag{9.5.19}$$

Then the matrix is expressed separately as follows.

If $i = k$,

$$D_{NMik} = (\mu/8)\int_{-1}^{1}\int_{-1}^{1}[\{(A_{N1} + B_{N1}{}^1\xi_1 + B_{N1}{}^2\xi_2)(A_{M1} + B_{M1}{}^1\xi_1 + B_{M2}{}^2\xi_2) + \{(A_{N2} + B_{N2}{}^1\xi_1 + B_{N2}{}^2\xi_2)(A_{M2} + B_{M2}{}^1\xi_1 + B_{M2}{}^2\xi_2) + (A_{Nk} + B_{Nk}{}^1\xi_1 + B_{Nk}{}^2\xi_2)(A_{Mi} + B_{Mi}{}^1\xi_1 + B_{Mi}{}^2\xi_2)\}/(\alpha_0 + \alpha_1\xi_1 + \alpha_2\xi_2)]\,d\xi\, d\eta \tag{9.5.20}$$

otherwise, that is, $i \neq k$,

$$D_{NMik} = (\mu/8)\int_{-1}^{1}\int_{-1}^{1}\{(A_{Nk} + B_{Nk}{}^1\xi_1 + B_{Nk}{}^2\xi_2)(A_{Mi} + B_{Mi}{}^1\xi_1 + B_{Mi}{}^2\xi_2)/(\alpha_0 + \alpha_1\xi_1 + \alpha_2\xi_2)\}\,d\xi\, d\eta \tag{9.5.21}$$

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