Finite Element Formulation of a Linear Viscoelastic Material

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Abstract

Rock bodies are assumed to be composed of a two elements Maxwell and a generalized Maxwell model. The finite element formulation of these two viscoelastic materials is that viscous term within constitutive equation of the materials is regarded as initial strain and the unknown displacements are calculated for suitable time interval by the stiffness equation which is derived from the variational principle of elasticity. The resultant computer program of the two elements Maxwell model is adopted to a simple problem of stress response in order to be tested. Numerical and analytical solutions fit well together. It is interesting that tensile stress is observed parallel to contractive strain in this test.

Introduction

Time affects strongly the formation of all the geologic structures. Therefore any analysis on tectonics must necessarily include the time as a variable. From this point of view, a geologic rock material should be regarded as a time dependent material, e. g., a viscoelastic material or a viscous fluid. Lee and other workers solved some viscoelastic problems with simple boundary conditions using the correspondence principle (Lee, 1955 ; Lee et al., 1959). The correspondence principle is described in the next section.

Since many geologic bodies have the complicated structures and shapes, the analysis of them is too difficult to solve analytically. But the finite element method is a powerful technique throughout the analysis of continua (Zienkiewicz and Cheung, 1967). The finite element method for the viscoelastic problem is more difficult to be formulated than the elastic problem, because the constitutive equation of viscoelasticity is more complicated than that of elasticity.

The writer has extended the FEM computer program of elasticity into that of linear viscoelasticity by the increment method using the FACOM 230-60 computer of the Hokkaido University Computing Center. The program is of the Maxwell type because this type is more convenient to treat with stress than the Voigt type.

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The FEM formulation is similar to the method developed by Zienkiewicz and others though they treated with the Voigt type (Zienkiewicz et al., 1968). The formulation developed by the present writer is briefly explained that a viscous term of the constitutive equation is considered to be an initial strain, and the stiffness equation including the initial strain and being the differential form is used to calculate the unknown displacements repeatedly for the suitable time interval.

When the assumption is given that a rock body is composed of a viscoelastic material, marvellous phenomena appear. One of them is that tensile stress occur parallel to contractive strain field. In this case, tension cracks can not more signify field of tensile stress simultaneously.

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Theory of Linear Viscoelasticity

The fundamental equations of linear viscoelasticity are identical to those of elasticity except for a constitutive equation and a dependence on time. They are composed of the equilibrium equation (eq. 1), constitutive equation (eq. 2) and boundary conditions (eq. 3) as follows.

$$\frac{\partial \sigma_{ij}(x_k,t)}{\partial x_j} + f_i(x_k,t) = 0 \tag{1}$$

$$\sigma'_{ij}(x_k,t) = G_1(t) * d\varepsilon'_{ij}(x_k,t) \qquad (i \neq j) \tag{2.1}$$

$$\sigma_{ii}(x_k,t) = G_2(t) * d\varepsilon_{ii}(x_k,t) \tag{2.2}$$

$$\varepsilon'_{ij}(x_k,t) = J_1(t) * d\sigma'_{ij}(x_k,t) \qquad (i \neq j) \tag{2.3}$$

$$\varepsilon_{ii}(x_k,t) = J_2(t) * d\sigma_{ii}(x_k,t) \tag{2.4}$$

$$\sigma_{ij}(x_k,t)\, n_j = P_i(x_k,t) \qquad \text{on} \quad S_\sigma \tag{3.1}$$

$$u_i(x_k,t) = Q_i(x_k,t) \qquad \text{on} \quad S_u \tag{3.2}$$

where symbols and notations of the formulae are referred to Table 1. The equilibrium equation and boundary conditions are not necessarily explained here because they have the same meaning as on elasticity except for time dependence.

Constitutive Equation of the Linear Viscoelasticity

Stress and strain tensors of the second order, $\sigma_{ij}(x_k, t)$ and $\varepsilon_{ij}(x_k, t)$, are defined in the time region $-\infty < t < \infty$ at the point x_k, where the subscripts i and j run through 1 to 3. So far as the displacement $u_i(x_k,t)$ is infinitesimal, the strain tensor $\varepsilon_{ij}(x_k,t)$ is derived from the differentiation of $u_i(x_k,t)$ as follows.

$$\varepsilon_{ij}(x_k,t) = \frac{1}{2}\left\{\frac{\partial u_i(x_k,t)}{\partial x_j} + \frac{\partial u_j(x_k,t)}{\partial x_i}\right\} \tag{4}$$

Table 1 Notations and symbols.

x_i ; Cartesian coordinate
t ; time
s ; parameter transformed from t by the Laplace transform
$u_i(x_k, t)$; displacement
$f_i(x_k, t)$; body force
n_i ; unit normal vector
$\sigma_{ij}(x_k, t)$; stress tensor
$\sigma'_{ij}(x_k, t)$; deviatoric stress tensor
$\varepsilon_{ij}(x_k, t)$; strain tensor
$\varepsilon'_{ij}(x_k, t)$; deviatoric strain tensor
δ_{ij}; Kronecker's delta
$P_i(x_k, t)$; given traction
S ; closed surface
S_σ ; part of S where external force is given
$Q_i(x_k, t)$; given displacement
S_u ; part of S where displacement is given
$S = S_\sigma + S_u$
λ, μ ; Lame's constants
E ; Young's modulus
ν ; Poisson's ratio
κ ; bulk modulus
G_{ijkl} ; tensorial relaxation function
J_{ijkl} ; tensorial creep function

$\boldsymbol{u}(t)$; displacement in vector form
$\boldsymbol{f}(t)$; external force in vector form
$\boldsymbol{\sigma}(t)$; stress in vector form
$\boldsymbol{\varepsilon}(t)$; strain in vector form
D ; stress-strain matrix
C ; stress-strain rate matrix
B ; strain-nodal point displacement matrix
K ; stiffness matrix
v ; three dimensional domain
$\boldsymbol{\sigma}_k(t)$; kth element stress vector of the generalised Maxwell model
D_k ; kth element stress-strain matrix
C_k ; kth element stress-strain rate matrix

$*$; composite product
$-$; Laplace transform
df ; total differential of f (f; certain function)
$\dot{\boldsymbol{f}} = d\boldsymbol{f}/dt$ ($\boldsymbol{f}(t)$; certain vector)
F^{-1} ; inverse matrix of F (F ; certain matrix)
${}^t(a, b, c)$; column vector of (a, b, c)
${}^tB$; transposed matrix of B

which is called the strain-displacement relation. The linear viscoelasticity is defined by the stress and strain tensors using the convolutional or hereditary integral as follows.

$$\sigma_{ij}(x_k,t)=\int_{-\infty}^{t} G_{ijkl}(x_k,t-\tau)\frac{\partial\varepsilon_{kl}(x_k,\tau)}{\partial\tau}d\tau \tag{5.1}$$

$$\varepsilon_{ij}(x_k,t)=\int_{-\infty}^{t} J_{ijkl}(x_k,t-\tau)\frac{\partial\sigma_{kl}(x_k,\tau)}{\partial\tau}d\tau \tag{5.2}$$

where k and l take the number 1 to 3. The equation 5.1 is called the relaxation type, while eq. 5.2 the creep type.

If movement occurs at $t=0$, $\sigma_{ij}(x_k,t)=0$ and $\varepsilon_{ij}(x_k,t)=0$ holds within $t<0$, eqs. 5.1 and 5.2 are written as

$$\sigma_{ij}(x_k,t)=G_{ijkl}(x_k,t)\,\varepsilon_{kl}(x_k,+0)+\int_{+0}^{t} G_{ijkl}(x_k,t-\tau)\frac{\partial\varepsilon_{kl}(x_k,\tau)}{\partial\tau}d\tau \tag{6.1}$$

$$\varepsilon_{ij}(x_k,t)=J_{ijkl}(x_k,t)\,\sigma_{kl}(x_k,+0)+\int_{+0}^{t} J_{ijkl}(x_k,t-\tau)\frac{\partial\sigma_{kl}(x_k,\tau)}{\partial\tau}d\tau \tag{6.2}$$

A composite product $*$ has the following property

$$\varphi(t)*d\psi(t)=\varphi(t)\,\psi(0)+\int_{0}^{t}\varphi(t-\tau)\frac{d\psi(\tau)}{d\tau}d\tau \tag{7}$$

where $\varphi(t)$ and $\psi(t)$ are certain functions. Using this property, eq. 6 is simplified to

$$\sigma_{ij}(x_k,t)=G_{ijkl}(x_k,t)*d\varepsilon_{kl}(x_k,t) \tag{8.1}$$

$$\varepsilon_{ij}(x_k,t)=J_{ijkl}(x_k,t)*d\sigma_{kl}(x_k,t) \tag{8.2}$$

If the stress and strain tensors are symmetric, the following relations hold.

$$G_{ijkl}(x_k,t)=G_{jikl}(x_k,t)=G_{ijlk}(x_k,t) \tag{9.1}$$

$$J_{ijkl}(x_k,t)=J_{jikl}(x_k,t)=J_{ijlk}(x_k,t) \tag{9.2}$$

Furthermore, if the tensors G_{ijkl} and J_{ijkl} of the fourth order are isotropic, they are described by the functions $G_1(t)$, $G_2(t)$, $J_1(t)$ and $J_2(t)$, which depend only on time as follows.

$$G_{ijkl}(t)=\frac{G_2(t)-G_1(t)}{3}\delta_{ij}\,\delta_{kl}+G_1(t)\frac{\delta_{ik}\,\delta_{jl}+\delta_{il}\,\delta_{jk}}{2} \tag{10.1}$$

$$J_{ijkl}(t)=\frac{J_2(t)-J_1(t)}{3}\delta_{ij}\,\delta_{kl}+J_1(t)\frac{\delta_{ik}\,\delta_{jl}+\delta_{il}\,\delta_{jk}}{2} \tag{10.2}$$

Therefore, the constitutive equation of linear viscoelasticity is represented by substituting eq. 10 into eq. 8 as follows, where $\sigma_{ij}(x_k,t)$ and $\varepsilon_{ij}(x_k,t)$ are the symmetric tensors, and G_{ijkl} and J_{ijkl} are the isotropic tensors.

$$\sigma'_{ij}(x_k,t)=G_1(t)*d\varepsilon'_{ij}(x_k,t) \qquad (i\neq j) \tag{11.1}$$

$$\sigma_{ii}(x_k,t)=G_2(t)*d\varepsilon_{ii}(x_k,t) \tag{11.2}$$

$$\varepsilon'_{ij}(x_k,t)=J_1(t)*d\sigma'_{ij}(x_k,t) \qquad (i\neq j) \tag{11.3}$$

$$\varepsilon_{ii}(x_k, t) = J_2(t) * d\sigma_{ii}(x_k, t) \tag{11.4}$$

Laplace Transform of the Fundamental Equations of the Linear Viscoelasticity

Some equations of linear viscoelasticity are solved analytically by the corres-spondance principle. The method based on the principle is that if the fundamental equations of linear viscoelasticity are transformed from the time field into the other field (s-field) by the Laplace transform, the resultant equations in the s-field are identical to the fundamental equations of linear elasticity. Consequently, if the solution of linear elasticity can be transformed into the original time field, the transformed solution is the solution of the original linear viscoelastic equations. For convenience sake, the variables x_k and t are omitted below.

The fundamental equations eqs. 1, 2 and 3, are transformed into the s-field by the Laplace transform as follows.

$$\frac{\partial \bar{\sigma}_{ij}}{\partial x_j} + \bar{f}_i = 0 \tag{12}$$

$$\bar{\sigma}'_{ij} = s\bar{G}_1\, \bar{\varepsilon}'_{ij} \qquad (13.1)^*$$

$$\bar{\sigma}_{ii} = s\bar{G}_2\, \bar{\varepsilon}_{ii} \qquad (13.2)^*$$

$$\bar{\varepsilon}'_{ij} = s\bar{J}_1\, \bar{\sigma}'_{ij} \qquad (13.3)^*$$

$$\bar{\varepsilon}_{ii} = s\bar{J}_2\, \bar{\sigma}_{ii} \qquad (13.4)^*$$

$$\bar{\sigma}_{ij}\, n_j = \bar{P}_i \quad \text{on} \quad \bar{S}_\sigma \tag{14.1}$$

$$\bar{u}_i = \bar{Q}_i \quad \text{on} \quad \bar{S}_u \tag{14.2}$$

where the Laplace transform is represented by a bar above the function concerned. In the case, the next relations hold between $\bar{\mu}$ and $\bar{G}_1$ and between $\bar{\kappa}$ and $\bar{G}_2$.

$$\bar{\mu} = \frac{s\bar{G}_1}{2} \tag{15.1}$$

$$\bar{\kappa} = \frac{s\bar{G}_2}{3} \tag{15.2}$$

The following equations are derived from substituting eq. 15 into the constitutive equations 13.

$$\bar{\sigma}'_{ij} = 2\,\bar{\mu}\,\bar{\varepsilon}'_{ij} \tag{16.1}$$

$$\bar{\sigma}_{ii} = 3\,\bar{\kappa}\,\bar{\varepsilon}_{ii} \tag{16.2}$$

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*Composite product has a remarkable property for the Laplace transform.

$$\overline{\varphi(t) * d\psi(t)} = s\bar{\varphi}(s)\,\bar{\psi}(s) \tag{18}$$

where $\varphi(t)$ and $\psi(t)$ are certain functions.

Obviously, the coefficients $\bar{\mu}$ and $\bar{\kappa}$ have the similar relation of the elastic coefficients μ and κ Furthermore, the other coefficients $\bar{E}$, $\bar{\nu}$ and $\bar{\lambda}$ correspond to the Young's modulus, Poisson's ratio and Lame's constant, respectively and are represented by $\bar{G}_1$ and $\bar{G}_2$.

$$\bar{E}=\frac{3\,s\bar{G}_2\,\bar{G}_1}{2\,\bar{G}_2+\bar{G}_1} \tag{17.1}$$

$$\bar{\nu}=\frac{\bar{G}_2-\bar{G}_1}{2\,\bar{G}_2+\bar{G}_1} \tag{17.2}$$

$$\bar{\lambda}=\frac{s(\bar{G}_2-\bar{G}_1)}{3} \tag{17.3}$$

Consequently, the transformed fundamental equations of linear viscoelasticity eqs. 12, 13 and 14, are identical to those of linear elasticity in the s-field.

Finite Element Formulation of Linear Viscoelasticity

Mathematically, the method based on the correspondence principle is simple but practically the final numerical Laplace transform to the original time field is considerably difficult. From this reason the other method is adopted here.

This method is known as the expanded finite element technique of elasticity. In the method the constitutive equation is not represented by the form of the composite product but of the original differential equation where the viscous term regards as the initial strain. Plane problems on the two elements Maxwell and the generalized Maxwell model are described here. Stress and strain tensors $\sigma_{ij}(x_k,t)$ and $\varepsilon_{ij}(x_k,t)$, are written by the vector form $\boldsymbol{\sigma}(t)={}^t(\sigma_{11}(t), \sigma_{22}(t), \sigma_{12}(t))$ and $\boldsymbol{\varepsilon}(t)={}^t(\varepsilon_{11}(t), \varepsilon_{22}(t), 2\,\varepsilon_{12}(t))$, respectively.

Two Elements Maxwell Model

The constitutive equation represented by the vector form is,

$$\dot{\boldsymbol{\varepsilon}}(t)=D^{-1}\,\dot{\boldsymbol{\sigma}}(t)+C^{-1}\,\boldsymbol{\sigma}(t) \tag{19}$$

where the dots above $\boldsymbol{\varepsilon}(t)$ and $\boldsymbol{\sigma}(t)$ mean the time differential, and D^{-1} and C^{-1} are the inverse matrices of D and C, respectively. D is called the stress-strain matrix which describes the relation between stress and strain, and also C called the stress-strain rate matrix indicating the relation between stress and strain rate. The forms of these matrices on the plane strain state are represented in detail as follows.

$$D=\begin{bmatrix} \kappa+\frac{4}{3}\mu & \kappa-\frac{2}{3}\mu & 0 \\ \kappa-\frac{2}{3}\mu & \kappa+\frac{4}{3}\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \qquad D^{-1}=\frac{1+\nu}{E}\begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$C=\begin{bmatrix} \frac{4}{3}\eta & -\frac{2}{3}\eta & 0 \\ -\frac{2}{3}\eta & \frac{4}{3}\eta & 0 \\ 0 & 0 & \eta \end{bmatrix} \qquad C^{-1}=\begin{bmatrix} \frac{1}{\eta} & \frac{1}{2\eta} & 0 \\ \frac{1}{2\eta} & \frac{1}{\eta} & 0 \\ 0 & 0 & \frac{1}{\eta} \end{bmatrix}$$

The other form of eq. 19 is

$$\dot{\boldsymbol{\sigma}}(t)=D\dot{\boldsymbol{\varepsilon}}(t)-DC^{-1}\boldsymbol{\sigma}(t) \tag{19'}$$

where the matrix DC^{-1} being the product of D and C^{-1} is indicated as

$$DC^{-1}=\begin{bmatrix} \frac{3\kappa+2\mu}{2\eta} & \frac{3\kappa}{2\eta} & 0 \\ \frac{3\kappa}{2\eta} & \frac{3\kappa+2\mu}{2\eta} & 0 \\ 0 & 0 & \frac{\mu}{\eta} \end{bmatrix}$$

The stiffness equation derived from the principle of virtual work is written as follows.

$$\dot{\boldsymbol{f}}(t)=\int_v {}^tB\dot{\boldsymbol{\sigma}}(t)\,dv \tag{20}$$

where B is defined as $\boldsymbol{\varepsilon}(t)=B\boldsymbol{u}(t)$ and is called the strain-nodal point displacement matrix. Substitute eq. 19′ into eq. 20,

$$\dot{\boldsymbol{f}}(t)+\int_v {}^tBDC^{-1}\boldsymbol{\sigma}(t)\,dv=\left\{\int_v {}^tBDBdv\right\}\dot{\boldsymbol{u}}(t)=K\dot{\boldsymbol{u}}(t) \tag{21}$$

is given, where K is called the stiffness matrix. Therefore, the numerical treatment based on eq. 21 is stated as follows.

(1) Obtain the initial elastic stress $\boldsymbol{\sigma}(0)$ from the stiffness equation $\boldsymbol{f}(0)=K\boldsymbol{u}(0)$.

(2) Obtain the value $\dot{\boldsymbol{u}}(t)$ from eq. 17 using the value $\boldsymbol{\sigma}(t)$ and the displacement boundary condition.

(3) Substitute the equation $\dot{\boldsymbol{\varepsilon}}(t)=B\dot{\boldsymbol{u}}(t)$ into the equation 19′ and obtain the following equation.

$$\dot{\boldsymbol{\sigma}}(t)=DB\dot{\boldsymbol{u}}(t)-DC^{-1}\boldsymbol{\sigma}(t) \tag{22}$$

The explicit representation of this equation is described in detail as follows.

$$\dot{\sigma}_{11}(t)=a_1-\frac{3\kappa+2\mu}{2\eta}\sigma_{11}(t)-\frac{3\kappa}{2\eta}\sigma_{22}(t) \tag{22.1'}$$

$$\dot{\sigma}_{22}(t)=a_2-\frac{3\kappa}{2\eta}\sigma_{11}(t)-\frac{3\kappa+2\mu}{2\eta}\sigma_{22}(t) \tag{22.2'}$$

$$\dot{\sigma}_{12}(t)=a_3-\frac{\mu}{\eta}\sigma_{12}(t) \tag{22.3'}$$

where the vector $\boldsymbol{a}$ is defined as $\boldsymbol{a}={}^t(a_1,a_2,a_3)=DB\dot{\boldsymbol{u}}$. Obtain the value $\boldsymbol{\sigma}(t)$ by solving the equation 22 in a suitable time interval.

(4) Repeat the operation from (2) hereafter.

Generalized Maxwell Model

The constitutive equations are

$$\dot{\boldsymbol{\varepsilon}}(t)=D_k^{-1}\,\dot{\boldsymbol{\sigma}}_k(t)+C_k^{-1}\,\boldsymbol{\sigma}_k(t) \qquad (k=1,\ 2,\ \cdots,\ n) \tag{23}$$

where the subscript k means the element number of model. Changing the terms $\dot{\boldsymbol{\varepsilon}}(t)$ and $\dot{\boldsymbol{\sigma}}(t)$ mutually and multiplying D_k to them,

$$\dot{\boldsymbol{\sigma}}_k(t)=D_k\,\dot{\boldsymbol{\varepsilon}}(t)-D_k\,C_k^{-1}\,\boldsymbol{\sigma}_k(t) \qquad (k=1,\ 2,\cdots,\ n) \tag{24}$$

is obtained. Since in the Maxwell type the total stress $\boldsymbol{\sigma}(t)$ is the sum of all the element stresses $\boldsymbol{\sigma}_i(t)$, $\boldsymbol{\sigma}(t)$ is

$$\boldsymbol{\sigma}(t)=\sum_{i=1}^{n}\boldsymbol{\sigma}_i(t) \tag{25}$$

Differentiating eq. 25 with t, eq. 25 becomes to

$$\dot{\boldsymbol{\sigma}}(t)=\sum_{i=1}^{n}\dot{\boldsymbol{\sigma}}_i(t) \tag{26}$$

Substituting eq. 24 into eq. 26,

$$\dot{\boldsymbol{\sigma}}(t)=\sum_{i=1}^{n}\left\{D_i\,\dot{\boldsymbol{\varepsilon}}(t)\right\}-\sum_{i=1}^{n}\left\{D_i\,C_i^{-1}\boldsymbol{\sigma}_i(t)\right\} \tag{27}$$

The stiffness equation of the model is identical to that of the two elements Maxwell model, i. e., eq. 20, so the extended stiffness equation is derived from rearranging eq. 20 using eq. 27 as follows.

$$\begin{aligned}\dot{\boldsymbol{f}}(t)+\int_v {}^tB\left\{\sum_{i=1}^{n}D_i\,C_i^{-1}\boldsymbol{\sigma}_i(t)\right\}dv&=\int_v {}^tB\left\{\sum_{i=1}^{n}D_i\,\dot{\boldsymbol{\varepsilon}}(t)\right\}dv\\&=\int_v {}^tB\left(\sum_{i=1}^{n}D_i\right)\dot{\boldsymbol{\varepsilon}}(t)\,dv\\&=\left\{\int_v {}^tB\left(\sum_{i=1}^{n}D_i\right)Bdv\right\}\dot{\boldsymbol{u}}(t)\\&=K\dot{\boldsymbol{u}}(t)\end{aligned} \tag{28}$$

The numerical procedure based on eq. 28 is explained as follows.

(1) Obtain the value $\boldsymbol{\sigma}(0)$ calculating

$$\boldsymbol{f}(t)=\left\{\int_v {}^tB\left(\sum_{i=1}^{n}D_i\right)Bdv\right\}\boldsymbol{u}(t)$$

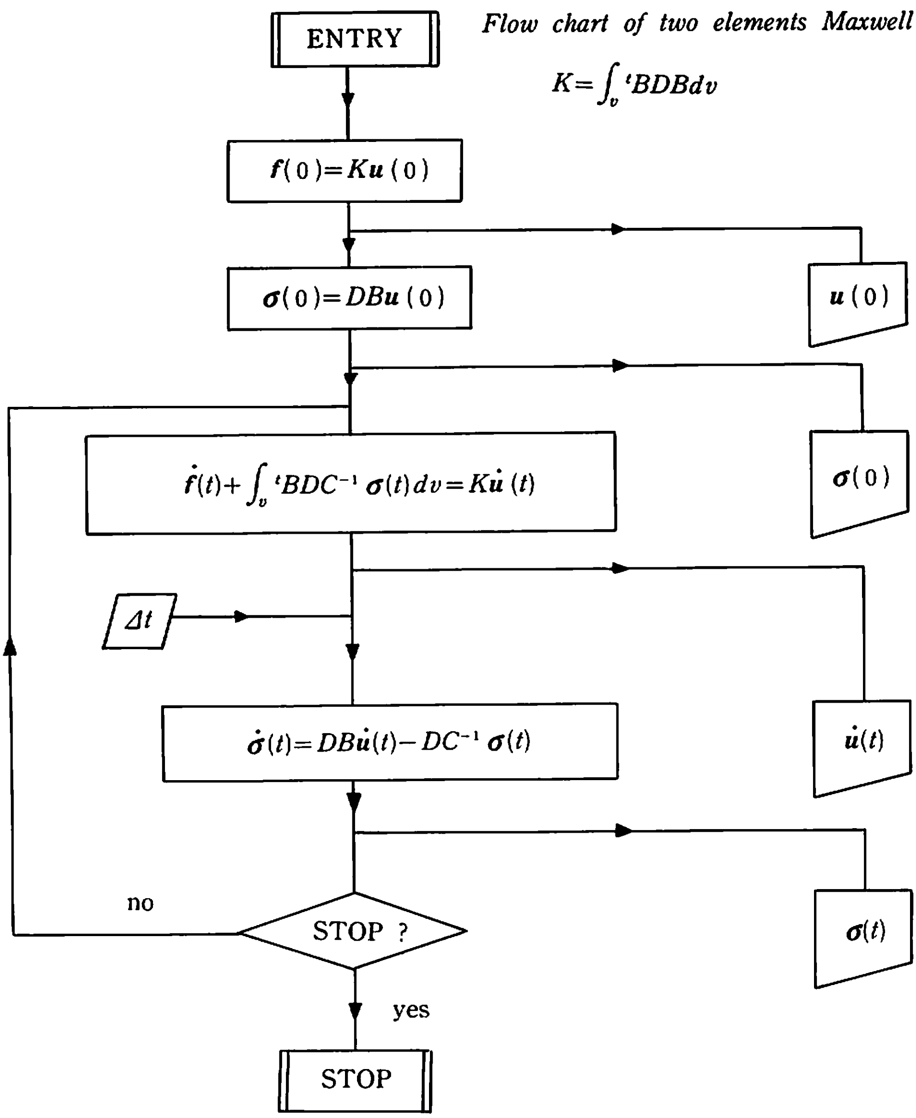
(2) Obtain the value $\boldsymbol{\sigma}_k(0)$ using

$$\boldsymbol{\sigma}_k(0)=\frac{D_k\boldsymbol{\sigma}(0)}{\sum_{i=1}^{n}D_i}$$

(3) Obtain the value $\dot{u}(t)$ calculating eq. 28, substituting the values $\sigma_k(t)$ and using the displacement boundary condition.
(4) Solve the equation 24 for all the elements of the model and obtain the value $\sigma_k(t)$.
(5) Obtain the value $\sigma(t)$ from eq. 25, adding all the element stresses.
(6) Repeat from (3) hereafter.

Program Flow Chart

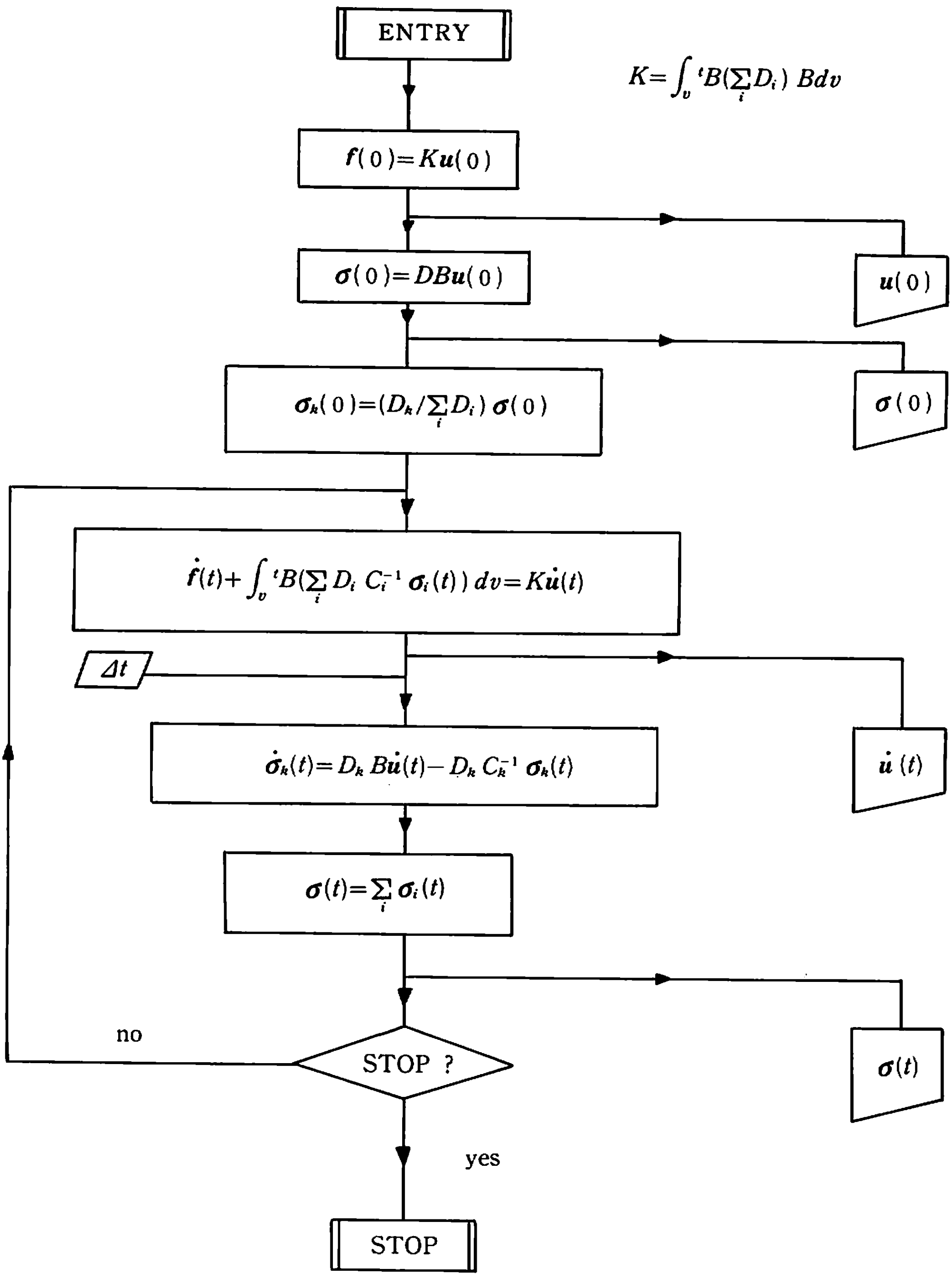
The technique of the linear viscoelastic stress calculation described above, is concluded as a flow chart. The flow charts of the two elements Maxwell and generalized Maxwell model are presented, but the programs themselves do not indicated here. Because the computer program of the numerical calculation can be constructed easily, referring the flow charts. The variables and symbols are referred to Table 1.



Flow chart of two elements Maxwell model

$$K=\int_v {}^tBDBdv$$

Flow chart of generalized Maxwell model



Test of Finite Element Computer Program

Test of the program is performed on the plane strain state using the model A and B explained after. This is the stress response test against input of strain of periodic form. Sinusoidal function is taken to be the form of strain, because of its simplicity in analysis. The test is carried out under one and two dimensions where gravity is not considered in order to compare with the analytical solution.

Analytical Solution of the Two Elements Maxwell Model

The constitutive equation of the two elements Maxwell model have been already written as eq. 19. The other form of eq. 19 is

$$\dot{\sigma}(t)+DC^{-1}\sigma(t)=D\dot{\varepsilon}(t) \tag{29}$$

This equation is expanded to the component type,

$$\dot{\sigma}_{11}(t)+a_0\sigma_{11}(t)+b_0\sigma_{22}(t)=d_0\dot{\varepsilon}_{11}(t)+e_0\dot{\varepsilon}_{22}(t) \tag{29.1'}$$

$$\dot{\sigma}_{22}(t)+b_0\sigma_{11}(t)+a_0\sigma_{22}(t)=e_0\dot{\varepsilon}_{11}(t)+d_0\dot{\varepsilon}_{22}(t) \tag{29.2'}$$

$$\dot{\sigma}_{12}(t)+c_0\sigma_{12}(t)=2\mu\dot{\varepsilon}_{12}(t)=\mu\dot{\gamma}_{12}(t) \tag{29.3'}$$

where $a_0=\dfrac{3\varkappa+2\mu}{\eta}$, $b_0=\dfrac{3\varkappa}{2\eta}$, $c_0=\dfrac{\mu}{\eta}$, $d_0=\varkappa+\dfrac{4}{3}\mu$, $e_0=\varkappa-\dfrac{2}{3}\mu$ and $\gamma_{12}(t)\equiv 2\varepsilon_{12}(t)$.

If the strain $\varepsilon_{11}(t)$, $\varepsilon_{22}(t)$ and $\gamma_{12}(t)$ are given in a simple form, eq. 29 is solved analytically by the Laplace transform. Considering them, the sinusoidal function is taken to be the strain function here.

$$\varepsilon_{11}(t)=\varepsilon_1\sin\omega t \tag{30.1}$$

$$\varepsilon_{22}(t)=\varepsilon_2\sin\omega t \tag{30.2}$$

$$\gamma_{12}(t)=\gamma_{12}\sin\omega t \tag{30.3}$$

In this condition, the solutions of eq. 29 are given as follows.

$$\sigma_{11}(t)=\left\{\frac{\sigma_{11}(0)+\sigma_{22}(0)}{2}-A_0\right\}exp\left(-\frac{3\varkappa+\mu}{3}t\right)+\left\{\frac{\sigma_{11}(0)-\sigma_{22}(0)}{2}-B_0\right\}exp\left(-\frac{t}{\tau_\varepsilon}\right)$$
$$+\sqrt{(A_0+B_0)^2+(C_0+D_0)^2}\ \sin(\omega t+\delta_{11}) \tag{31.1}$$

$$\sigma_{22}(t)=\left\{\frac{\sigma_{11}(0)+\sigma_{22}(0)}{2}-A_0\right\}exp\left(-\frac{3\varkappa+\mu}{\eta}t\right)-\left\{\frac{\sigma_{11}(0)-\sigma_{22}(0)}{2}-B_0\right\}exp\left(-\frac{t}{\tau_\varepsilon}\right)$$
$$+\sqrt{(A_0-B_0)^2+(C_0-D_0)^2}\ \sin(\omega t+\delta_{22}) \tag{31.2}$$

$$\sigma_{12}(t)=\{\sigma_{12}(0)-E_0\}\ exp\left(-\frac{t}{\tau_\varepsilon}\right)+F_0\sin(\omega t+\delta_{12}) \tag{31.3}$$

where

$$\tau_\varepsilon=\frac{\eta}{\mu}\ ,\quad A_0=\frac{1}{3}\frac{(3\kappa+\mu)^2\,\omega\eta}{(3\kappa+\mu)^2+(\omega\eta)^2}\,(\varepsilon_1+\varepsilon_2)\ ,\quad B_0=\frac{\mu\omega\tau_\varepsilon}{1+(\omega\tau_\varepsilon)^2}\,(\varepsilon_1-\varepsilon_2)\ ,$$

$$C_0=\frac{1}{3}\frac{(3\kappa+\mu)(\omega\eta)^2}{(3\kappa+\mu)^2+(\omega\eta)^2}\,(\varepsilon_1+\varepsilon_2)\ ,\quad D_0=\frac{\mu(\omega\tau_\varepsilon)^2}{1+(\omega\tau_\varepsilon)^2}\,(\varepsilon_1-\varepsilon_2)\ ,$$

$$\delta_{11}=arctan\ \frac{A_0+B_0}{C_0+D_0}\ ,\quad \delta_{22}=arctan\ \frac{A_0-B_0}{C_0-D_0}\ ,\quad \delta_{12}=arctan\ \frac{1}{\omega\tau_\varepsilon}\,.$$

Test on One Dimensional State

Consider the model A which is a square with 10 m length, 10 m width and a unit thickness as shown in Fig. 1. Material constants of the model are ; $E=176$kb, $\nu=0.36$, $\eta=10^{22}$ poises, $\tau_\varepsilon=4{,}897$ years. These constants except for viscosity are taken from the measurement of the cordierite migmatite of which the Oshirabetsu dome is composed, that is, the constants belong to the real geologic rock bock (Hayashi and Kizaki, 1972).

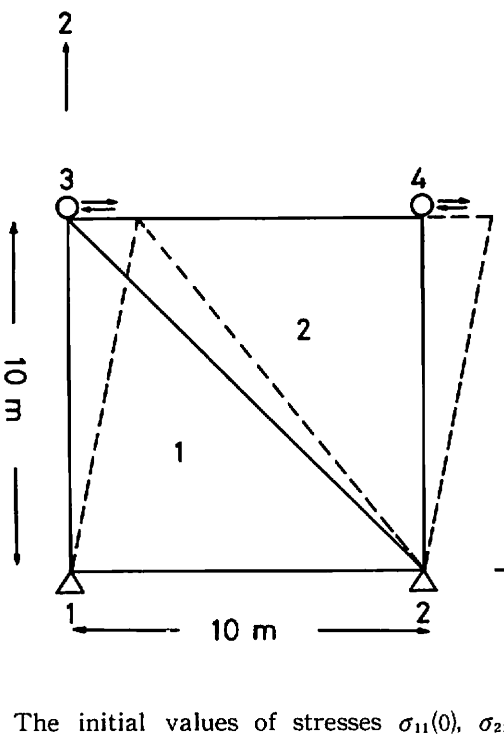
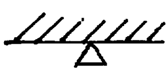
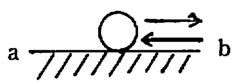


Fig. 1 Model A and boundary condition.


; perfectly restricted



; free along ab, periodic displacement is given.

The initial values of stresses $\sigma_{11}(0)$, $\sigma_{22}(0)$ and $\sigma_{12}(0)$ are 0 bars. The displacement $u(t)$ is given at the nodal points 3 and 4 as $u_0 \sin \omega t$ where $\omega=0.018$ deg/year and $u_0=0.154545$ m which occurs 1 kb shearing stress within the model. In order to use the finite element method, the model is divided into two elements of triangle and has four nodal points. The calculation is performed 200 times where the time interval is 100 years. The calculated solutions are shown in Table 3 and Fig. 2.

On the other hand, analytical solution of the same model is given by eq. 31. 3

Table 2 Material constants (E, ν, η, τ_ε), boundary condition (u_0, γ_0, ω), initial condition ($\sigma_{11}(0)$, $\sigma_{22}(0)$, $\sigma_{12}(0)$) and coefficients of eq. 31.3 (E_0, F_0, δ_{12}) on the model A.

E	176 kb
ν	0.36
η	10^{22} poises
τ_ε	4897 years
u_0	0.154545 m
γ_0	1.54545×10^{-2}
ω	0.018 deg/year
$\sigma_{11}(0)$	0 bars
$\sigma_{22}(0)$	0 bars
$\sigma_{12}(0)$	0 bars
E_0	457 bars
F_0	838 bars
δ_{12}	0.576 rad = 33.0 deg

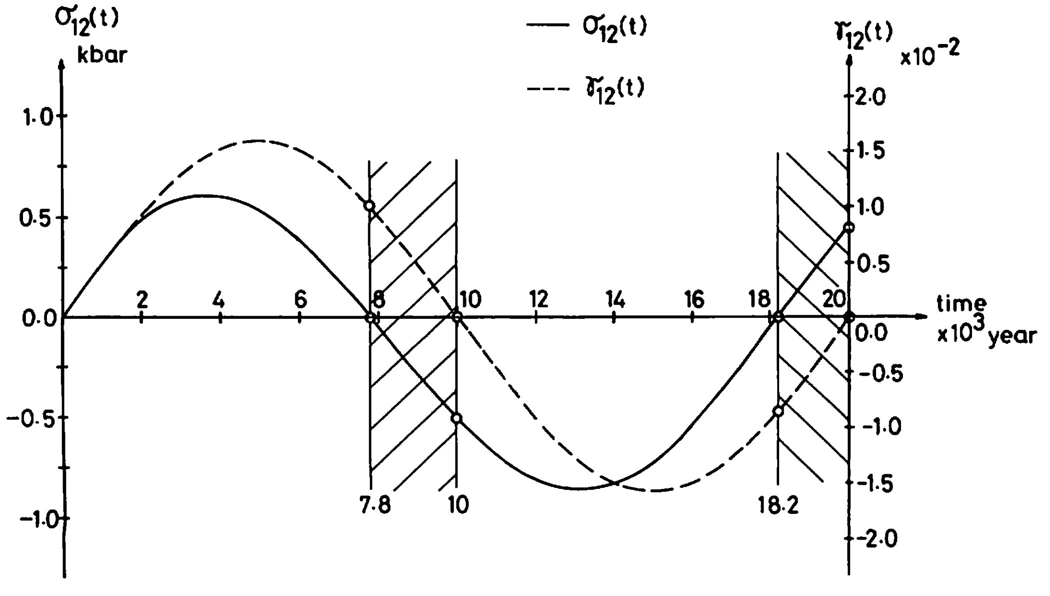


Fig. 2 Input strain curve $\gamma_{12}(t)$ and the stress response curve $\sigma_{12}(t)$ on the model A. Stress and strain have an opposite sign in shaded area each other.

Table 3 Input values of shearing strain $\gamma_{12}(t)$ and the analytical and numerical solutions of shearing stress on the model A.

time ($\times 10^3$ year)	$\gamma_{12}(t)$ ($\times 10^{-2}$)	$\sigma_{12}(t)$ (bar) Analytical solution	Numerical solution
0	0.000	0	0
1	0.478	279	279
2	0.908	479	476
3	1.250	590	584
4	1.470	608	600
5	1.545	538	527
6	1.470	393	380
7	1.250	191	176
8	0.908	−46	−61
9	0.478	−290	−304
10	0.000	−516	−529
11	−0.478	−700	−710
12	−0.908	−822	−828
13	−1.250	−869	−871
14	−1.470	−836	−833
15	−1.545	−724	−718
16	−1.470	−545	−535
17	−1.250	−314	−302
18	−0.908	−55	−42
19	−0.478	208	220
20	0.000	449	460

where $\gamma_0 = 1.54545 \times 10^{-2}$, $E_0 = 457$ bars, $F_0 = 838$ bars and $\delta_{12} = 0.576$ rad $= 33.0$ deg. Material constants, initial condition, boundary condition and the coefficients of eq. 31.3 are summerized in Table 2. The solution is also indicated in Table 3 and Fig. 2.

It is clear from Table 3 and Fig. 2 that the numerical solution well fits to that of the analytical. Theoretically, the shearing stress $\sigma_{12}(t)$ has a faster phase in a stational state about 1,800 years than the shearing strain $\gamma_{12}(t)$ according to $\delta_{12} = 33$ deg. In fact, about 1,800 years difference of the phase between stress and strain is read from Fig. 2, though the state is not stationary but transient. Two interesting cases are recognised, one is the case where $\gamma_{12}(t)$ has positive values (anticlockwise) but $\sigma_{12}(t)$ negative (clockwise) during 7,800 to 10,000 years. The other similar case occurs during 18,200 to 20,000 years where $\gamma_{12}(t)$ is negative but $\sigma_{12}(t)$ positive.

Test on Two Dimensional State

The model B shown in Fig. 3 has the same material properties and initial values of stress as well as the model A. The displacement $u(t)$ is given at the nodal points 2 and 4 as $u_0 \sin\omega t$ where $\omega=0.018$deg/year and $u_0=(0.2\text{m}, -0.1\text{m})$ which produces 4,252bars along x_1 axis and -370bars along x_2 axis within the model. In this condition, the numerical solutions are illustrated in Figs. 4, 5 and Table 5.

The analytical solutions of the model B is given by eqs. 31.1 and 31.2 where $\varepsilon_1=0.02$, $\varepsilon_2=-0.01$, $A_0=325$bars, $B_0=887$bars, $C_0=47$bars, $D_0=1{,}365$bars, $\delta_{11}=0.710$rad $=40.7$deg and $\delta_{22}=-2.739$rad $=-156.9$deg. The material constants, initial condition, boundary condition and the coefficients of eqs. 31.1 and 31.2 are summerized in Table 4. Results are also indicated in Table 5 and Figs. 4 and 5.

Showing these table and figures, the stress $\sigma_{11}(t)$ has the faster phase than the strain $\varepsilon_{11}(t)$ as well as one dimensional state. Although the difference of the phase is 2,300 years theoretically in a stationary state, that is 2,200 years numerically from Fig.4, because the present state is not stationary but transient. It is impossible that the stress and strain are parallel each other and have the opposite sign within elastic material. This occurs only within viscoelastic material because of the difference of phase between stress and strain. The tensile stress are presented in the contractive strain field during 17,800 to 20,000 years. Therefore, it is concluded from the model A and B that the tensional strain does not always mean the tensile stress and contrarily the contractive strain does not always signify the compressive stress.

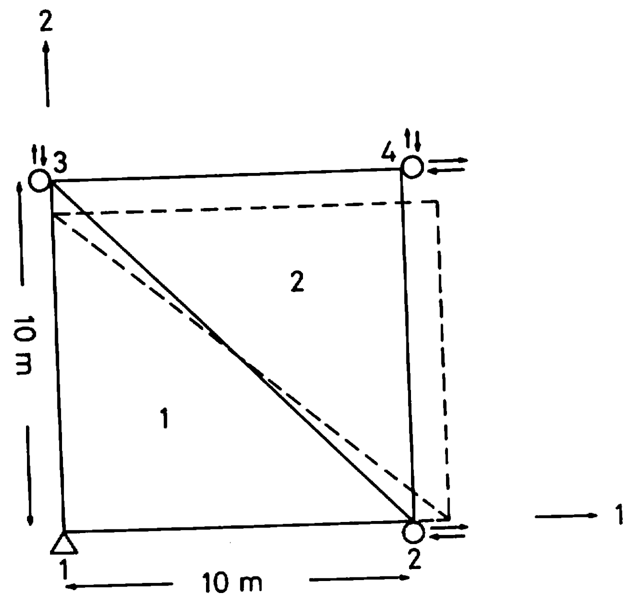


Fig. 3 Model B and boundary condition.

Table 4 Material constants (E, ν, η, τ_ε), boundary condition (u_0, ε_1, ε_2, ω), initial condition ($\sigma_{11}(0)$, $\sigma_{22}(0)$, $\sigma_{12}(0)$) and coefficients of eqs. 31.1 and 31.2 (A_0, B_0, C_0, D_0, δ_{11}, δ_{22}) on the model B.

E	176 kb
ν	0.36
η	10^{22} poises
τ_ε	4897 years
u_0	0.2 m (x_1 component)
	−0.1 m (x_2 component)
ε_1	0.02
ε_2	−0.01
ω	0.018 deg/year
$\sigma_{11}(0)$	0 bars
$\sigma_{22}(0)$	0 bars
$\sigma_{12}(0)$	0 bars
A_0	325 bars
B_0	887 bars
C_0	47 bars
D_0	1365 bars
δ_{11}	0.710 rad = 40.7 deg
δ_{22}	−2.739 rad = −156.9 deg

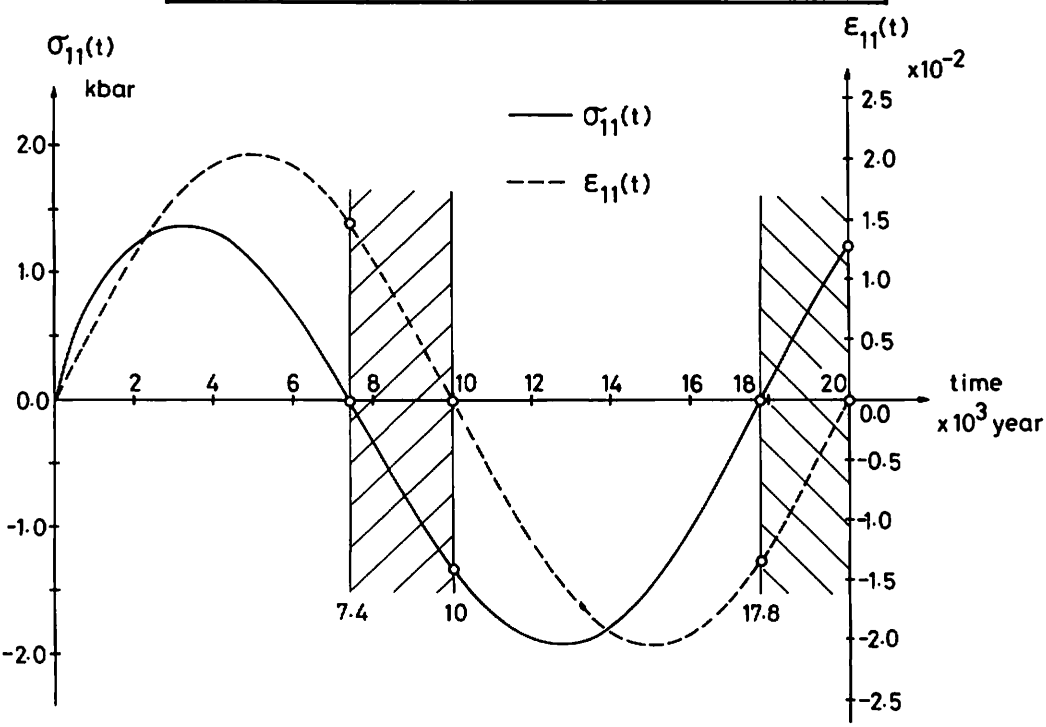


Fig. 4 Input strain curve $\varepsilon_{11}(t)$ and the stress response curve $\sigma_{11}(t)$ on the model B.

Table 5 Input values of normal strains $\varepsilon_{11}(t)$, $\varepsilon_{22}(t)$, and the analytical and numerical solutions of normal stresses $\sigma_{11}(t)$, $\sigma_{22}(t)$ on the model B.

time ($\times 10^3$ year)	$\varepsilon_{11}(t)$ ($\times 10^{-2}$)	$\sigma_{11}(t)$ (bar)		time ($\times 10^3$ year)	$\varepsilon_{22}(t)$ ($\times 10^{-2}$)	$\sigma_{22}(t)$ (bar)	
		Analytical solution	Numerical solution			Analytical solution	Numerical solution
0	0.000	0	0	0	0.000	0	0
1	0.618	829	827	1	−0.309	− 255	− 254
2	1.176	1217	1209	2	−0.588	− 644	− 641
3	1.618	1373	1359	3	−0.809	− 916	− 910
4	1.902	1325	1304	4	−0.951	−1035	−1024
5	2.000	1092	1065	5	−1.000	− 998	− 982
6	1.902	707	676	6	−0.951	− 819	− 798
7	1.618	217	− 184	7	−0.809	− 524	− 499
8	1.176	− 324	− 357	8	−0.588	− 147	− 121
9	0.618	− 858	− 888	9	−0.309	268	294
10	0.000	−1327	−1352	10	0.000	677	700
11	−0.618	−1683	−1700	11	0.309	1035	1055
12	−1.176	−1887	−1895	12	0.588	1306	1319
13	−1.618	−1917	−1916	13	0.809	1459	1465
14	−1.902	−1768	−1758	14	0.951	1478	1477
15	−2.000	−1453	−1435	15	1.000	1359	1351
16	−1.902	−1002	− 978	16	0.951	1114	1099
17	−1.618	− 457	− 429	17	0.809	764	745
18	−1.176	129	157	18	0.588	343	321
19	−0.618	698	725	19	0.309	− 109	− 131
20	0.000	1197	1219	20	0.000	− 547	− 567

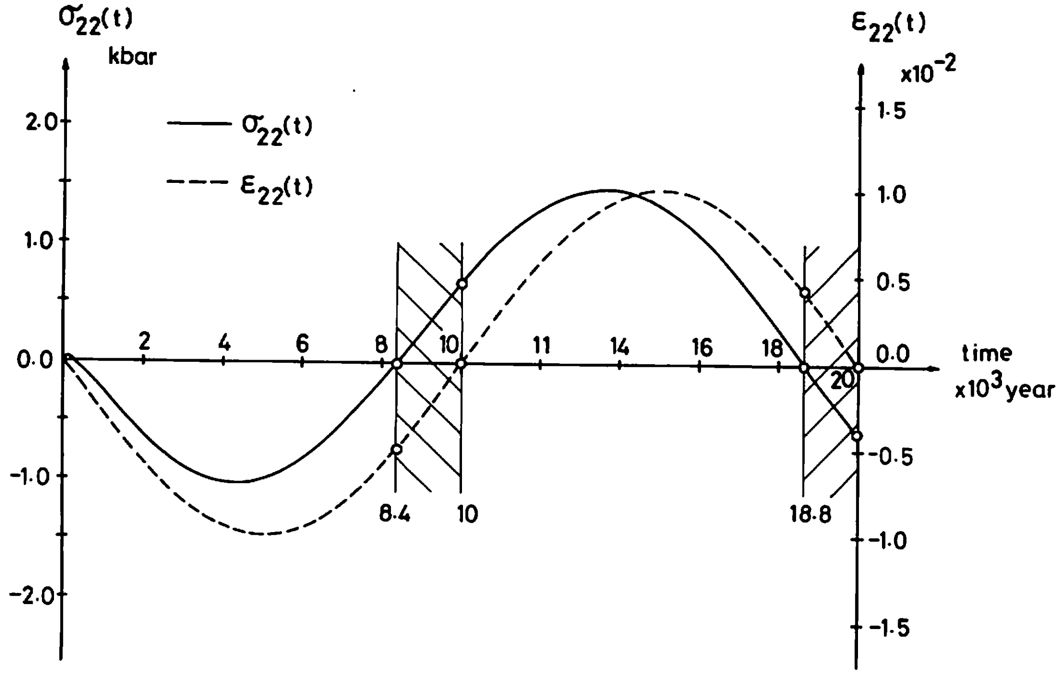


Fig. 5 Input strain curve $\varepsilon_{22}(t)$ and the stress response curve $\sigma_{22}(t)$ on the model B. Stress and strain have an opposite sign in shaded area each other.

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