FINITE ELEMENT FORMULATION OF VISCOUS FLUID BASED ON A VARIATIONAL PRINCIPLE

FINITE ELEMENT FORMULATION OF VISCOUS FLUID BASED ON A VARIATIONAL PRINCIPLE

DAIGORO HAYASHI*

Abstract

The paper describes a finite element formulation of the incompressible viscous fluid without inertia term in detail that is based on a variational principle of viscous fluid and is formulated by the direct method of variational calculus. Although the variational principles of two and three dimensional viscous flows are explained, the FEM formulation is performed with regard to only the two dimensional flow. One of the examples of the calculation using the method has already been published and given a fruitful result (Hayashi, 1975).

INTRODUCTION

There are a number of methods to analyse the geological structures. One of them, for example, is the minor fault analysis which recovers a field of principal stresses (e. g. Kinugasa et al., 1969). The other is the measurement of the direction of crystal axes of minerals using universal stage (Turner and Weiss, 1963). The third is the mesoscopic measurement of fabrics and lineations in fields.

However, in order to explain how the geological structures are formed, it is not sufficient to interpret the problem with such methods but the other more quantitative techniques are required. They are known as the scale model experiment or the numerical experiment.

The paper describes a numerical formulation of the flow of viscous fluid under low speed instead of the incomplete method derived from the finite element technique that was developed by Dieterich and others for the analysis of the mechanism of folding (Dieterich and Onat, 1969; Dieterich and Carter, 1969). The present formulation is based on a variational principle which was originally found out by Millikan (1929).

One of the examples calculated by the method were performed concerning to the diapir problem with FACOM 230-75 computer of the Hokkaido University Computing Center and the result is well fitted to the experimental data (Hayashi, 1975).

Symbols and notations used in the following sections are referred to Table 1.

VISCOUS FLUID

In order to understand the variational principle of viscous fluid, it is necessary to note the theory of continuum dynamics. Two basic equations of the incompressible viscous fluid without inertia term are derived from the conservative laws of continuum dynamics, one is the continuity equation and the other is the equation of motion.

Received : April 20, 1979

*Department of Marine Sciences, University of the Ryukyus

Table 1; Notations

x_i	; Cartesian coordinate
t	; time
ρ	; density
D	; domain
∂D	; closed surface surrounding D
v_i	; velocity vector
f_i	; body force vector per unit mass
n_i	; unit normal vector
σ_{ij}	; stress tensor
D/Dt	; Lagrangian differentiation
δ_{ij}	; Kronecker's delta
p	; pressure
μ^*	; coefficient of viscosity
f	; certain function
$I[f]$	; functional with respect to f
$\delta I[f]$	; first variation of $I[f]$
df	; total differential of f
S	; two dimensional closed surface or area of finite element
v	; three dimensional closed domain
e_{ij}	; strain rate tensor
ξ_i	; temporary coordinate
x_{ij}	; value of x_j component at the nodal point i
ξ_{ij}	; value of ξ_j component at the nodal point i
$\bar{u}_i$	; value of u (ξ_{ij}) at the nodal point i
ξ_{ij}^{-1}	; ij-component of inverse matrix of ξ_{ij}
Δ	; determinant of ξ_{ij} matrix
Δ_{ij}	; cofactor of Δ
$\nabla_{ij}=\xi_{ji}^{-1}=\Delta_{ij}/\Delta$	
$\bar{u}_{ik}$	; value of $u_k(\xi_{ij})$ at the nodal point i
f	; certain function
g_i	; certain vector
$Df/Dt=f_{,t}+v_jf_{,j}$	
$f'(x)=df/dx$; $f_{,i}=\partial f/\partial x_i$; $f_{,t}=\partial f/\partial t$;	
$f_{,x}=\partial f/\partial x$; $f_{,g_i}=\partial f/\partial g_i$	
$g_{i,j}=\partial g_i/\partial x_j$; $g_{i,i}=g_{k,k}=\mathrm{div}\ \boldsymbol{g}$ ($\boldsymbol{g}$ means a vector g_i)	

Further studies are referred to the works of Truesdell (1952), and Landau and Lifshitz (1959) and others.

Conservative law of mass

The mass that flows per unit time through the boundary $d(\partial D)$ into the region ∂D in Fig. 1 is given by $-\rho v_i n_i d(\partial D)$. Surface integral of this formula with respect to ∂D is equal to the increment of mass flowing into ∂D per unit time, that is

$$\frac{\partial}{\partial t}\int_D \rho dD = -\int_{\partial D} \rho v_i n_i d(\partial D) \tag{1}$$

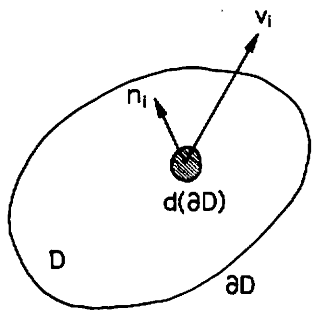

Fig. 1 ; Conservative law of mass and momentum

Transform the surface integral of the right hand of (1) into a volume integral by the Gauss's divergence theorem,

$$\frac{\partial}{\partial t}\int_D \rho dD = -\int_D (\rho v_i)_{,i} dD \tag{2}$$

is obtained.

(2) is more simplified to

$$\int_D \left\{\rho_{,t} + (\rho v_i)_{,i}\right\} dD = 0 \tag{3}$$

Since ∂D is taken arbitrarily, the integrand of this volume integral is equal to zero and the final form of the conservative law of mass of general continuum becomes

$$\rho_{,t} + (\rho v_i)_{,i} = 0 \tag{4}$$

Conservative law of momentum

A domain D is surrounded with a certain closed surface ∂D as shown in Fig. 1. Applying the conservative law of momentum to the mass that occupies the domain D, we obtain for any x_i component of momentum,

$$\frac{\partial}{\partial t}\int_D \rho v_i dD = \int_{\partial D} \sigma_{ij} n_j d(\partial D) - \int_{\partial D} (\rho v_i) v_j n_j d(\partial D) + \int_D \rho f_i dD \tag{5}$$

When the Gauss's theorem is applied to the former two terms of the right hand side of (5), (5) changes into

$$\frac{\partial}{\partial t}\int_D \rho v_i dD=\int_D (\sigma_{ij})_{,j}\, dD-\int_D (\rho v_i v_j)_{,j} dD+\int_D \rho f_i dD \tag{6}$$

Since ∂D is taken arbitrarily, the integrands of both sides of (6) must be equal together, that is,

$$(\rho v_i)_{,t}=\sigma_{ij,j}-(\rho v_i v_j)_{,j}+\rho f_i \tag{7}$$

The term $(\rho v_i)_{,t}+\rho(v_i v_j)_{,j}$ is expanded as

$$\left\{\rho_{,t}+(\rho v_j)_{,j}\right\}v_i+\rho(v_{i,t}+v_j v_{i,j}) \tag{8}$$

Let (4) substitute into (8), (8) becomes

$$\rho(v_{i,t}+v_j v_{i,j}) \tag{9}$$

Since (9) is the definition of $\rho\frac{Dv_i}{Dt}$, the conservative law of momentum of general continuum becomes finally

$$\rho\frac{Dv_i}{Dt}=\sigma_{ij,j}+\rho f_i \tag{10}$$

Fundamental equations of the incompressible viscous fluid without inertia term

The conservative law of mass (4) is transformed into

$$\rho_{,t}+\rho_{,i}v_i+\rho v_{i,i}=0 \tag{4$'$}$$

Applying the definition of the Lagrangian differentiation, (4)′ is written by

$$\frac{D\rho}{Dt}+\rho v_{i,i}=0 \tag{4$''$}$$

If the viscous fluid is incompressible, the Lagrangian differentiation of ρ becomes zero, so that (4)″ reduces to

$$v_{i,i}=0 \quad (11) \qquad \text{or} \qquad \operatorname{div} v=0 \quad (11)'$$

These are the continuity equation of the incompressible viscous fluid.

It is well known that the constitutive equation of the incompressible viscous fluid is defined as

$$\sigma_{ij}=-p\delta_{ij}+\mu^*(v_{i,j}+v_{j,i}) \tag{12}$$

When (12) is differentiated with respect to x_j,

$$\sigma_{ij,j}=-p_{,i}+\mu^*\left\{(v_{i,j})_{,j}+(v_{k,k})_{,i}\right\} \tag{13}$$

is obtained.

Substitute (11) into (13), (13) becomes

$$\sigma_{ij,j} = -p_{,i} + \mu^*(v_{i,j})_{,j} \tag{14}$$

Substitute (14) into (10) and neglect an inertia term,

$$-p_{,i} + \mu^*(v_{i,j})_{,j} + \rho f_i = 0 \tag{15}$$

is obtained, which is called the equilibrium equation of the incompressible viscous fluid under low speed.

VARIATIONAL PRINCIPLE

If a numerical formulation of a continuum should be accomplished by the finite element method, there must be existed a corresponding functional to the continuum. After finding to the functional, one can construct a system of linear simultaneous equation instead of the complicated system of the partial differential equations. In the section the author describes in detail the functionals corresponding to the incompressible viscous fluid without inertia term in two and three dimensions.

Euler's equation of the variational calculus

The Euler's equation of the most general type is the equation represented by multiple variables and multiple functions. The u_i is the function of n variables x_i, and f depends on x_i, u_i and $u_{i,j}$. The functional of $f\ (x_i, u_i, u_{i,j})$ is defined as $I\,[u_i] = \int_D f(x_i,\ u_i,\ u_{i,j})\ dD$. The first variation of $I\ [u_i]$ is writter by $\delta I\ [u_i] = \int_D df\ (x_i,\ u_i,\ u_{i,j})\ dD$, where the total differential of $f\,(x_i,\ u_i,\ u_{i,j})$ is represented by

$$df(x_i,\ u_i,\ u_{i,j}) = \sum_i \varepsilon_i\left(\eta_i\frac{\partial f}{\partial u_i} + \eta_{i,1}\frac{\partial f}{\partial u_{i,1}} + \eta_{i,2}\frac{\partial f}{\partial u_{i,2}} + \cdots\cdots\cdots + \eta_{i,n}\frac{\partial f}{\partial u_{i,n}}\right).$$

where ε_i are the small real numbers, and the functions of $\eta_i(x_i)$ and $\eta_i'\,(x_i)$ are continuous within the domain D and $\eta_i(x_i) = 0$ on ∂D.

Let $\delta I[u_i]$ be zero, the Euler's equation of $I[u_i]$ becomes,

$$\int_D\left(\eta_i\frac{\partial f}{\partial u_i} + \eta_i\frac{\partial f}{\partial u_{i,1}} + \eta_{i,2}\frac{\partial f}{\partial u_{i,2}} + \cdots\cdots\cdots + \eta_{i,n}\frac{\partial f}{\partial u_{i,n}}\right)dD = 0 \tag{16}$$

The integral of left hand side without the first term results in

$$\int_D\left\{\frac{\partial}{\partial x_1}\left(\eta_i\frac{\partial f}{\partial u_{i,1}}\right) + \frac{\partial}{\partial x_2}\left(\eta_i\frac{\partial f}{\partial u_{i,2}}\right) + \cdots\cdots\cdots + \frac{\partial}{\partial x_n}\left(\eta_i\frac{\partial f}{\partial u_{i,n}}\right)\right\}dD$$

$$-\int_D \eta_i\left\{\frac{\partial f}{\partial x_1}\left(\frac{\partial f}{\partial u_{i,1}}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial u_{i,2}}\right) + \cdots\cdots\cdots + \frac{\partial}{\partial x_n}\left(\frac{\partial f}{\partial u_{i,n}}\right)\right\}dD \tag{17}$$

From the Gauss's theorem of n dimensions, the first integral of (17) is given by

$$\int_{\partial D}\eta_i\left(n_1\frac{\partial f}{\partial u_{i,1}} + n_2\frac{\partial f}{\partial u_{i,2}} + \cdots\cdots\cdots + n_n\frac{\partial f}{\partial u_{i,n}}\right)d(\partial D) \tag{18}$$

Since the contribution of $\eta_i(x_i)$ vanishes on ∂D, (18) is equal to zero. Consequently, the original equation (16) becomes

$$\int_D \eta_i \left\{ \frac{\partial f}{\partial u_i} - \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial u_{i,1}}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial u_{i,2}}\right) - \cdots\cdots\cdots - \frac{\partial}{\partial x_n}\left(\frac{\partial f}{\partial u_{i,n}}\right)\right\} dD = 0 \quad (19)$$

Applying the fundamental auxiliary theorem to (19), we have

$$\frac{\partial f}{\partial u_i} - \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial u_{i,1}}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial u_{i,2}}\right) - \cdots\cdots\cdots - \frac{\partial}{\partial x_n}\left(\frac{\partial f}{\partial u_{i,n}}\right) = 0 \quad (20)$$

Variational principle of two dimensional viscous fluid

The system of fundamental equations of the incompressible viscous fluid in low speed has been already described as (4) and (15). (4) and (15) are expanded into each of x_i components as follows.

$$\begin{cases} v_{1,1} + v_{2,2} = 0 & (21.1) \\ -p_{,1} + \mu^*(v_{1,11} + v_{1,22}) + \rho f_1 = 0 & (21.2) \\ -p_{,2} + \mu^*(v_{2,11} + v_{2,22}) + \rho f_2 = 0 & (21.3) \end{cases}$$

A corresponding functional to (21) is taken as

$$\Pi[v_i, p] = \int_S F(x_i,\ v_i,\ p,\ v_{i,j})\, dS \quad (22)$$

in which the function F is defined as

$$F = \mu^* e_{ij} e_{ij} - p v_{k,k} - \rho f_i v_i \quad (23)$$

Expanding (23) into two dimensional form explicitly, we have

$$F = \mu^* \left\{ v_{1,1}^2 + \frac{(v_{1,2} + v_{2,2})^2}{2} + v_{2,2}^2 \right\} - p\,(v_{1,1} + v_{2,2}) - p\,(f_1 v_1 + f_2 v_2) \quad (24)$$

Since F is the function of two variables and three functions, we can derive the Euler's equation of $\Pi[v_i, p]$ from (20) as follows.

$$\begin{cases} -\dfrac{\partial F}{\partial p} = 0 & (25.1) \\ \dfrac{\partial}{\partial x_1}\left(\dfrac{\partial F}{\partial v_{1,1}}\right) + \dfrac{\partial}{\partial x_2}\left(\dfrac{\partial F}{\partial v_{1,2}}\right) - \dfrac{\partial F}{\partial v_1} = 0 & (25.2) \\ \dfrac{\partial}{\partial x_1}\left(\dfrac{\partial F}{\partial v_{2,1}}\right) + \dfrac{\partial}{\partial x_2}\left(\dfrac{\partial F}{\partial v_{2,2}}\right) - \dfrac{\partial F}{\partial v_2} = 0 & (25.3) \end{cases}$$

Substitute (24) into (25) and take account of (21. 1), we have

$$\begin{cases} v_{1,1} + v_{2,2} = 0 \\ -p_{,1} + \mu^*(v_{1,11} + v_{1,22}) + \rho f_1 = 0 \\ -p_{,2} + \mu^*(v_{2,11} + v_{2,22}) + \rho f_2 = 0 \end{cases}$$

They are equal to (21), so that the functional of the incompressible viscous fluid in low speed exists as (22).

Variatinal principle of three dimensional viscous fluid

The variational principle of three dimensional case is discussed in a similar way as of the two dimensional case. The expanded form of the fundamental equations of this case are as follows.

$$\begin{cases} v_{1,1}+v_{2,2}+v_{3,3}=0 & (26.1) \\ -p_{,1}+\mu^*(v_{1,11}+v_{1,22}+v_{1,33})+\rho f_1=0 & (26.2) \\ -p_{,2}+\mu^*(v_{2,11}+v_{2,22}+v_{2,33})+\rho f_2=0 & (26.3) \\ -p_{,3}+\mu^*(v_{3,11}+v_{3,22}+v_{3,33})+\rho f_3=0 & (26.4) \end{cases}$$

A functional corresponding to (26) is represented by

$$\Pi[v_i,\ p]=\int_v F(x_i,\ v_i,\ p,\ v_{i,j})dv \tag{27}$$

where F is defined as

$$F=\mu^*\left[v_{1,1}^2+v_{2,2}^2+v_{3,3}^2+\frac{1}{2}\left\{(v_{2,3}+v_{3,2})^2+(v_{3,1}+v_{1,3})^2+(v_{1,2}+v_{2,1})^2\right\}\right]$$
$$-p(v_{1,1}+v_{2,2}+v_{3,3})-\rho\,(f_1 v_1+f_2 v_2+f_3 v_3) \tag{28}$$

Since F is of three variables and four functions, we can obtain the Euler's equation of $\Pi[v_i,\ p]$ from (20) as follows.

$$\begin{cases} -\dfrac{\partial F}{\partial p}=0 & (29.1) \\ \dfrac{\partial}{\partial x_1}\left(\dfrac{\partial F}{\partial v_{1,1}}\right)+\dfrac{\partial}{\partial x_2}\left(\dfrac{\partial F}{\partial v_{1,2}}\right)+\dfrac{\partial}{\partial x_3}\left(\dfrac{\partial F}{\partial v_{1,3}}\right)-\dfrac{\partial F}{\partial v_1}=0 & (29.2) \\ \dfrac{\partial}{\partial x_1}\left(\dfrac{\partial F}{\partial v_{2,1}}\right)+\dfrac{\partial}{\partial x_2}\left(\dfrac{\partial F}{\partial v_{2,2}}\right)+\dfrac{\partial}{\partial x_3}\left(\dfrac{\partial F}{\partial v_{2,3}}\right)-\dfrac{\partial F}{\partial v_2}=0 & (29.3) \\ \dfrac{\partial}{\partial x_1}\left(\dfrac{\partial F}{\partial v_{3,1}}\right)+\dfrac{\partial}{\partial x_2}\left(\dfrac{\partial F}{\partial v_{3,2}}\right)+\dfrac{\partial}{\partial x_3}\left(\dfrac{\partial F}{\partial v_{3,3}}\right)-\dfrac{\partial F}{\partial v_3}=0 & (29.4) \end{cases}$$

Substitute (28) into (29) and consider (26. 1), we have

$$\begin{cases} v_{1,1}+v_{2,2}+v_{3,3}=0 \\ -p_{,1}+\mu^*(v_{1,11}+v_{1,22}+v_{1,33})+\rho f_1=0 \\ -p_{,2}+\mu^*(v_{2,11}+v_{2,22}+v_{2,33})+\rho f_2=0 \\ -p_{,3}+\mu^*(v_{3,11}+v_{3,22}+v_{3,33})+\rho f_3=0 \end{cases}$$

They are also identical with (26).

FINITE ELEMENT FORMULATION OF TWO DIMENSIONAL VISCOUS FLUID

In order to disperse a continuous function, it is necessary to derive a temporary function which has coordinates as variables. The temporary function is called the trial function in the Ritz's method. Two kinds of the trial function are stated here, one is of two variables and one function, and the other is of two variables and two functions.

Trial function of two variables and one function

If $u(x_1, x_2)$ is linear with respect to x_1 and x_2, we can describe the function by three coefficients a_1, a_2 and a_3 in two dimensions as follows.

$$u(x_1, x_2)=a_1+a_2x_1+a_3x_2 \tag{30}$$

Let ξ_1, ξ_2 and ξ_3 be identical to 1, x_1 and x_2, then (30) can be written by

$$u(x_1, x_2)=u(\xi_1, \xi_2, \xi_3)=a_i\xi_i \tag{31}$$

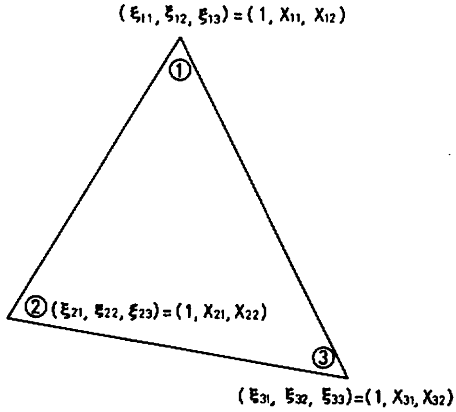

Fig. 2 ; Definition of ξ-coordinate

As shown in Fig. 2, ξ_{i1}, ξ_{i2} and ξ_{i3} are taken to be 1, x_{i1} and x_{i2}, respectively. Since $\bar{u}_i$ $(\xi_{i1}, \xi_{i2}, \xi_{i3})$ which is often abbreviated to $\bar{u}_i$ denotes the value of $u(\xi_1, \xi_2, \xi_3)$ at the nodal point i, the following three relations hold.

$$\begin{cases} \bar{u}_1=u(\xi_{11}, \xi_{12}, \xi_{13}) \\ \bar{u}_2=u(\xi_{21}, \xi_{22}, \xi_{23}) \\ \bar{u}_3=u(\xi_{31}, \xi_{32}, \xi_{33}) \end{cases} \tag{32}$$

According to (32) and (31), $\bar{u}_1$, $\bar{u}_2$ and $\bar{u}_3$ are represented as follows.

$$\bar{u}_1=a_j\xi_{1j} \quad ; \quad \bar{u}_2=a_j\xi_{2j} \quad ; \quad \bar{u}_3=a_j\xi_{3j}$$

The expanded forms of $\bar{u}_1$, $\bar{u}_2$ and $\bar{u}_3$ are simplified into the following tensor representation.

$$\bar{u}_i=a_j\xi_{ij} \tag{33}$$

Multiplying an inverse matrix of ξ_{ij} to (33) and exchanging i and j mutually, (33) becomes

$$a_i = \xi_{ji}^{-1}\bar{u}_j \tag{34}$$

Substitute (34) into (31), we have

$$u(\xi_1,\ \xi_2,\ \xi_3) = \xi_{ji}^{-1}\ \bar{u}_j\xi_i \tag{35}$$

Notice that if we define Δ and Δ_{ij} to be $\det(\xi_{ij})$ and its cofactor respectively, the relation $\xi_{ji}^{-1} = \Delta_{ij}/\Delta$ holds according to a theory of linear algebra. Rewriting ξ_{ji}^{-1} in (35) to F_{ij}, (35) is

$$u(\xi_1,\ \xi_2,\ \xi_3) = F_{ij}\bar{u}_j\xi_i \tag{35$'$}$$

When we wish to represent (35)′ in the x-system, exchanging ξ_1, ξ_2 and ξ_3 into 1, x_1 and x_2 respectively, we have

$$u(x_1,\ x_2) = F_{1j}\bar{u}_j + F_{2j}\bar{u}_jx_1 + F_{3j}\bar{u}_jx_2 \tag{36}$$

The partial differential coefficients of (36) with respect to x_1 and x_2, therefore, become

$$u_{,1}(x_1,\ x_2) = F_{2j}\bar{u}_j \quad ; \quad u_{,2}(x_1,\ x_2) = F_{3j}\bar{u}_j \tag{37}$$

Trial function of two variables and two functions

When $u_1\ (x_1,\ x_2)$ and $u_2\ (x_1,\ x_2)$ are linear with respect to x_1 and x_2, we can also derive $u_k(\xi_1,\ \xi_2,\ \xi_3)$ from (36) as

$$u_k(\xi_1,\ \xi_2,\ \xi_3) = F_{ij}\bar{u}_{jk}\xi_i \tag{38}$$

where i and j run through 1 to 3 and k takes 1 and 2. The term $\bar{u}_{jk}$ is defined as the value of $u_k\ (\xi_1,\ \xi_2,\ \xi_3)$ at the nodal point j, then the following relations hold.

$$\bar{u}_{1k} = u_k(\xi_{11},\ \xi_{12},\ \xi_{13})\ ;\ \bar{u}_{2k} = u_k(\xi_{21},\ \xi_{22},\ \xi_{23})\ ;\ \bar{u}_{3k} = u_k(\xi_{31},\ \xi_{32},\ \xi_{33}) \tag{39}$$

In order to transform (38) into the x-system, exchanging ξ_1, ξ_2 and ξ_3 to 1, x_1 and x_2, respectively, we obtain

$$u_k(x_1,\ x_2) = F_{1j}\bar{u}_{jk} + F_{2j}\bar{u}_{jk}x_1 + F_{3j}\bar{u}_{jk}x_2 \tag{40}$$

All the partial differential coefficients of (40) with respect to x_1 and x_2 are written as follows.

$$u_{1,1}(x_1,\ x_2) = F_{2j}\bar{u}_{j1}\ ;\ u_{1,2}(x_1,\ x_2) = F_{3j}\bar{u}_{j1}\ ;\ u_{2,1}(x_1,\ x_2) = F_{2j}\bar{u}_{j2}\ ;$$

$$u_{2,2}(x_1,\ x_2) = F_{3j}\bar{u}_{j2} \tag{41}$$

Finite element formulation of two dimensional viscous fluid

The functional of two dimensional incompressible viscous fluid without inertia term has already been given by (22).

Two variables v_1 and v_2 in (22) are considered to the function which is approximated by the trial function of two variables and two functions, and p is approximated by the trial function of two variables and one function. In this case, therefore, we use the following two kinds of trial functions,

$$\begin{cases} v_k(x_1,\ x_2) = \nabla_{1i}\bar{v}_{ik} + \nabla_{2i}\bar{v}_{ik}x_1 + \nabla_{3i}\bar{v}_{ik}x_2 \\ p(x_1,\ x_2) = \nabla_{1i}\bar{p}_i + \nabla_{2i}\bar{p}_i x_1 + \nabla_{3i}\bar{p}_i x_2 \end{cases} \tag{42}$$

Four partial derivatives of $v_k(x_1,\ x_2)$ have been already given by (41).

According to the direct method of variational principle, we consider $\Pi[v_1, v_2, p]$ of (22) to be the function of $\bar{v}_{11}$, $\bar{v}_{12}$, $\bar{v}_{21}$, $\bar{v}_{22}$, $\bar{v}_{31}$, $\bar{v}_{32}$, $\bar{p}_1$, $\bar{p}_2$ and $\bar{p}_3$. Substitute (42) and (41) into (22), we have

$$\Pi[\bar{v}_{11},\ \bar{v}_{12},\ \bar{v}_{21},\ \bar{v}_{22},\ \bar{v}_{31},\ \bar{v}_{32},\ \bar{p}_1,\ \bar{p}_2,\ \bar{p}_3] = \int_S\Big[\mu^*\Big\{(\nabla_{2i}\bar{v}_{i1})^2 + \frac{1}{2}(\nabla_{3i}\bar{v}_{i1} + \nabla_{2i}\bar{v}_{i2})^2 + (\nabla_{3i}\bar{v}_{i2})^2\Big\} - (\nabla_{1i}\bar{p}_i + \nabla_{2i}\bar{p}_i x_1 + \nabla_{3i}\bar{p}_i x_2)(\nabla_{2i}\bar{v}_{i1} + \nabla_{3i}\bar{v}_{i2}) - \rho f_1(\nabla_{1i}\bar{v}_{i1} + \nabla_{2i}\bar{v}_{i1}x_1 + \nabla_{3i}\bar{v}_{i1}x_2) - \rho f_2(\nabla_{1i}\bar{v}_{i2} + \nabla_{2i}\bar{v}_{i2}x_1 + \nabla_{3i}\bar{v}_{i2}x_2)\Big]dS \tag{43}$$

To differentiate (43) with respect to $\bar{v}_{11}$ and to be zero, we obtain

$$\frac{\partial\Pi}{\partial\bar{v}_{11}} = \int_S\Big[\mu^*\Big\{2\nabla_{21}\nabla_{2i}\bar{v}_{i1} + \nabla_{31}(\nabla_{3i}\bar{v}_{i1} + \nabla_{2i}\bar{v}_{i2})\Big\} - \nabla_{21}(\nabla_{1i}\bar{p}_i + \nabla_{2i}\bar{p}_i x_1 + \nabla_{3i}\bar{p}_i x_2) - \rho f_1(\nabla_{11} + \nabla_{21}x_1 + \nabla_{31}x_2)\Big]dS = 0 \tag{44}$$

Since μ^* is a constant and ρf_1 is independent of all the variables within element, the simpler form of (44) that excludes an integral symbol is given as follows.

$$\frac{\partial\Pi}{\partial\bar{v}_{11}} = S\Big\{\mu^*(2\nabla_{21}\nabla_{2i} + \nabla_{31}\nabla_{3i})\bar{v}_{i1} + \mu^*\nabla_{31}\nabla_{2i}\bar{v}_{i2} - \frac{1}{3}\nabla_{21}\bar{p} - \frac{1}{3}\rho f_1\Big\} = 0 \tag{45}$$

where $\bar{p} = \frac{1}{3}(\bar{p}_1 + \bar{p}_2 + \bar{p}_3)$ and S is the area of element in question. A further simplification can now be made by substituting firstly S and ∇_{ij} into $\frac{1}{2}\varDelta$ and $\frac{\varDelta_{ij}}{\varDelta}$, respectively and secondly $\frac{\mu^*}{2\varDelta}$ into A. The resultant simpler form of (45) becomes

$$\frac{\partial\Pi}{\partial\bar{v}_{11}} = A\Big\{(2\varDelta_{21}\varDelta_{2i} + \varDelta_{31}\varDelta_{3i})\bar{v}_{i1} + \varDelta_{31}\varDelta_{2i}\bar{v}_{i2}\Big\} - \frac{1}{6}\varDelta_{21}\bar{p} - \frac{1}{6}\varDelta\rho f_1 = 0 \tag{46.1}$$

Similarly we have the other eight equations as follows.

$$\frac{\partial\Pi}{\partial\bar{v}_{12}} = A\Big\{\varDelta_{21}\varDelta_{3i}\bar{v}_{i1} + (\varDelta_{21}\varDelta_{2i} + 2\varDelta_{31}\varDelta_{3i})\bar{v}_{i2}\Big\} - \frac{1}{6}\varDelta_{31}\bar{p} - \frac{1}{6}\varDelta\rho f_2 = 0 \tag{46.2}$$

$$\frac{\partial\Pi}{\partial\bar{p}_1} = -\frac{1}{6}(\varDelta_{2i}\bar{v}_{i1} + \varDelta_{3i}\bar{v}_{i2}) = 0 \tag{46.3}$$

$$\frac{\partial\Pi}{\partial\bar{v}_{21}} = A\Big\{(2\varDelta_{22}\varDelta_{2i} + \varDelta_{32}\varDelta_{3i})\bar{v}_{i1} + \varDelta_{32}\varDelta_{2i}\bar{v}_{i2}\Big\} - \frac{1}{6}\varDelta_{22}\bar{p} - \frac{1}{6}\varDelta\rho f_1 = 0 \tag{46.4}$$

$$\frac{\partial \Pi}{\partial \bar{v}_{22}} = A\left\{ \varDelta_{22}\varDelta_{3i}\bar{v}_{i1} + (\varDelta_{22}\varDelta_{2i} + 2\varDelta_{32}\varDelta_{3i})\,\bar{v}_{i2} \right\} - \frac{1}{6}\varDelta_{32}\bar{p} - \frac{1}{6}\varDelta\rho f_2 = 0 \qquad (46.5)$$

$$\frac{\partial \Pi}{\partial \bar{p}_2} = -\frac{1}{6}(\varDelta_{2i}\bar{v}_{i1} + \varDelta_{3i}\bar{v}_{i2}) = 0 \qquad (46.6)$$

$$\frac{\partial \Pi}{\partial \bar{v}_{31}} = A\left\{ (2\varDelta_{23}\varDelta_{2i} + \varDelta_{33}\varDelta_{3i})\,\bar{v}_{i1} + \varDelta_{33}\varDelta_{2i}\bar{v}_{i2} \right\} - \frac{1}{6}\varDelta_{23}\bar{p} - \frac{1}{6}\varDelta\rho f_1 = 0 \qquad (46.7)$$

$$\frac{\partial \Pi}{\partial \bar{v}_{32}} = A\left\{ \varDelta_{23}\varDelta_{3i}\bar{v}_{i1} + (\varDelta_{23}\varDelta_{2i} + 2\varDelta_{33}\varDelta_{3i})\,\bar{v}_{i2} \right\} - \frac{1}{6}\varDelta_{33}\bar{p} - \frac{1}{6}\varDelta\rho f_2 = 0 \qquad (46.8)$$

$$\frac{\partial \Pi}{\partial \bar{p}_3} = -\frac{1}{6}(\varDelta_{2i}\bar{v}_{i1} + \varDelta_{3i}\bar{v}_{i2}) = 0 \qquad (46.9)$$

The nine equations of (46) are shown in a matrix form in Table 2, where $\varDelta_{21}$, $\varDelta_{31}$, $\varDelta_{22}$, $\varDelta_{32}$, $\varDelta_{23}$ and $\varDelta_{33}$ are replaced by $\varDelta_1$, $\varDelta_2$, $\varDelta_3$, $\varDelta_4$, $\varDelta_5$ and $\varDelta_6$, respectively.

Table 2 ; Stiffness matrix of the incompressible Newtonian fluid without inertia term in two dimensions

$A(2\triangle_1^2+\triangle_2^2)$	$A\triangle_2\triangle_1$	$-\triangle_1/6$	$A(2\triangle_1\triangle_3+\triangle_2\triangle_4)$	$A\triangle_2\triangle_3$	$-\triangle_1/6$	$A(2\triangle_1\triangle_5+\triangle_2\triangle_6)$	$A\triangle_2\triangle_5$	$-\triangle_1/6$
	$A(\triangle_1^2+2\triangle_2^2)$	$-\triangle_2/6$	$A\triangle_1\triangle_4$	$A(\triangle_1\triangle_3+2\triangle_2\triangle_4)$	$-\triangle_2/6$	$A\triangle_1\triangle_6$	$A(\triangle_1\triangle_5+2\triangle_2\triangle_6)$	$-\triangle_2/6$
		0	$-\triangle_3/6$	$-\triangle_4/6$	0	$-\triangle_5/6$	$-\triangle_6/6$	0
			$A(2\triangle_3^2+\triangle_4^2)$	$A\triangle_4\triangle_3$	$-\triangle_3/6$	$A(2\triangle_3\triangle_5+\triangle_4\triangle_6)$	$A\triangle_4\triangle_5$	$-\triangle_3/6$
				$A(\triangle_3^2+2\triangle_4^2)$	$-\triangle_4/6$	$A\triangle_3\triangle_6$	$A(\triangle_3\triangle_5+2\triangle_4\triangle_6)$	$-\triangle_4/6$
					0	$-\triangle_5/6$	$-\triangle_6/6$	0
						$A(2\triangle_5^2+\triangle_6^2)$	$A\triangle_6\triangle_5$	$-\triangle_5/6$
							$A(\triangle_5^2+2\triangle_6^2)$	$-\triangle_6/6$
symmetric								0

Calculation of strain rate, pressure and stress

Three components of a strain rate e_1, e_2 and e_3 are constant within every element. After we can solve the simultaneous equation of (46), the strain rate is calculated by substituting the known values of $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, $\bar{v}_4$, $\bar{v}_5$ and $\bar{v}_6$ into the following formulae.

$$
\begin{aligned}
e_1 &= v_{1,1} = \frac{\Delta_{2i}\bar{v}_{i1}}{\Delta} = \frac{\Delta_1\bar{v}_1 + \Delta_3\bar{v}_3 + \Delta_5\bar{v}_5}{\Delta} \\
e_2 &= v_{2,2} = \frac{\Delta_{3i}\bar{v}_{i2}}{\Delta} = \frac{\Delta_2\bar{v}_2 + \Delta_4\bar{v}_4 + \Delta_6\bar{v}_6}{\Delta} \\
e_3 &= \frac{v_{1,2} + v_{2,1}}{2} = \frac{\Delta_2\bar{v}_1 + \Delta_1\bar{v}_2 + \Delta_4\bar{v}_3 + \Delta_3\bar{v}_4 + \Delta_6\bar{v}_5 + \Delta_5\bar{v}_6}{2\Delta}
\end{aligned}
\tag{47}
$$

where $\bar{v}_{21}$, $\bar{v}_{31}$, $\bar{v}_{22}$, $\bar{v}_{32}$, $\bar{v}_{23}$ and $\bar{v}_{33}$ are replaced by $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, $\bar{v}_4$, $\bar{v}_5$ and $\bar{v}_6$, respectively.

Secondly, pressures at the center of every element are obtained by

$$
p_g = -\frac{\Delta_{1i}\bar{p}_i}{\Delta} = -\frac{\bar{p}_1 + \bar{p}_2 + \bar{p}_3}{3} \tag{48}
$$

Finally, three components of the stress at the center of element σ_1, σ_2 and σ_3 are carried out by the following formulae,

$$
\sigma_1 = p_g + 2\mu^* e_1 \quad ; \quad \sigma_2 = p_g + 2\mu^* e_2 \quad ; \quad \sigma_3 = 2\mu^* e_3 \tag{49}
$$

REFERENCES

Dieterich, J. H. and Onat, E. T., 1969; Slow finite deformations of viscous solids. Jour. Geophys. Res., **74**, 2081–2088.

Dieterich, J. H. and Carter, N. L., 1969; Stress-history of folding. Am. Jour. Sci., **267**, 129–154.

Hayashi, D., 1975; Rising of a granitic mass as an incompressible Newtonian fluid. Jour. Geol. Soc. Japan, **81**, 769–782. (in Japanese with English abstract)

Kinugasa, Y., Kakimi, T. and Hirayama, J., 1969; The minorfault systems on the coastal area of the eastern Boso Peninsula. Bull. Geol. Surv. Japan, **20**, 13–38.
(in Japanese with English abstract)

Landau, L. D. and Lifshitz, E. M., 1959; Fluid Mechanics. Pergamon, 536p.

Millikan, C. B., 1929; On the steady motion of viscous incompressible fluids, with particular reference to a variation principle. Phil. Mag., **7**, 641–662.

Truesdell, C., 1952; The mechanical foundations of elasticity and fluid dynamics. J. Rat. Mech. and Analysis, **1**, 125–300.

Turner, F. J. and Weiss, L. E., 1963; Structural analysis of metamorphic tectonites. McGraw-Hill, 545p.