# Inversions and Möbius invariants

Ryukyu Math. J., 20 (2007), 9-23

# Inversions and Möbius invariants

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**Abstract**

Two $n$-point-sets in Euclidean space are said to be inversion-equivalent if one set can be transformed into the other set by applying inversions of the space. All 3-point-sets are inversion-equivalent to each other. For each four points $x, y, z, w$ in an $n$-point-set, $n \geq 4$, the ratio $(xy \cdot zw)/(xw \cdot yz)$ is invariant under inversions, which is called a Möbius invariant of the $n$-point-set. We prove that for $4 \leq n \leq d+2$, the minimum number of Möbius invariants necessary to determine all Möbius invariants for every $n$-point-set in Euclidean $d$-space is equal to $n(n-3)/2$, and discuss the case of planar $n$-point-sets in some detail. We also characterize those fractional functions that are invariant under inversions.

## 1 Introduction

Let $S$ be a sphere in the $d$-dimensional Euclidean space $\mathbb{R}^d$ with center $p$ and radius $r$. The *inversion* of $\mathbb{R}^d$ with respect to $S$ is the transformation of $\mathbb{R}^d$ that sends each point $x\ (\neq p)$ to a point $x'$ on the ray $\overrightarrow{px}$ such that $px \cdot px' = r^2$, where $px$ denotes the distance between $p$ and $x$. The point $p$ and the radius $r$ are called the *center* and the *radius* of the inversion, respectively. Note that in an inversion, the image of its center is not defined. One of the typical features of an inversion is that it transforms a sphere into another sphere, with regarding a hyperplane as a sphere of infinite radius. For more about inversions, see, e.g. [2,5,8].

In this paper, we consider to transform a finite point-set by inversions. Note that *an inversion can be applied to a point-set only when*

Received November 30, 2007.

*its center does not belong to the point-set,* since the image of the center of an inversion is not defined. (Usually, to avoid such restriction, a point at infinity is added to $\mathbb{R}^d$ as to be the image of the center for every inversion. Another less usual way is to define the image of the center of an inversion to be the center itself. In [7], transformations of a finite point-set by 'center-fixing-inversions' with centers in the point-set are investigated.)

A pair of $n$-point-sets are called *inversion-equivalent* if one set can be transformed into the other set by applying a series of inversions. This relation is clearly an equivalence relation. An *ordered $n$-point-set* is an $n$-point-set whose points are ordered, which is denoted by the juxtaposition of the $n$ points in order like $a_1a_2a_3\dots a_n$ (the same notation as used for a polygon or a polygonal curve). Two ordered $n$-point-sets are called *inversion-equivalent* if they are inversion-equivalent with keeping the order. It turns out that all ordered triples are mutually inversion-equivalent.

For a quadruplet (that is, an ordered 4-point-set) $\boldsymbol{abcd}$, let us define $[\boldsymbol{abcd}]$ by

$$[\boldsymbol{abcd}] = (\boldsymbol{ab}\cdot\boldsymbol{cd})/(\boldsymbol{ad}\cdot\boldsymbol{bc}),$$

which is called a *Möbius invariant* of the quadruplet. This is indeed invariant under any inversion (Corollary 2.1, see also [5, p.92], [3, p.310]). Hence, for example, a quadruplet $\boldsymbol{abcd}$ with $[\boldsymbol{abcd}] \neq 1$ is never inversion-equivalent to the vertex set of a regular tetrahedron.

For a quadruplet $\boldsymbol{a}_i\boldsymbol{a}_j\boldsymbol{a}_k\boldsymbol{a}_l$ taken from an ordered $n$-point-set $\alpha = \boldsymbol{a}_1\boldsymbol{a}_2\dots\boldsymbol{a}_n$, $n \geq 4$, the Möbius invariant $[\boldsymbol{a}_i\boldsymbol{a}_j\boldsymbol{a}_k\boldsymbol{a}_l]$ is simply denoted by $[ijkl]$, and its 'value' at $\alpha$ is denoted by $[ijkl]_\alpha$. Let us call

$$[ijkl] \quad (i,j,k,l \text{ are all different, } 1 \leq i,j,k,l \leq n)$$

the Möbius invariants for an ordered $n$-point-set. We show the following:

- *For $n \geq 4$, two ordered $n$-point-sets $\alpha, \beta$ are inversion-equivalent if and only if $[ijkl]_\alpha = [ijkl]_\beta$ holds for every $[ijkl]$.*

Applying this result, it is proved that every quadruplet is inversion equivalent to the vertex set of a (possibly degenerate) parallelogram.

Notice that $[ijkl] = [klij] = [jilk] = [lkji]$. Hence there are at most $6\binom{n}{4}$ distinct Möbius invariants for an ordered $n$-point-set.

They are not independent as 'variables'. Rather a few of them determine all others. Let $R(d,n)$ denote the minimum cardinality of a set of Möbius invariants whose values determine the values of all Möbius invariants for every ordered $n$-point-set in $\mathbb{R}^d$. We prove the following.

- $R(d,n) = n(n-3)/2$ for $4 \leq n \leq d+2$.

If $d \ll n$ then $R(d,n)$ would be much smaller than $n(n-3)/2$ by the dimensional restriction. However, it is not easy to determine the value of $R(d,n)$ even in the planar case $d = 2$, where the bound I could prove is $R(2,n) \leq 3n - 10$. In Section 5, we will discuss the planar case in some detail.

Besides Möbius invariants, there are many fractional functions, such as $(\boldsymbol{ab} \cdot \boldsymbol{ac} \cdot \boldsymbol{de})/(\boldsymbol{ad} \cdot \boldsymbol{ae} \cdot \boldsymbol{bc})$ and $(\boldsymbol{ab} \cdot \boldsymbol{cd} \cdot \boldsymbol{ef})/(\boldsymbol{bc} \cdot \boldsymbol{de} \cdot \boldsymbol{fa})$ that are invariant under inversions. They are characterized in the following way:

- *A fractional function is invariant under inversions if and only if, for each point-symbol, the number of times it appears in the numerator is equal to the number of times it appears in the denominator.*

# 2 A few basic facts on inversions

**Lemma 2.1.** *Let $\boldsymbol{a}', \boldsymbol{b}'$ denote the images of $\boldsymbol{a}, \boldsymbol{b}$ under the inversion with center $\boldsymbol{p}$ and radius $r$. Then $\boldsymbol{a}'\boldsymbol{b}' = (r^2 \cdot \boldsymbol{ab})/(\boldsymbol{pa} \cdot \boldsymbol{pb})$.*

*Proof.* Since $\boldsymbol{pa}' \cdot \boldsymbol{pa} = r^2 = \boldsymbol{pb}' \cdot \boldsymbol{pb}$ and $\angle \boldsymbol{apb} = \angle \boldsymbol{a}'\boldsymbol{pb}'$, the two (possibly degenerate) triangles $\boldsymbol{pa}'\boldsymbol{b}'$ and $\boldsymbol{pba}$ are similar. Hence $\boldsymbol{a}'\boldsymbol{b}'/\boldsymbol{ba} = \boldsymbol{pa}'/\boldsymbol{pb} = r^2/(\boldsymbol{pa} \cdot \boldsymbol{pb})$, and the lemma follows. □

The following corollaries follow from this by simple calculations.

**Corollary 2.1.** *Suppose an inversion sends $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ to $\boldsymbol{a}', \boldsymbol{b}', \boldsymbol{c}', \boldsymbol{d}'$, respectively. Then $[\boldsymbol{a}'\boldsymbol{b}'\boldsymbol{c}'\boldsymbol{d}'] = [\boldsymbol{abcd}]$.* □

**Corollary 2.2.** *Let $f_1$, $f_2$ be the inversions with the same center $\boldsymbol{p}$ and radii $r_1$, $r_2$, respectively. Then the composition $f_2 \circ f_1$ is a homothety with center $\boldsymbol{p}$ and similitude ratio $(r_2/r_1)^2$.* □

**Lemma 2.2.** *Let $H$ be a hyperplane in $\mathbb{R}^d$, $p \in \mathbb{R}^d \setminus H$, and $q \in \mathbb{R}^d$ be the point that is symmetric to $p$ with respect to $H$. Let $g_1$ be the inversion of $\mathbb{R}^d$ with respect to the sphere with center $p$ and radius $pq$, and let $g_2$ be the inversion of $\mathbb{R}^d$ with center $q$ and radius $pq$. Then the composition $f := g_1 \circ g_2 \circ g_1$ is the reflection of $\mathbb{R}^d$ with respect to the hyperplane $H$.*

*Proof.* It will be enough to consider the plane case $d = 2$. We may suppose that $p = (-1,0), q = (1,0)$ and $H$ is the $y$-axis. Note that $g_1(q) = q$, $g_2(p) = p$. Let $C_q$ be the circle with center $q$ and radius $pq$. For a point $u$ on the $y$-axis, let $v$ be the intersection of the ray $\overrightarrow{pu}$ and $C_q$ other than $p$. Then, since $g_1$ sends the $y$-axis to $C_q$, we have $g_1(u) = v$. Since $g_2(v) = v$, we have $f(u) = u$. Thus, $f$ fixes all points on the $y$-axis. It is also clear that $f$ sends the $x$-axis to itself.

For any line $\ell$, $g_1(\ell)$ is either a line passing through $p$ or a circle passing through $p$, and hence, $g_2(g_1(\ell))$ is either a circle passing through $p$ or a line passing through $p$. Therefore, $g_1(g_2(g_1(\ell)))$ is always a line. Thus, $f$ sends every line to a line, and sends every pair of parallel lines to a pair of parallel lines.

Now, since $g(x,0) = (\frac{3-x}{x+1}, 0)$, $g_2(x,0) = (\frac{x+3}{x-1}, 0)$ as easily verified, $f(x,0) = (-x,0)$ follows by a simple calculation. Then, for a given point $(x_0, y_0)$, the line "$x = x_0$" (which is parallel to the $y$-axis) is sent to the line passing through $(-x_0, 0)$ and parallel to the $y$-axis, that is, the line "$x = -x_0$". Similarly, the line "$y = y_0$" is sent to itself by $f$. Therefore the intersection $(x_0, y_0)$ of the two lines $x = x_0$ and $y = y_0$ is sent to $(-x_0, y_0)$, that is, $f(x_0, y_0) = (-x_0, y_0)$. This proves the lemma. □

Note that in Corollary 2.2, the center of a homothety can be chosen independently from the similitude ratio, and in Lemma 2.2, the point $p \in \mathbb{R}^d \setminus H$ can be chosen arbitrarily. Since all reflections of $\mathbb{R}^d$ generate all isometries of $\mathbb{R}^d$, we have the following corollary from Corollary 2.2 and Lemma 2.2.

**Corollary 2.3.** *If two n-point-sets are similar, then they are inversion-equivalent.* □

**Lemma 2.3.** *Let $\sigma$ be a finite point-set containing three points $a, b, c$. Then, for every $\lambda > \lambda_0$ (where $\lambda_0$ is a constant depending on $\sigma$), there exists an inversion with center $p \notin \sigma$ that transforms $a, b, c$ into $a', b', c'$ such that $a'b' = \lambda$, $b'c' = 1$, $a'c' = 1 + \lambda$.*

*Proof.* Let $\Gamma$ be the circle (or line) passing through $a, b, c$. Let $p$ be a point on $\Gamma \backslash$(the arc $\widehat{abc}$). Let $f$ be an inversion with center $p$ and some radius $r$, and let $a' = f(a), b' = f(b), c' = f(c)$. Then, $a', b', c'$ are collinear in this order, and by Lemma 2.1, we have

$$\frac{a'b'}{b'c'} = \left(\frac{r^2 \cdot ab}{pa \cdot pb}\right)\left(\frac{pb \cdot pc}{r^2 \cdot bc}\right) = \frac{ab \cdot pc}{bc \cdot pa}.$$

Let $\varepsilon$ be the distance from $a$ to the nearest point in $\sigma - \{a\}$. Then, $\varepsilon > 0$. Let $\lambda_0 = (ab/bc)(ac/\varepsilon + 1)$. If $pa = \varepsilon$, we have

$$\frac{ab \cdot pc}{bc \cdot pa} \leq \frac{ab(pa + ac)}{bc \cdot pa} = \frac{ab}{bc}\left(\frac{ac}{\varepsilon} + 1\right) = \lambda_0.$$

Since $pc/pa$ continuously tends to infinity as $pa$ continuously tends to 0, $(ab \cdot pc)/(bc \cdot pa)$ can take every value $\lambda \geq \lambda_0$. Thus, for every $\lambda > \lambda_0$, we can choose $p$ on $\Gamma$ so that $pa < \varepsilon$, $a'b'/b'c' = (ab/bc)(pc/pa) = \lambda$, and we can choose $r > 0$ so that $b'c' = 1$. □

An ordered triple $abc$ is called a *linear triple* if $a, b, c$ are collinear in this order.

**Corollary 2.4.** *Every pair of ordered triples are inversion-equivalent.* □

# 3 Möbius invariants

For three points $a, b, p \in \mathbb{R}^d$ $(d \geq 2)$, the locus of the points $x$ satisfying $ax/bx = ap/bp$ is called the *Apollonian sphere* (*Apollonian circle* if $d = 2$) determined by $ab$ and $p$, which is denoted by $A(ab, p)$. If $ap \neq bp$, then $A(ab, p)$ is indeed a sphere with center at the extension of the line segment $ab$ (beyond $a$ or $b$), but if $ap = bp$, then $A(ab, p)$ is a hyperplane that bisects the line segment $ab$ perpendicularly. Since

$$A(ab, p) \ni q \Leftrightarrow \frac{ap}{bp} = \frac{aq}{bq} \Leftrightarrow \frac{pa}{qa} = \frac{pb}{qb} \Leftrightarrow A(pq, a) \ni b$$

holds, $q \in A(ab, p) \cap A(ac, p)$ implies that $\{a, b, c\} \subset A(pq, a)$. Therefore, *if* $abc$ *is a linear triple, then* $A(ab, p)$ *and* $A(ac, p)$ *are different spheres.* For more about Apollonian circles, see Coxeter [5].

For an ordered $n$-point-set $\alpha$, we denote the distance between the $i$-th point and the $j$-th point by $d_{ij}$ or $d_{ij}(\alpha)$.

**Lemma 3.1.** *Let $n \geq 4$ and $\alpha = a_1 a_2 \dots a_n$ be an ordered n-point-set in which $a_1 a_2 a_3$ is a linear triple. Then the two distances $d_{12}$, $d_{23}$ and the values of the Möbius invariants*

$$[j123], [j213],\ j = 4, 5, \dots, n, \text{ and } [jk12],\ 4 \leq j < k \leq n, \qquad (1)$$

*determine all the distances $d_{ij}$ in $\alpha$.*

*Proof.* Suppose that three points $a_1, a_2, a_3$ are fixed so that $d_{12} = s, d_{23} = t, d_{13} = s + t$. Since $[4213]_\alpha = (a_4 a_2 \cdot (s+t))/(a_4 a_3 \cdot s)$, we have $a_2 a_4 / a_3 a_4 = [4213]_\alpha \cdot s/(s+t)$. Hence the value of $[4213]$ determines the Apollonian sphere $A(a_2 a_3, a_4)$. Similarly, the value of $[4123]$ determines the Apollonian sphere $A(a_1 a_3, a_4)$. Since $a_1 a_2 a_3$ is a linear triple, these two Apollonian spheres are different with centers on the line $a_1 a_2$. Hence, all intersection points of the two Apollonian spheres are at the same distance from the line $a_1 a_2$, and hence each of the distances $d_{41}, d_{42}, d_{43}$ are uniquely determined. Similarly, for each $4 \leq j \leq n$, the values of $[j213]$ and $[j123]$ determine the distances $d_{j1}, d_{j2}, d_{j3}$ uniquely. Then, the values of $[jk12]$ determine the distances $d_{jk}$ for $4 \leq j < k \leq n$. □

Let $\alpha = a_1 \dots a_n$ be an ordered $n$-point-set in the plane $\mathbb{R}^2$, $n \geq 4$. Suppose $a_1 = (-s, 0), a_2 = (0,0), a_3 = (t, 0)$ and $a_4 = (u, v)$, $uv \neq 0$. Let $f$ be the inversion with center $(0, w)$, radius $r$, and let $g$ be the inversion with the center $(0, -w)$ and radius $r$, where $w \neq 0$. Then the triples $f(a_1 a_2 a_3)$, $g(a_1 a_2 a_3)$ are congruent, and every Möbius invariant takes the same value at $f(\alpha)$ and at $g(\alpha)$. But $f(\alpha)$ and $g(\alpha)$ are not congruent. Let us state this fact as a remark.

**Remark 3.1.** If the first three points of an ordered $n$-point-set ($n \geq 4$) in $\mathbb{R}^2$ are not collinear, then the three distances $d_{12}, d_{23}, d_{31}$ and the values of all Möbius invariants are not enough to determine all distances among the $n$ points.

**Theorem 3.1.** *For every $n \geq 4$, a pair of ordered n-point-sets $\alpha$ and $\beta$ in $\mathbb{R}^d$ are inversion-equivalent if and only if $[ijkl]_\alpha = [ijkl]_\beta$ holds for every $[ijkl]$.*

*Proof.* The *if* part is obvious since Möbius invariants are invariant under inversions.

Let $\alpha = \boldsymbol{a}_1 \ldots \boldsymbol{a}_n$, $\beta = \boldsymbol{b}_1 \ldots \boldsymbol{b}_n$. By Lemma 2.3, we can apply inversions to $\alpha$ and $\beta$ independently, so that $\boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{a}_3$ and $\boldsymbol{b}_1\boldsymbol{b}_2\boldsymbol{b}_3$ become congruent linear triples. Hence, we may assume from the first that $\boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{a}_3$ and $\boldsymbol{b}_1\boldsymbol{b}_2\boldsymbol{b}_3$ are congruent linear triples, $d_{12}(\alpha) = d_{12}(\beta) = s$, $d_{23}(\alpha) = d_{23}(\beta) = t$. Since $[ijkl]_\alpha = [ijkl]_\beta$ for every $[ijkl]$, it follows that $d_{ij}(\alpha) = d_{ij}(\beta)$ holds for all $i, j$ by Lemma 3.1. Hence the two ordered $n$-point-sets are congruent to each other. Therefore, they are inversion-equivalent by Corollary 2.3. □

**Theorem 3.2.** *Every quadruplet in $\mathbb{R}^2$ is inversion-equivalent to the vertex set of a (possibly degenerate) parallelogram.*

*Proof.* Let $\boldsymbol{abcd}$ be a quadruplet, and put $a = [\boldsymbol{abcd}], b = [\boldsymbol{acbd}]$. (Then, $[\boldsymbol{abdc}] = a/b$, $[\boldsymbol{acdb}] = b/a$, $[\boldsymbol{adbc}] = 1/b$, $[\boldsymbol{adcb}] = 1/a$ as easily verified.) By generalized Ptolemy's inequality (see, e.g. [1]) we have

$$\begin{aligned} \boldsymbol{ab}\cdot\boldsymbol{cd} &\leq \boldsymbol{ac}\cdot\boldsymbol{bd} + \boldsymbol{bc}\cdot\boldsymbol{ad}, \\ \boldsymbol{ac}\cdot\boldsymbol{bd} &\leq \boldsymbol{ab}\cdot\boldsymbol{cd} + \boldsymbol{ad}\cdot\boldsymbol{bc}, \\ \boldsymbol{ad}\cdot\boldsymbol{bc} &\leq \boldsymbol{ab}\cdot\boldsymbol{cd} + \boldsymbol{ac}\cdot\boldsymbol{bd}. \end{aligned}$$

Since $[\boldsymbol{abcd}] = (\boldsymbol{ab}\cdot\boldsymbol{cd})/(\boldsymbol{bc}\cdot\boldsymbol{ad})$ and $[\boldsymbol{acbd}] = (\boldsymbol{ad}\cdot\boldsymbol{bd})/(\boldsymbol{ad}\cdot\boldsymbol{bc})$, it follows that $a \leq b+1, b \leq a+1$ and $1 \leq a+b$. Therefore $|a-1| \leq b \leq a+1$. Let

$$p = \pm\frac{1}{2}\sqrt{(a+1)^2 - b^2},\ q = \frac{1}{2}\sqrt{b^2 - (a-1)^2}$$

(we may choose either sign $\pm$ for $p$) and put

$$\boldsymbol{a}' = (0,0),\ \boldsymbol{b}' = (p,q),\ \boldsymbol{c}' = (p+1,q),\ \boldsymbol{d}' = (1,0) \in \mathbb{R}^2.$$

Then $\boldsymbol{a}'\boldsymbol{b}'\boldsymbol{c}'\boldsymbol{d}'$ is a parallelogram, and

$$[\boldsymbol{a}'\boldsymbol{b}'\boldsymbol{c}'\boldsymbol{d}'] = a,\ [\boldsymbol{a}'\boldsymbol{c}'\boldsymbol{b}'\boldsymbol{d}'] = b.$$

Hence $\boldsymbol{abcd}$ and $\boldsymbol{a}'\boldsymbol{b}'\boldsymbol{c}'\boldsymbol{d}'$ are inversion-equivalent. □

**Corollary 3.1.** *The vertex-sets of two parallelograms are not inversion-equivalent unless the two parallelograms are similar to each other.* □

Since a parallelogram $\boldsymbol{abcd}$ is a rhombus if and only if $[\boldsymbol{abcd}] = 1$, we have the following.

**Corollary 3.2.** *A quadruplet $\boldsymbol{abcd}$ is inversion-equivalent to the vertex set of a rhombus if and only if $[\boldsymbol{abcd}] = 1$.* □

# 4 Number of necessary invariants

For a quadruplet $a_1a_2a_3a_4$ in $\mathbb{R}^d$ $(d > 0)$, let $x = d_{12}$, $y = d_{23}$, $z = d_{34}$, $w = d_{41}$. Then $[1234] = xz/(yw)$, which is the ratio of the two products of opposite edges in the (possibly self-intersecting) quadrilateral. By changing the order of the vertices cyclically, we get two distinct Möbius invariants, namely, $[1234] = [3412] = xz/(yw)$ and $[2341] = [4123] = yw/(xz) = [1234]^{-1}$. Since four points produce three distinct quadrilaterals, it follows that there are $6 \times \binom{n}{4}$ distinct Möbius invariants for $n$ points. Since $[2341] = [1234]^{-1}$, half of the $6\binom{n}{4}$ Möbius invariants are reciprocals of the other half. So, $6\binom{n}{4}$ Möbius invariants are determined by $3\binom{n}{4}$ members, probably much fewer members. Recall that $R(d,n)$ denotes the minimum cardinality of a set of Möbius invariants whose values determine the values of all Möbius invariants for every ordered $n$-point-set in $\mathbb{R}^d$.

**Lemma 4.1.** *For $d > n-2 \geq 2$, $R(d,n) = R(n-2,n)$.*

*Proof.* Since every $n$ points in $\mathbb{R}^d$ lie on an $(n-1)$-dimensional flat, $R(d,n) = R(n-1,n)$ holds. Every $n$ points in $\mathbb{R}^{n-1}$ lie on a sphere or on a hyperplane in $\mathbb{R}^{n-1}$, and every sphere can be transformed into a hyperplane (that is, an $(n-2)$-dimensional flat) by an inversion of $\mathbb{R}^{n-1}$. Hence $R(n-1,n) = R(n-2,n)$. □

**Theorem 4.1.** *For $4 \leq n \leq d+2$, $R(d,n) = n(n-3)/2$.*

*Proof.* Every ordered $n$-point-set is inversion-equivalent to an ordered $n$-point-set in which the first three points are collinear with fixed distances $d_{12} = \lambda, d_{23} = 1, d_{13} = \lambda + 1$. Then, as in Lemma 3.1, the Möbius invariants in (1) determine all the distances between the $n$ points, and hence determine all Möbius invariants. The number of Möbius invariants in (1) is equal to $2(n-3) + \binom{n-3}{2} = n(n-3)/2$. Hence, $R(d,n) \leq n(n-3)/2$.

Next, we show that $R(d,n) \geq n(n-3)/2$. By Lemma 4.1, it is enough to show $R(n-1,n) \geq n(n-3)/2$. Let $\alpha = a_1a_2\ldots a_n$ be an ordered $n$-point-set in $\mathbb{R}^{n-1}$ that span an $(n-1)$-dimensional simplex. Then, every small perturbations of the distances $d_{ij}$ in $\alpha$ also determine a simplex in $\mathbb{R}^{n-1}$. Hence there is a neighborhood $U$ of the point $(\ldots, d_{ij}(\alpha), \ldots)$ in $\mathbb{R}^{\binom{n}{2}}$ such that every $(\ldots, d_{ij}, \ldots) \in U$

can be attained by an ordered $n$-point-set in $\mathbb{R}^{n-1}$. Let $x_{ij} = \log d_{ij}$, and $a_{ijkl}(\alpha) = \log[ijkl]_\alpha$. Then

$$\log[ijkl] = \log\left(\frac{d_{ij}d_{kl}}{d_{il}d_{jk}}\right) = x_{ij} + x_{kl} - x_{il} - x_{jk}.$$

Since the value of each $d_{ij}$ can be changed by moving $a_i$ continuously with keeping the values of other distances fixed, the $\binom{n}{2}$ variables $d_{ij}$ are independent in the sense that the value of each $d_{ij}$ is not determined by the values of all other variables. Hence the $\binom{n}{2}$ variables $x_{ij} = \log d_{ij}$ are also independent. Since $a_{ijkl}(\alpha) = \log[ijkl]_\alpha$, if we regard $x_{ij}$s as unknowns, the simultaneous linear equations

$$x_{ij} + x_{kl} - x_{il} - x_{jk} = a_{ijkl}(\alpha),\ 1 \leq i,j,k,l \leq n \tag{2}$$

($i,j,k,l$ are all different) has a solution. Therefore the coefficient matrix and the 'enlarged' coefficient matrix of (2) have the same rank, say, $r$. Let us show that $R(n-1,n) \geq r$.

To see this, suppose that $R(n-1,n) = m < r$. We may suppose that the first $m$ equations in the linear system (2) correspond to the $m$ Möbius invariants. Then the coefficient matrix of the first $m$ equations of (2) must have full rank $m$ (for otherwise, in the first $m$ equations of (2), some equations are obtained from others, which implies that a smaller number of Möbius invariants determine all other Möbius invariants, contradicting $R(n-1,n) = m$). Hence, by adding to these $m$ equations $r-m$ other equations chosen from the remaining equations in (2), we can make a new system of $r$ linear equations that has rank $r$. Then the new system of $r$ linear equations determine an *onto* linear map $f : \mathbb{R}^{\binom{n}{2}} \to \mathbb{R}^r$ by $(\dots, x_{ij}, \dots) \mapsto (\dots, a_{ijkl}, \dots)$. Since every onto linear map is an *open map*, $f$ sends every neighborhood of $(\dots, x_{ij}(\alpha), \dots) \in \mathbb{R}^{\binom{n}{2}}$, to a neighborhood of $(\dots, a_{ijkl}(\alpha), \dots) \in \mathbb{R}^r$. Hence, there is a neighborhood $V$ of $(\dots, a_{ijkl}(\alpha), \dots) \in \mathbb{R}^r$ such that every $(\dots, a_{ijkl}, \dots)$ in $V$ can be attained by an $n$-point-set in $\mathbb{R}^{n-1}$. Thus, there is an $\varepsilon > 0$ such that, in the system of $r$ linear equations, if we replace the constant $a_{ijkl}(\alpha)$ of the last equation (that is, the lastly added equation) with $a_{ijkl}(\alpha) + \varepsilon$, then the system of the $r$ linear equations still have a solution that can be attained by an ordered $n$-point-set in $\mathbb{R}^{n-1}$. This implies that the $m$ Möbius invariants cannot determine the Möbius invariant corresponding to the last equation, a contradiction. Hence, we have $R(n-1,n) \geq r$.

Now, the coefficient vectors of the $n(n-3)/2$ equations

$$\begin{aligned} x_{j1} + x_{23} - x_{j3} - x_{12} &= \alpha_{j123},\ j = 4,5,\ldots,n \\ x_{j2} + x_{13} - x_{j3} - x_{12} &= \alpha_{j213},\ j = 4,5,\ldots,n \\ x_{jk} + x_{12} - x_{j2} - x_{k1} &= \alpha_{jk12},\ \leq j < k \leq n \end{aligned}$$

are linearly independent. This is shown in the following way:

Let $\vec{v}(j123), \vec{v}(j213), \vec{v}(jk12)$ be the corresponding coefficient vectors, and suppose that

$$\sum_{j=4}^{n} s_j \vec{v}(j123) + \sum_{j=4}^{n} t_j \vec{v}(j213) + \sum_{4 \leq j < k \leq n} u_{jk} \vec{v}(jk12) = 0.$$

Since each variable $x_{jk}$ $(4 \leq j < k \leq n)$ appears in just one of the last $\binom{n-3}{2}$ equations, we must have $u_{jk} = 0$. Then, since $x_{j1}$ and $x_{j2}$ are independent variables, we have similarly $s_j = t_j = 0$. Therefore the rank of the coefficient vectors of (2) is at least $n(n-3)/2$. Thus $R(n-1,n) \geq n(n-3)/2$. □

## 5 Planar case

**Lemma 5.1.** *A quadruplet $a_1 a_2 a_3 a_3$ lie on a circle (or a line) if and only if it satisfies* $[4123] - [4213] = \pm 1$.

*Proof.* By Ptolemy's theorem, $a_1 a_2 a_3 a_3$ lie on a circle (or a line) if and only if $d_{41}d_{23} = d_{13}d_{42} + d_{43}d_{12}$ or $d_{42}d_{13} = d_{43}d_{12} + d_{41}d_{23}$. These equalities are equivalent to $[4123] = [4213] + 1$ or $[4213] = 1 + [4123]$, and hence, equivalent to $[4123] - [4213] = \pm 1$. □

**Theorem 5.1.** *For $n \geq 4$, $R(2,n) \leq 3n - 10$.*

*Proof.* Let $\alpha_n = a_1 a_2 \ldots a_n$ denote an ordered $n$-point-set in the plane. Applying inversions, we may suppose that $a_1 a_2 a_3$ is a linear triple with $d_{12} = \lambda, d_{23} = 1$ for a fixed $\lambda$. If $n = 4$, then $F_2 = \{[4123], [4213]\}$ determines $\alpha_4$ up to congruence, and the theorem holds. If $n = 5$, then $F_5 = \{[4123], [4213], [5123], [5213], [5412]\}$ determines $\alpha_5$ up to congruence, and hence determines all Möbius invariants. Since $5 = 3 \cdot 5 - 10$, the theorem holds.

Suppose that there is a set $F_{n-1}$ consisting of at most $3(n-1) - 10$ Möbius invariants such that $(i)$ $F_{n-1}$ determines $\alpha_{n-1}$ up to congruence, and $(ii)$ $F_{n-1}$ contains $[j123], [j213]$, $4 \leq j \leq n-1$. Then for each $4 \leq j \leq n-1$, we can check, by $(ii)$ and Lemma 5.1, whether $a_j$ lies on the line $a_1 a_2$ or not. If there is an $a_j$ $(j \leq n-1)$ that does not lie on the line $a_1 a_2$, then put $F_n = F_{n-1} \cup \{[n123], [n213], [nj12]\}$, otherwise, put $F_n = F_{n-1} \cup \{[n123], [n213]\}$. Then, $F_n$ determines $\alpha_n$ up to congruence, and $F_n$ contains $[j123], [j213]$, $4 \leq j \leq n$. Since $F_n$ contains at most $3n - 10$ members, the proof is done. □

If the first three points in an ordered $n$-point-set $\alpha_n$ $(n \geq 4)$ in $\mathbb{R}^2$ are not collinear, then the three distances $d_{12}, d_{23}, d_{31}$ and the values of all Möbius invariants are not enough to determine all distances in $\alpha_n$ as pointed out in Remark 3.1.

**Lemma 5.2.** *If* $[4123] - [4213] \neq \pm 1$ *holds in an ordered n-point-set* $(n \geq 4)$ *in* $\mathbb{R}^2$*, then the four distances* $d_{12}, d_{23}, d_{31}, d_{14}$ *and the values of all Möbius invariants determine all the distances* $d_{ij}$ *in the n-point-set uniquely.*

*Proof.* The four distances $d_{12}, d_{23}, d_{31}, d_{14}$ and $[4123], [4213]$ determine all distances among the first four points. Hence the lemma is true for $n = 4$. To show the lemma for $n > 4$, let $\alpha, \beta$ be two $n$-point-sets in $\mathbb{R}^2$ that have the same first four points $a_1 a_2 a_3 a_4$. Then it will be enough to show that if $[4123] - [4213] \neq \pm 1$ holds in $a_1 a_2 a_3 a_4$ and $[ijkl]_\alpha = [ijkl]_\beta$ for every $[ijkl]$, then $\alpha = \beta$. Let $\Gamma$ be a circle passing through $a_1, a_2, a_3$. Let $f$ be an inversion with center $p \in \Gamma \setminus (\alpha \cup \beta)$ and some radius $r$. Then, since the first three points in $f(\alpha) \cap f(\beta)$ are collinear, $d_{ij}(f(\alpha)) = d_{ij}(f(\beta))$ for all $ij$ by Lemma 3.1. Hence $f(\alpha)$ and $f(\beta)$ are congruent, and since they have the same first four points that are not collinear, $f(\alpha)$ and $f(\beta)$ coincide with each other. This implies $\alpha = \beta$. □

**Lemma 5.3.** *Let* $n \geq 5$*, and let* $\mathcal{M}$ *be a set of Möbius invariants that determine the values of all Möbius invariants for every ordered n-point-set in* $\mathbb{R}^2$*. Then, each* $i$ $(1 \leq i \leq n)$ *appears in at least three members of* $\mathcal{M}$.

*Proof.* Suppose that $n$ appears in at most two members of $\mathcal{M}$, say, in only $[nabc], [nijk] \in \mathcal{M}$. Let $\alpha = a_1 a_2 \dots a_{n-1}$ be a fixed $(n-1)$-point-set in $\mathbb{R}^2$ such that $a_1 a_2 a_3$ is a linear triple and the first four points are not collinear. Let us extend $\alpha$ to an ordered $n$-point-set in $\mathbb{R}^2$ by adding a point so that $[nabc] = s$ and $[nijk] = t$, for some $s, t > 0$. Then, we may choose any point $x$ as the $n$th point as far as $x$ satisfies that

$$\frac{a_a x}{a_c x} = s \frac{a_a a_b}{a_b a_c}, \quad \frac{a_i x}{a_k x} = t \frac{a_i a_j}{a_j a_k}.$$

These two equations determine two Apollonian circles, and we may assume that $s$ and $t$ are chosen so that these two Apollonian circles intersect in two points. Then, we can get two ordered $n$-point-sets $\beta$ and $\gamma$ as extensions of $\alpha$. Note that each member of $\mathcal{M}$ has the same value at $\beta$ and at $\gamma$. But $\beta$ and $\gamma$ are not congruent since the $n-1$ points in $\alpha$ are not collinear. Therefore, the values of some Möbius invariant ($\notin \mathcal{M}$) takes different values at $\beta$ and at $\gamma$ by Lemma 5.2. This is a contradiction. □

**Corollary 5.1.** $R(2,5) \geq 4$.

*Proof.* For any three Möbius invariants, one of $1,2,3,4,5$ cannot appear in all of the three. □

We have $R(2,4) = 2$ by Theorem 4.1, and $R(2,5) = 4$ or $5$ by Corollary 5.1 and Theorem 5.1. It seems that the set of four Möbius invariants $[4123], [5431], [5142], [5213]$ determine all values of Möbius invariants for any ordered 5-poin-set in $\mathbb{R}^2$, but I could not prove it.

**Problem.** Determine $R(2,5)$.

A set of $n$ points in $\mathbb{R}^d$ are called *generic* if the $dn$ coordinates of the $n$ points are algebraically independent over the rationals. Among the $\binom{n}{2}$ distances between generic $n$ points in $\mathbb{R}^d$, how many distances are necessary to determine all distances? If $n \leq d+2$, then all $\binom{n}{2}$ distances are necessary. If $n \gg d$, then the necessary number would be very small compared with $\binom{n}{2}$ by the dimensional restriction. However, to find the exact minimum necessary number is a difficult problem, see Connelly [4], or Jackson-Jordan-Szabadka [6]. Let us state the problem more precisely.

Let $n \geq 4$ and $G = (V, E)$ denote a graph with vertex set $V = \{1,2,3,\dots,n\}$. Then the problem is to characterize the graph G that satisfies the following condition:

($\diamond$) For any two ordered sets $\alpha$, $\beta$ of generic $n$ points in $\mathbb{R}^d$, $d_{ij}(\alpha) = d_{ij}(\beta)$ $(ij \in E)$ implies that $\alpha$ and $\beta$ are congruent in order-preserving fashion.

Recently it was proved (Connelly [4], Jackson *et al* [6]) that in the planar case $d = 2$, every graph $G = (V, E)$ satisfying the condition ($\diamond$) is obtained from the complete graph $K_4$ by a sequence of Henneberg 1-extension operations and edge additions. The *Henneberg 1-extension operation* on a graph is the following: Remove an edge $xy$ from the graph and add a new vertex $z$ and new edges $zx, zy, zw$, for some vertex $w$ of the graph other than $x, y$. Thus, in the planar case $d = 2$, the minimum cardinality of $E$ is $2n - 2$.

**Theorem 5.2 (Connelly [4] and Jackson** *et al* **[6]).** *For $n \geq 4$, the minimum number of distances necessary to determine all other distances among generic n points in the plane is $2n - 2$.* □

Put $F_4 = \{[4123], [4213]\}$, and for each $n \geq 5$, define $F_n$ inductively in the following way:

$F_5 = (F_4 \setminus \{[4213]\}) \cup \{[5431], [5142], [5213]\}$,
$F_6 = (F_5 \setminus \{[5213]\}) \cup \{[6531], [6152], [6213]\}$,
$F_7 = (F_6 \setminus \{[6213]\}) \cup \{[7631], [7162], [7213]\}$,
...
$F_n = (F_{n-1} \setminus \{[(n-1)213]\}) \cup \{[n(n-1)31], [n1(n-1)2], [n213]\}$.

Then it seems that $F_n$ determines the values of all Möbius invariants for every *generic* ordered $n$-point-set in $\mathbb{R}^2$, though I could not prove it. Note that $|F_n| = 2(n-4) + 3 = 2n - 6$.

Suppose that $\mathcal{M}$ is a minimal set of Möbius invariants that determine the values of all Möbius invariants for every *generic* ordered $n$-point-set $\alpha_n$ in $\mathbb{R}^2$. Since $[4123] - [4213] \neq \pm 1$ always holds for a generic ordered $n$-point-set, the distances $d_{12}, d_{23}, d_{31}, d_{14}$ and $\mathcal{M}$ determine all distances in $\alpha_n$ in $\mathbb{R}^2$ by Lemma 5.2. From the values of the four distances and the values of the members of $\mathcal{M}$, we obtain $4 + |\mathcal{M}|$ equations for unknowns $d_{ij}$. Since the minimum number of distances in $\alpha_n$ that determine all distances in $\alpha_n$ is $2n-2$ by Theorem 5.2, it would be natural to expect that the number of equations $4+|\mathcal{M}|$ is at least $2n-2$. This suggests that $|\mathcal{M}| \geq 2n-6$.

**Conjecture.** For a generic ordered $n$-point-set ($n \geq 4$) in $\mathbb{R}^2$, the minimum number of Möbius invariants necessary to determine the values of all Möbius invariants is equal to $2n-6$.

# 6 Invariant fractions

Möbius invariant is generalized as follows. By a *segment*, we mean a distance represented by a pair of points. A *segment-product* is a product of a number of segments. For example, $\boldsymbol{ab} \cdot \boldsymbol{cd} \cdot \boldsymbol{ae}$ is a segment-product.

**Theorem 6.1.** *A fraction of segment-products is invariant under inversions if and only if the following condition holds:*

($*$) *For each point-symbol, the number of times it appears in the numerator is equal to the number of times it appears in the denominator.*

For example, the fraction $(\boldsymbol{ab} \cdot \boldsymbol{ac} \cdot \boldsymbol{de})/(\boldsymbol{ad} \cdot \boldsymbol{ae} \cdot \boldsymbol{bc})$ is invariant under inversions, but the fraction $(\boldsymbol{ab} \cdot \boldsymbol{cd})/(\boldsymbol{bc} \cdot \boldsymbol{de})$ is not.

*Proof.* Let us show that ($*$) implies that the fraction is invariant under inversions. Instead the general case, we consider, for example, the fraction $(\boldsymbol{ab} \cdot \boldsymbol{ac} \cdot \boldsymbol{de})/(\boldsymbol{ad} \cdot \boldsymbol{ae} \cdot \boldsymbol{bc})$. Let $\boldsymbol{a}', \ldots, \boldsymbol{e}'$ be the images of $\boldsymbol{a}, \ldots, \boldsymbol{e}$ by an inversion with center $\boldsymbol{p}$ and radius $r$. We show that

$$\frac{\boldsymbol{ab} \cdot \boldsymbol{ac} \cdot \boldsymbol{de}}{\boldsymbol{ad} \cdot \boldsymbol{ae} \cdot \boldsymbol{bc}} = \frac{\boldsymbol{a'b'} \cdot \boldsymbol{a'c'} \cdot \boldsymbol{d'e'}}{\boldsymbol{a'd'} \cdot \boldsymbol{a'e'} \cdot \boldsymbol{b'c'}}. \tag{3}$$

By Lemma 2.1, we have

$$\boldsymbol{a'b'} = \frac{r^2\boldsymbol{ab}}{\boldsymbol{pa} \cdot \boldsymbol{pb}},\ \boldsymbol{a'c'} = \frac{r^2\boldsymbol{ac}}{\boldsymbol{pa} \cdot \boldsymbol{pc}},\ \boldsymbol{d'e'} = \frac{r^2\boldsymbol{de}}{\boldsymbol{pd} \cdot \boldsymbol{pe}}$$
$$\boldsymbol{a'd'} = \frac{r^2\boldsymbol{ad}}{\boldsymbol{pa} \cdot \boldsymbol{pd}},\ \boldsymbol{a'e'} = \frac{r^2\boldsymbol{ae}}{\boldsymbol{pa} \cdot \boldsymbol{pe}},\ \boldsymbol{b'c'} = \frac{r^2\boldsymbol{bc}}{\boldsymbol{pb} \cdot \boldsymbol{pc}}.$$

If these are substituted in the right hand side fraction of (3), then $1/\boldsymbol{pa}$ will appear in the numerator the same number of times as $\boldsymbol{a}'$

appears in the numerator, and also, $1/pa$ will appear in the denominator the same number of times as $a'$ appears in the denominator. Similar things will happen for $1/pb, 1/pc, 1/pd, 1/pe$ and $r^2$. Since $(*)$ holds for this fraction, all $1/pa, \ldots, 1/pe$ and $r^2$ will be cancelled out, and we get the equality (3).

To see the *only if* part, suppose that for some point-symbol, say, $x$, the number of times it appears in the numerator is not equal to the number of times it appears in the denominator. Then we cannot cancel out $1/px$, and the fraction would not be invariant under inversions. □

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