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# Inversions and Möbius invariants

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## Abstract

Two  $n$ -point-sets in Euclidean space are said to be inversion-equivalent if one set can be transformed into the other set by applying inversions of the space. All 3-point-sets are inversion-equivalent to each other. For each four points  $x, y, z, w$  in an  $n$ -point-set,  $n \geq 4$ , the ratio  $(xy \cdot zw)/(xw \cdot yz)$  is invariant under inversions, which is called a Möbius invariant of the  $n$ -point-set. We prove that for  $4 \leq n \leq d + 2$ , the minimum number of Möbius invariants necessary to determine all Möbius invariants for every  $n$ -point-set in Euclidean  $d$ -space is equal to  $n(n - 3)/2$ , and discuss the case of planar  $n$ -point-sets in some detail. We also characterize those fractional functions that are invariant under inversions.

## 1 Introduction

Let  $S$  be a sphere in the  $d$ -dimensional Euclidean space  $\mathbb{R}^d$  with center  $p$  and radius  $r$ . The *inversion* of  $\mathbb{R}^d$  with respect to  $S$  is the transformation of  $\mathbb{R}^d$  that sends each point  $x$  ( $\neq p$ ) to a point  $x'$  on the ray  $\overrightarrow{px}$  such that  $px \cdot px' = r^2$ , where  $px$  denotes the distance between  $p$  and  $x$ . The point  $p$  and the radius  $r$  are called the *center* and the *radius* of the inversion, respectively. Note that in an inversion, the image of its center is not defined. One of the typical features of an inversion is that it transforms a sphere into another sphere, with regarding a hyperplane as a sphere of infinite radius. For more about inversions, see, e.g. [2,5,8].

In this paper, we consider to transform a finite point-set by inversions. Note that *an inversion can be applied to a point-set only when*

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its center does not belong to the point-set, since the image of the center of an inversion is not defined. (Usually, to avoid such restriction, a point at infinity is added to  $\mathbb{R}^d$  as to be the image of the center for every inversion. Another less usual way is to define the image of the center of an inversion to be the center itself. In [7], transformations of a finite point-set by ‘center-fixing-inversions’ with centers in the point-set are investigated.)

A pair of  $n$ -point-sets are called *inversion-equivalent* if one set can be transformed into the other set by applying a series of inversions. This relation is clearly an equivalence relation. An *ordered  $n$ -point-set* is an  $n$ -point-set whose points are ordered, which is denoted by the juxtaposition of the  $n$  points in order like  $a_1 a_2 a_3 \dots a_n$  (the same notation as used for a polygon or a polygonal curve). Two ordered  $n$ -point-sets are called *inversion-equivalent* if they are inversion-equivalent with keeping the order. It turns out that all ordered triples are mutually inversion-equivalent.

For a quadruplet (that is, an ordered 4-point-set)  $abcd$ , let us define  $[abcd]$  by

$$[abcd] = (ab \cdot cd) / (ad \cdot bc),$$

which is called a *Möbius invariant* of the quadruplet. This is indeed invariant under any inversion (Corollary 2.1, see also [5, p.92], [3, p.310]). Hence, for example, a quadruplet  $abcd$  with  $[abcd] \neq 1$  is never inversion-equivalent to the vertex set of a regular tetrahedron.

For a quadruplet  $a_i a_j a_k a_l$  taken from an ordered  $n$ -point-set  $\alpha = a_1 a_2 \dots a_n$ ,  $n \geq 4$ , the Möbius invariant  $[a_i a_j a_k a_l]$  is simply denoted by  $[ijkl]$ , and its ‘value’ at  $\alpha$  is denoted by  $[ijkl]_\alpha$ . Let us call

$$[ijkl] \quad (i, j, k, l \text{ are all different, } 1 \leq i, j, k, l \leq n)$$

the Möbius invariants for an ordered  $n$ -point-set. We show the following:

- For  $n \geq 4$ , two ordered  $n$ -point-sets  $\alpha, \beta$  are inversion-equivalent if and only if  $[ijkl]_\alpha = [ijkl]_\beta$  holds for every  $[ijkl]$ .

Applying this result, it is proved that every quadruplet is inversion equivalent to the vertex set of a (possibly degenerate) parallelogram.

Notice that  $[ijkl] = [klij] = [jilk] = [lkji]$ . Hence there are at most  $6 \binom{n}{4}$  distinct Möbius invariants for an ordered  $n$ -point-set.

They are not independent as ‘variables’. Rather a few of them determine all others. Let  $R(d, n)$  denote the minimum cardinality of a set of Möbius invariants whose values determine the values of all Möbius invariants for every ordered  $n$ -point-set in  $\mathbb{R}^d$ . We prove the following.

- $R(d, n) = n(n - 3)/2$  for  $4 \leq n \leq d + 2$ .

If  $d \ll n$  then  $R(d, n)$  would be much smaller than  $n(n - 3)/2$  by the dimensional restriction. However, it is not easy to determine the value of  $R(d, n)$  even in the planar case  $d = 2$ , where the bound I could prove is  $R(2, n) \leq 3n - 10$ . In Section 5, we will discuss the planar case in some detail.

Besides Möbius invariants, there are many fractional functions, such as  $(ab \cdot ac \cdot de)/(ad \cdot ae \cdot bc)$  and  $(ab \cdot cd \cdot ef)/(bc \cdot de \cdot fa)$  that are invariant under inversions. They are characterized in the following way:

- *A fractional function is invariant under inversions if and only if, for each point-symbol, the number of times it appears in the numerator is equal to the number of times it appears in the denominator.*

## 2 A few basic facts on inversions

**Lemma 2.1.** *Let  $a', b'$  denote the images of  $a, b$  under the inversion with center  $p$  and radius  $r$ . Then  $a'b' = (r^2 \cdot ab)/(pa \cdot pb)$ .*

*Proof.* Since  $pa' \cdot pa = r^2 = pb' \cdot pb$  and  $\angle apb = \angle a'pb'$ , the two (possibly degenerate) triangles  $pa'b'$  and  $pba$  are similar. Hence  $a'b'/ba = pa'/pb = r^2/(pa \cdot pb)$ , and the lemma follows.  $\square$

The following corollaries follow from this by simple calculations.

**Corollary 2.1.** *Suppose an inversion sends  $a, b, c, d$  to  $a', b', c', d'$ , respectively. Then  $[a'b'c'd'] = [abcd]$ .*  $\square$

**Corollary 2.2.** *Let  $f_1, f_2$  be the inversions with the same center  $p$  and radii  $r_1, r_2$ , respectively. Then the composition  $f_2 \circ f_1$  is a homothety with center  $p$  and similitude ratio  $(r_2/r_1)^2$ .*  $\square$

**Lemma 2.2.** *Let  $H$  be a hyperplane in  $\mathbb{R}^d$ ,  $p \in \mathbb{R}^d \setminus H$ , and  $q \in \mathbb{R}^d$  be the point that is symmetric to  $p$  with respect to  $H$ . Let  $g_1$  be the inversion of  $\mathbb{R}^d$  with respect to the sphere with center  $p$  and radius  $pq$ , and let  $g_2$  be the inversion of  $\mathbb{R}^d$  with center  $q$  and radius  $pq$ . Then the composition  $f := g_1 \circ g_2 \circ g_1$  is the reflection of  $\mathbb{R}^d$  with respect to the hyperplane  $H$ .*

*Proof.* It will be enough to consider the plane case  $d = 2$ . We may suppose that  $p = (-1, 0)$ ,  $q = (1, 0)$  and  $H$  is the  $y$ -axis. Note that  $g_1(q) = q$ ,  $g_2(p) = p$ . Let  $C_q$  be the circle with center  $q$  and radius  $pq$ . For a point  $u$  on the  $y$ -axis, let  $v$  be the intersection of the ray  $\overrightarrow{pu}$  and  $C_q$  other than  $p$ . Then, since  $g_1$  sends the  $y$ -axis to  $C_q$ , we have  $g_1(u) = v$ . Since  $g_2(v) = v$ , we have  $f(u) = u$ . Thus,  $f$  fixes all points on the  $y$ -axis. It is also clear that  $f$  sends the  $x$ -axis to itself.

For any line  $\ell$ ,  $g_1(\ell)$  is either a line passing through  $p$  or a circle passing through  $p$ , and hence,  $g_2(g_1(\ell))$  is either a circle passing through  $p$  or a line passing through  $p$ . Therefore,  $g_1(g_2(g_1(\ell)))$  is always a line. Thus,  $f$  sends every line to a line, and sends every pair of parallel lines to a pair of parallel lines.

Now, since  $g(x, 0) = (\frac{3-x}{x+1}, 0)$ ,  $g_2(x, 0) = (\frac{x+3}{x-1}, 0)$  as easily verified,  $f(x, 0) = (-x, 0)$  follows by a simple calculation. Then, for a given point  $(x_0, y_0)$ , the line " $x = x_0$ " (which is parallel to the  $y$ -axis) is sent to the line passing through  $(-x_0, 0)$  and parallel to the  $y$ -axis, that is, the line " $x = -x_0$ ". Similarly, the line " $y = y_0$ " is sent to itself by  $f$ . Therefore the intersection  $(x_0, y_0)$  of the two lines  $x = x_0$  and  $y = y_0$  is sent to  $(-x_0, y_0)$ , that is,  $f(x_0, y_0) = (-x_0, y_0)$ . This proves the lemma.  $\square$

Note that in Corollary 2.2, the center of a homothety can be chosen independently from the similitude ratio, and in Lemma 2.2, the point  $p \in \mathbb{R}^d \setminus H$  can be chosen arbitrarily. Since all reflections of  $\mathbb{R}^d$  generate all isometries of  $\mathbb{R}^d$ , we have the following corollary from Corollary 2.2 and Lemma 2.2.

**Corollary 2.3.** *If two  $n$ -point-sets are similar, then they are inversion-equivalent.*  $\square$

**Lemma 2.3.** *Let  $\sigma$  be a finite point-set containing three points  $a, b, c$ . Then, for every  $\lambda > \lambda_0$  (where  $\lambda_0$  is a constant depending on  $\sigma$ ), there exists an inversion with center  $p \notin \sigma$  that transforms  $a, b, c$  into  $a', b', c'$  such that  $a'b' = \lambda$ ,  $b'c' = 1$ ,  $a'c' = 1 + \lambda$ .*

*Proof.* Let  $\Gamma$  be the circle (or line) passing through  $a, b, c$ . Let  $p$  be a point on  $\Gamma \setminus (\widehat{abc})$ . Let  $f$  be an inversion with center  $p$  and some radius  $r$ , and let  $a' = f(a), b' = f(b), c' = f(c)$ . Then,  $a', b', c'$  are collinear in this order, and by Lemma 2.1, we have

$$\frac{a'b'}{b'c'} = \left( \frac{r^2 \cdot ab}{pa \cdot pb} \right) \left( \frac{pb \cdot pc}{r^2 \cdot bc} \right) = \frac{ab \cdot pc}{bc \cdot pa}.$$

Let  $\varepsilon$  be the distance from  $a$  to the nearest point in  $\sigma - \{a\}$ . Then,  $\varepsilon > 0$ . Let  $\lambda_0 = (ab/bc)(ac/\varepsilon + 1)$ . If  $pa = \varepsilon$ , we have

$$\frac{ab \cdot pc}{bc \cdot pa} \leq \frac{ab(pa + ac)}{bc \cdot pa} = \frac{ab}{bc} \left( \frac{ac}{\varepsilon} + 1 \right) = \lambda_0.$$

Since  $pc/pa$  continuously tends to infinity as  $pa$  continuously tends to 0,  $(ab \cdot pc)/(bc \cdot pa)$  can take every value  $\lambda \geq \lambda_0$ . Thus, for every  $\lambda > \lambda_0$ , we can choose  $p$  on  $\Gamma$  so that  $pa < \varepsilon$ ,  $a'b'/b'c' = (ab/bc)(pc/pa) = \lambda$ , and we can choose  $r > 0$  so that  $b'c' = 1$ .  $\square$

An ordered triple  $abc$  is called a *linear triple* if  $a, b, c$  are collinear in this order.

**Corollary 2.4.** *Every pair of ordered triples are inversion-equivalent.*  $\square$

### 3 Möbius invariants

For three points  $a, b, p \in \mathbb{R}^d$  ( $d \geq 2$ ), the locus of the points  $x$  satisfying  $ax/bx = ap/bp$  is called the *Apollonian sphere* (*Apollonian circle* if  $d = 2$ ) determined by  $ab$  and  $p$ , which is denoted by  $A(ab, p)$ . If  $ap \neq bp$ , then  $A(ab, p)$  is indeed a sphere with center at the extension of the line segment  $ab$  (beyond  $a$  or  $b$ ), but if  $ap = bp$ , then  $A(ab, p)$  is a hyperplane that bisects the line segment  $ab$  perpendicularly. Since

$$A(ab, p) \ni q \Leftrightarrow \frac{ap}{bp} = \frac{aq}{bq} \Leftrightarrow \frac{pa}{qa} = \frac{pb}{qb} \Leftrightarrow A(pq, a) \ni b$$

holds,  $q \in A(ab, p) \cap A(ac, p)$  implies that  $\{a, b, c\} \subset A(pq, a)$ . Therefore, if  $abc$  is a linear triple, then  $A(ab, p)$  and  $A(ac, p)$  are different spheres. For more about Apollonian circles, see Coxeter [5].

For an ordered  $n$ -point-set  $\alpha$ , we denote the distance between the  $i$ -th point and the  $j$ -th point by  $d_{ij}$  or  $d_{ij}(\alpha)$ .

**Lemma 3.1.** *Let  $n \geq 4$  and  $\alpha = a_1 a_2 \dots a_n$  be an ordered  $n$ -point-set in which  $a_1 a_2 a_3$  is a linear triple. Then the two distances  $d_{12}$ ,  $d_{23}$  and the values of the Möbius invariants*

$$[j123], [j213], j = 4, 5, \dots, n, \text{ and } [jk12], 4 \leq j < k \leq n, \quad (1)$$

*determine all the distances  $d_{ij}$  in  $\alpha$ .*

*Proof.* Suppose that three points  $a_1, a_2, a_3$  are fixed so that  $d_{12} = s, d_{23} = t, d_{13} = s + t$ . Since  $[4213]_\alpha = (a_4 a_2 \cdot (s + t)) / (a_4 a_3 \cdot s)$ , we have  $a_2 a_4 / a_3 a_4 = [4213]_\alpha \cdot s / (s + t)$ . Hence the value of  $[4213]$  determines the Apollonian sphere  $A(a_2 a_3, a_4)$ . Similarly, the value of  $[4123]$  determines the Apollonian sphere  $A(a_1 a_3, a_4)$ . Since  $a_1 a_2 a_3$  is a linear triple, these two Apollonian spheres are different with centers on the line  $a_1 a_2$ . Hence, all intersection points of the two Apollonian spheres are at the same distance from the line  $a_1 a_2$ , and hence each of the distances  $d_{41}, d_{42}, d_{43}$  are uniquely determined. Similarly, for each  $4 \leq j \leq n$ , the values of  $[j213]$  and  $[j123]$  determine the distances  $d_{j1}, d_{j2}, d_{j3}$  uniquely. Then, the values of  $[jk12]$  determine the distances  $d_{jk}$  for  $4 \leq j < k \leq n$ .  $\square$

Let  $\alpha = a_1 \dots a_n$  be an ordered  $n$ -point-set in the plane  $\mathbb{R}^2$ ,  $n \geq 4$ . Suppose  $a_1 = (-s, 0), a_2 = (0, 0), a_3 = (t, 0)$  and  $a_4 = (u, v), uv \neq 0$ . Let  $f$  be the inversion with center  $(0, w)$ , radius  $r$ , and let  $g$  be the inversion with the center  $(0, -w)$  and radius  $r$ , where  $w \neq 0$ . Then the triples  $f(a_1 a_2 a_3), g(a_1 a_2 a_3)$  are congruent, and every Möbius invariant takes the same value at  $f(\alpha)$  and at  $g(\alpha)$ . But  $f(\alpha)$  and  $g(\alpha)$  are not congruent. Let us state this fact as a remark.

**Remark 3.1.** *If the first three points of an ordered  $n$ -point-set ( $n \geq 4$ ) in  $\mathbb{R}^2$  are not collinear, then the three distances  $d_{12}, d_{23}, d_{31}$  and the values of all Möbius invariants are not enough to determine all distances among the  $n$  points.*

**Theorem 3.1.** *For every  $n \geq 4$ , a pair of ordered  $n$ -point-sets  $\alpha$  and  $\beta$  in  $\mathbb{R}^d$  are inversion-equivalent if and only if  $[ijkl]_\alpha = [ijkl]_\beta$  holds for every  $[ijkl]$ .*

*Proof.* The *if* part is obvious since Möbius invariants are invariant under inversions.

Let  $\alpha = a_1 \dots a_n$ ,  $\beta = b_1 \dots b_n$ . By Lemma 2.3, we can apply inversions to  $\alpha$  and  $\beta$  independently, so that  $a_1 a_2 a_3$  and  $b_1 b_2 b_3$  become congruent linear triples. Hence, we may assume from the first that  $a_1 a_2 a_3$  and  $b_1 b_2 b_3$  are congruent linear triples,  $d_{12}(\alpha) = d_{12}(\beta) = s$ ,  $d_{23}(\alpha) = d_{23}(\beta) = t$ . Since  $[ijkl]_\alpha = [ijkl]_\beta$  for every  $[ijkl]$ , it follows that  $d_{ij}(\alpha) = d_{ij}(\beta)$  holds for all  $i, j$  by Lemma 3.1. Hence the two ordered  $n$ -point-sets are congruent to each other. Therefore, they are inversion-equivalent by Corollary 2.3.  $\square$

**Theorem 3.2.** *Every quadruplet in  $\mathbb{R}^2$  is inversion-equivalent to the vertex set of a (possibly degenerate) parallelogram.*

*Proof.* Let  $abcd$  be a quadruplet, and put  $a = [abcd]$ ,  $b = [acbd]$ . (Then,  $[abdc] = a/b$ ,  $[acdb] = b/a$ ,  $[adbc] = 1/b$ ,  $[adcb] = 1/a$  as easily verified.) By generalized Ptolemy's inequality (see, e.g. [1]) we have

$$\begin{aligned} ab \cdot cd &\leq ac \cdot bd + bc \cdot ad, \\ ac \cdot bd &\leq ab \cdot cd + ad \cdot bc, \\ ad \cdot bc &\leq ab \cdot cd + ac \cdot bd. \end{aligned}$$

Since  $[abcd] = (ab \cdot cd)/(bc \cdot ad)$  and  $[acbd] = (ad \cdot bd)/(ad \cdot bc)$ , it follows that  $a \leq b + 1$ ,  $b \leq a + 1$  and  $1 \leq a + b$ . Therefore  $|a - 1| \leq b \leq a + 1$ . Let

$$p = \pm \frac{1}{2} \sqrt{(a+1)^2 - b^2}, \quad q = \frac{1}{2} \sqrt{b^2 - (a-1)^2}$$

(we may choose either sign  $\pm$  for  $p$ ) and put

$$a' = (0, 0), \quad b' = (p, q), \quad c' = (p+1, q), \quad d' = (1, 0) \in \mathbb{R}^2.$$

Then  $a'b'c'd'$  is a parallelogram, and

$$[a'b'c'd'] = a, \quad [a'c'b'd'] = b.$$

Hence  $abcd$  and  $a'b'c'd'$  are inversion-equivalent.  $\square$

**Corollary 3.1.** *The vertex-sets of two parallelograms are not inversion-equivalent unless the two parallelograms are similar to each other.*  $\square$

Since a parallelogram  $abcd$  is a rhombus if and only if  $[abcd] = 1$ , we have the following.

**Corollary 3.2.** *A quadruplet  $abcd$  is inversion-equivalent to the vertex set of a rhombus if and only if  $[abcd] = 1$ .*  $\square$



## 4 Number of necessary invariants

For a quadruplet  $a_1 a_2 a_3 a_4$  in  $\mathbb{R}^d$  ( $d > 0$ ), let  $x = d_{12}$ ,  $y = d_{23}$ ,  $z = d_{34}$ ,  $w = d_{41}$ . Then  $[1234] = xz/(yw)$ , which is the ratio of the two products of opposite edges in the (possibly self-intersecting) quadrilateral. By changing the order of the vertices cyclically, we get two distinct Möbius invariants, namely,  $[1234] = [3412] = xz/(yw)$  and  $[2341] = [4123] = yw/(xz) = [1234]^{-1}$ . Since four points produce three distinct quadrilaterals, it follows that there are  $6 \times \binom{n}{4}$  distinct Möbius invariants for  $n$  points. Since  $[2341] = [1234]^{-1}$ , half of the  $6\binom{n}{4}$  Möbius invariants are reciprocals of the other half. So,  $6\binom{n}{4}$  Möbius invariants are determined by  $3\binom{n}{4}$  members, probably much fewer members. Recall that  $R(d, n)$  denotes the minimum cardinality of a set of Möbius invariants whose values determine the values of all Möbius invariants for every ordered  $n$ -point-set in  $\mathbb{R}^d$ .

**Lemma 4.1.** For  $d > n - 2 \geq 2$ ,  $R(d, n) = R(n - 2, n)$ .

*Proof.* Since every  $n$  points in  $\mathbb{R}^d$  lie on an  $(n - 1)$ -dimensional flat,  $R(d, n) = R(n - 1, n)$  holds. Every  $n$  points in  $\mathbb{R}^{n-1}$  lie on a sphere or on a hyperplane in  $\mathbb{R}^{n-1}$ , and every sphere can be transformed into a hyperplane (that is, an  $(n - 2)$ -dimensional flat) by an inversion of  $\mathbb{R}^{n-1}$ . Hence  $R(n - 1, n) = R(n - 2, n)$ .  $\square$

**Theorem 4.1.** For  $4 \leq n \leq d + 2$ ,  $R(d, n) = n(n - 3)/2$ .

*Proof.* Every ordered  $n$ -point-set is inversion-equivalent to an ordered  $n$ -point-set in which the first three points are collinear with fixed distances  $d_{12} = \lambda$ ,  $d_{23} = 1$ ,  $d_{13} = \lambda + 1$ . Then, as in Lemma 3.1, the Möbius invariants in (1) determine all the distances between the  $n$  points, and hence determine all Möbius invariants. The number of Möbius invariants in (1) is equal to  $2(n - 3) + \binom{n-3}{2} = n(n - 3)/2$ . Hence,  $R(d, n) \leq n(n - 3)/2$ .

Next, we show that  $R(d, n) \geq n(n - 3)/2$ . By Lemma 4.1, it is enough to show  $R(n - 1, n) \geq n(n - 3)/2$ . Let  $\alpha = a_1 a_2 \dots a_n$  be an ordered  $n$ -point-set in  $\mathbb{R}^{n-1}$  that span an  $(n - 1)$ -dimensional simplex. Then, every small perturbations of the distances  $d_{ij}$  in  $\alpha$  also determine a simplex in  $\mathbb{R}^{n-1}$ . Hence there is a neighborhood  $U$  of the point  $(\dots, d_{ij}(\alpha), \dots)$  in  $\mathbb{R}^{\binom{n}{2}}$  such that every  $(\dots, d_{ij}, \dots) \in U$

can be attained by an ordered  $n$ -point-set in  $\mathbb{R}^{n-1}$ . Let  $x_{ij} = \log d_{ij}$ , and  $a_{ijkl}(\alpha) = \log [ijkl]_\alpha$ . Then

$$\log [ijkl] = \log \left( \frac{d_{ij}d_{kl}}{d_{il}d_{jk}} \right) = x_{ij} + x_{kl} - x_{il} - x_{jk}.$$

Since the value of each  $d_{ij}$  can be changed by moving  $a_i$  continuously with keeping the values of other distances fixed, the  $\binom{n}{2}$  variables  $d_{ij}$  are independent in the sense that the value of each  $d_{ij}$  is not determined by the values of all other variables. Hence the  $\binom{n}{2}$  variables  $x_{ij} = \log d_{ij}$  are also independent. Since  $a_{ijkl}(\alpha) = \log [ijkl]_\alpha$ , if we regard  $x_{ij}$ s as unknowns, the simultaneous linear equations

$$x_{ij} + x_{kl} - x_{il} - x_{jk} = a_{ijkl}(\alpha), \quad 1 \leq i, j, k, l \leq n \quad (2)$$

( $i, j, k, l$  are all different) has a solution. Therefore the coefficient matrix and the 'enlarged' coefficient matrix of (2) have the same rank, say,  $r$ . Let us show that  $R(n-1, n) \geq r$ .

To see this, suppose that  $R(n-1, n) = m < r$ . We may suppose that the first  $m$  equations in the linear system (2) correspond to the  $m$  Möbius invariants. Then the coefficient matrix of the first  $m$  equations of (2) must have full rank  $m$  (for otherwise, in the first  $m$  equations of (2), some equations are obtained from others, which implies that a smaller number of Möbius invariants determine all other Möbius invariants, contradicting  $R(n-1, n) = m$ ). Hence, by adding to these  $m$  equations  $r - m$  other equations chosen from the remaining equations in (2), we can make a new system of  $r$  linear equations that has rank  $r$ . Then the new system of  $r$  linear equations determine an *onto* linear map  $f : \mathbb{R}^{\binom{n}{2}} \rightarrow \mathbb{R}^r$  by  $(\dots, x_{ij}, \dots) \mapsto (\dots, a_{ijkl}, \dots)$ . Since every onto linear map is an *open map*,  $f$  sends every neighborhood of  $(\dots, x_{ij}(\alpha), \dots) \in \mathbb{R}^{\binom{n}{2}}$ , to a neighborhood of  $(\dots, a_{ijkl}(\alpha), \dots) \in \mathbb{R}^r$ . Hence, there is a neighborhood  $V$  of  $(\dots, a_{ijkl}(\alpha), \dots) \in \mathbb{R}^r$  such that every  $(\dots, a_{ijkl}, \dots)$  in  $V$  can be attained by an  $n$ -point-set in  $\mathbb{R}^{n-1}$ . Thus, there is an  $\varepsilon > 0$  such that, in the system of  $r$  linear equations, if we replace the constant  $a_{ijkl}(\alpha)$  of the last equation (that is, the lastly added equation) with  $a_{ijkl}(\alpha) + \varepsilon$ , then the system of the  $r$  linear equations still have a solution that can be attained by an ordered  $n$ -point-set in  $\mathbb{R}^{n-1}$ . This

implies that the  $m$  Möbius invariants cannot determine the Möbius invariant corresponding to the last equation, a contradiction. Hence, we have  $R(n-1, n) \geq r$ .

Now, the coefficient vectors of the  $n(n-3)/2$  equations

$$\begin{aligned} x_{j1} + x_{23} - x_{j3} - x_{12} &= \alpha_{j123}, \quad j = 4, 5, \dots, n \\ x_{j2} + x_{13} - x_{j3} - x_{12} &= \alpha_{j213}, \quad j = 4, 5, \dots, n \\ x_{jk} + x_{12} - x_{j2} - x_{k1} &= \alpha_{jk12}, \quad 4 \leq j < k \leq n \end{aligned}$$

are linearly independent. This is shown in the following way:

Let  $\vec{v}(j123), \vec{v}(j213), \vec{v}(jk12)$  be the corresponding coefficient vectors, and suppose that

$$\sum_{j=4}^n s_j \vec{v}(j123) + \sum_{j=4}^n t_j \vec{v}(j213) + \sum_{4 \leq j < k \leq n} u_{jk} \vec{v}(jk12) = 0.$$

Since each variable  $x_{jk}$  ( $4 \leq j < k \leq n$ ) appears in just one of the last  $\binom{n-3}{2}$  equations, we must have  $u_{jk} = 0$ . Then, since  $x_{j1}$  and  $x_{j2}$  are independent variables, we have similarly  $s_j = t_j = 0$ . Therefore the rank of the coefficient vectors of (2) is at least  $n(n-3)/2$ . Thus  $R(n-1, n) \geq n(n-3)/2$ .  $\square$

## 5 Planar case

**Lemma 5.1.** *A quadruplet  $a_1 a_2 a_3 a_3$  lie on a circle (or a line) if and only if it satisfies  $[4123] - [4213] = \pm 1$ .*

*Proof.* By Ptolemy's theorem,  $a_1 a_2 a_3 a_3$  lie on a circle (or a line) if and only if  $d_{41}d_{23} = d_{13}d_{42} + d_{43}d_{12}$  or  $d_{42}d_{13} = d_{43}d_{12} + d_{41}d_{23}$ . These equalities are equivalent to  $[4123] = [4213] + 1$  or  $[4213] = 1 + [4123]$ , and hence, equivalent to  $[4123] - [4213] = \pm 1$ .  $\square$

**Theorem 5.1.** *For  $n \geq 4$ ,  $R(2, n) \leq 3n - 10$ .*

*Proof.* Let  $\alpha_n = a_1 a_2 \dots a_n$  denote an ordered  $n$ -point-set in the plane. Applying inversions, we may suppose that  $a_1 a_2 a_3$  is a linear triple with  $d_{12} = \lambda, d_{23} = 1$  for a fixed  $\lambda$ . If  $n = 4$ , then  $F_2 = \{[4123], [4213]\}$  determines  $\alpha_4$  up to congruence, and the theorem holds. If  $n = 5$ ,

then  $F_5 = \{[4123], [4213], [5123], [5213], [5412]\}$  determines  $\alpha_5$  up to congruence, and hence determines all Möbius invariants. Since  $5 = 3 \cdot 5 - 10$ , the theorem holds.

Suppose that there is a set  $F_{n-1}$  consisting of at most  $3(n-1) - 10$  Möbius invariants such that (i)  $F_{n-1}$  determines  $\alpha_{n-1}$  up to congruence, and (ii)  $F_{n-1}$  contains  $[j123], [j213]$ ,  $4 \leq j \leq n-1$ . Then for each  $4 \leq j \leq n-1$ , we can check, by (ii) and Lemma 5.1, whether  $a_j$  lies on the line  $a_1a_2$  or not. If there is an  $a_j$  ( $j \leq n-1$ ) that does not lie on the line  $a_1a_2$ , then put  $F_n = F_{n-1} \cup \{[n123], [n213], [nj12]\}$ , otherwise, put  $F_n = F_{n-1} \cup \{[n123], [n213]\}$ . Then,  $F_n$  determines  $\alpha_n$  up to congruence, and  $F_n$  contains  $[j123], [j213]$ ,  $4 \leq j \leq n$ . Since  $F_n$  contains at most  $3n - 10$  members, the proof is done.  $\square$

If the first three points in an ordered  $n$ -point-set  $\alpha_n$  ( $n \geq 4$ ) in  $\mathbb{R}^2$  are not collinear, then the three distances  $d_{12}, d_{23}, d_{31}$  and the values of all Möbius invariants are not enough to determine all distances in  $\alpha_n$  as pointed out in Remark 3.1.

**Lemma 5.2.** *If  $[4123] - [4213] \neq \pm 1$  holds in an ordered  $n$ -point-set ( $n \geq 4$ ) in  $\mathbb{R}^2$ , then the four distances  $d_{12}, d_{23}, d_{31}, d_{14}$  and the values of all Möbius invariants determine all the distances  $d_{ij}$  in the  $n$ -point-set uniquely.*

*Proof.* The four distances  $d_{12}, d_{23}, d_{31}, d_{14}$  and  $[4123], [4213]$  determine all distances among the first four points. Hence the lemma is true for  $n = 4$ . To show the lemma for  $n > 4$ , let  $\alpha, \beta$  be two  $n$ -point-sets in  $\mathbb{R}^2$  that have the same first four points  $a_1a_2a_3a_4$ . Then it will be enough to show that if  $[4123] - [4213] \neq \pm 1$  holds in  $a_1a_2a_3a_4$  and  $[ijkl]_\alpha = [ijkl]_\beta$  for every  $[ijkl]$ , then  $\alpha = \beta$ . Let  $\Gamma$  be a circle passing through  $a_1, a_2, a_3$ . Let  $f$  be an inversion with center  $p \in \Gamma \setminus (\alpha \cup \beta)$  and some radius  $r$ . Then, since the first three points in  $f(\alpha) \cap f(\beta)$  are collinear,  $d_{ij}(f(\alpha)) = d_{ij}(f(\beta))$  for all  $ij$  by Lemma 3.1. Hence  $f(\alpha)$  and  $f(\beta)$  are congruent, and since they have the same first four points that are not collinear,  $f(\alpha)$  and  $f(\beta)$  coincide with each other. This implies  $\alpha = \beta$ .  $\square$

**Lemma 5.3.** *Let  $n \geq 5$ , and let  $\mathcal{M}$  be a set of Möbius invariants that determine the values of all Möbius invariants for every ordered  $n$ -point-set in  $\mathbb{R}^2$ . Then, each  $i$  ( $1 \leq i \leq n$ ) appears in at least three members of  $\mathcal{M}$ .*

*Proof.* Suppose that  $n$  appears in at most two members of  $\mathcal{M}$ , say, in only  $[nabc], [nij k] \in \mathcal{M}$ . Let  $\alpha = a_1 a_2 \dots a_{n-1}$  be a fixed  $(n-1)$ -point-set in  $\mathbb{R}^2$  such that  $a_1 a_2 a_3$  is a linear triple and the first four points are not collinear. Let us extend  $\alpha$  to an ordered  $n$ -point-set in  $\mathbb{R}^2$  by adding a point so that  $[nabc] = s$  and  $[nij k] = t$ , for some  $s, t > 0$ . Then, we may choose any point  $x$  as the  $n$ th point as far as  $x$  satisfies that

$$\frac{a_n x}{a_c x} = s \frac{a_n a_b}{a_b a_c}, \quad \frac{a_i x}{a_k x} = t \frac{a_i a_j}{a_j a_k}.$$

These two equations determine two Apollonian circles, and we may assume that  $s$  and  $t$  are chosen so that these two Apollonian circles intersect in two points. Then, we can get two ordered  $n$ -point-sets  $\beta$  and  $\gamma$  as extensions of  $\alpha$ . Note that each member of  $\mathcal{M}$  has the same value at  $\beta$  and at  $\gamma$ . But  $\beta$  and  $\gamma$  are not congruent since the  $n-1$  points in  $\alpha$  are not collinear. Therefore, the values of some Möbius invariant ( $\notin \mathcal{M}$ ) takes different values at  $\beta$  and at  $\gamma$  by Lemma 5.2. This is a contradiction.  $\square$

**Corollary 5.1.**  $R(2,5) \geq 4$ .

*Proof.* For any three Möbius invariants, one of 1, 2, 3, 4, 5 cannot appear in all of the three.  $\square$

We have  $R(2,4) = 2$  by Theorem 4.1, and  $R(2,5) = 4$  or 5 by Corollary 5.1 and Theorem 5.1. It seems that the set of four Möbius invariants  $[4123], [5431], [5142], [5213]$  determine all values of Möbius invariants for any ordered 5-point-set in  $\mathbb{R}^2$ , but I could not prove it.

**Problem.** Determine  $R(2,5)$ .

A set of  $n$  points in  $\mathbb{R}^d$  are called *generic* if the  $dn$  coordinates of the  $n$  points are algebraically independent over the rationals. Among the  $\binom{n}{2}$  distances between generic  $n$  points in  $\mathbb{R}^d$ , how many distances are necessary to determine all distances? If  $n \leq d+2$ , then all  $\binom{n}{2}$  distances are necessary. If  $n \gg d$ , then the necessary number would be very small compared with  $\binom{n}{2}$  by the dimensional restriction. However, to find the exact minimum necessary number is a difficult problem, see Connelly [4], or Jackson-Jordan-Szabadka [6]. Let us state the problem more precisely.

Let  $n \geq 4$  and  $G = (V, E)$  denote a graph with vertex set  $V = \{1, 2, 3, \dots, n\}$ . Then the problem is to characterize the graph  $G$  that satisfies the following condition:

- ( $\diamond$ ) For any two ordered sets  $\alpha, \beta$  of generic  $n$  points in  $\mathbb{R}^d$ ,  $d_{ij}(\alpha) = d_{ij}(\beta)$  ( $ij \in E$ ) implies that  $\alpha$  and  $\beta$  are congruent in order-preserving fashion.

Recently it was proved (Connelly [4], Jackson *et al* [6]) that in the planar case  $d = 2$ , every graph  $G = (V, E)$  satisfying the condition ( $\diamond$ ) is obtained from the complete graph  $K_4$  by a sequence of Henneberg 1-extension operations and edge additions. The *Henneberg 1-extension operation* on a graph is the following: Remove an edge  $xy$  from the graph and add a new vertex  $z$  and new edges  $zx, zy, zw$ , for some vertex  $w$  of the graph other than  $x, y$ . Thus, in the planar case  $d = 2$ , the minimum cardinality of  $E$  is  $2n - 2$ .

**Theorem 5.2 (Connelly [4] and Jackson *et al* [6]).** *For  $n \geq 4$ , the minimum number of distances necessary to determine all other distances among generic  $n$  points in the plane is  $2n - 2$ .*  $\square$

Put  $F_4 = \{[4123], [4213]\}$ , and for each  $n \geq 5$ , define  $F_n$  inductively in the following way:

$$F_5 = (F_4 \setminus \{[4213]\}) \cup \{[5431], [5142], [5213]\},$$

$$F_6 = (F_5 \setminus \{[5213]\}) \cup \{[6531], [6152], [6213]\},$$

$$F_7 = (F_6 \setminus \{[6213]\}) \cup \{[7631], [7162], [7213]\},$$

...

$$F_n = (F_{n-1} \setminus \{[(n-1)213]\}) \cup \{[n(n-1)31], [n1(n-1)2], [n213]\}.$$

Then it seems that  $F_n$  determines the values of all Möbius invariants for every *generic* ordered  $n$ -point-set in  $\mathbb{R}^2$ , though I could not prove it. Note that  $|F_n| = 2(n-4) + 3 = 2n - 6$ .

Suppose that  $\mathcal{M}$  is a minimal set of Möbius invariants that determine the values of all Möbius invariants for every *generic* ordered  $n$ -point-set  $\alpha_n$  in  $\mathbb{R}^2$ . Since  $[4123] - [4213] \neq \pm 1$  always holds for a generic ordered  $n$ -point-set, the distances  $d_{12}, d_{23}, d_{31}, d_{14}$  and  $\mathcal{M}$  determine all distances in  $\alpha_n$  in  $\mathbb{R}^2$  by Lemma 5.2. From the values of the four distances and the values of the members of  $\mathcal{M}$ , we obtain  $4 + |\mathcal{M}|$  equations for unknowns  $d_{ij}$ . Since the minimum number of

distances in  $\alpha_n$ , that determine all distances in  $\alpha_n$  is  $2n - 2$  by Theorem 5.2, it would be natural to expect that the number of equations  $4 + |\mathcal{M}|$  is at least  $2n - 2$ . This suggests that  $|\mathcal{M}| \geq 2n - 6$ .

**Conjecture.** For a generic ordered  $n$ -point-set ( $n \geq 4$ ) in  $\mathbb{R}^2$ , the minimum number of Möbius invariants necessary to determine the values of all Möbius invariants is equal to  $2n - 6$ .

## 6 Invariant fractions

Möbius invariant is generalized as follows. By a *segment*, we mean a distance represented by a pair of points. A *segment-product* is a product of a number of segments. For example,  $ab \cdot cd \cdot ae$  is a segment-product.

**Theorem 6.1.** *A fraction of segment-products is invariant under inversions if and only if the following condition holds:*

(\*) *For each point-symbol, the number of times it appears in the numerator is equal to the number of times it appears in the denominator.*

For example, the fraction  $(ab \cdot ac \cdot de)/(ad \cdot ae \cdot bc)$  is invariant under inversions, but the fraction  $(ab \cdot cd)/(bc \cdot de)$  is not.

*Proof.* Let us show that (\*) implies that the fraction is invariant under inversions. Instead of the general case, we consider, for example, the fraction  $(ab \cdot ac \cdot de)/(ad \cdot ae \cdot bc)$ . Let  $a', \dots, e'$  be the images of  $a, \dots, e$  by an inversion with center  $p$  and radius  $r$ . We show that

$$\frac{ab \cdot ac \cdot de}{ad \cdot ae \cdot bc} = \frac{a'b' \cdot a'c' \cdot d'e'}{a'd' \cdot a'e' \cdot b'c'}. \quad (3)$$

By Lemma 2.1, we have

$$\begin{aligned} a'b' &= \frac{r^2 ab}{pa \cdot pb'}, & a'c' &= \frac{r^2 ac}{pa \cdot pc'}, & d'e' &= \frac{r^2 de}{pd \cdot pe'} \\ a'd' &= \frac{r^2 ad}{pa \cdot pd'}, & a'e' &= \frac{r^2 ae}{pa \cdot pe'}, & b'c' &= \frac{r^2 bc}{pb \cdot pc'}. \end{aligned}$$

If these are substituted in the right hand side fraction of (3), then  $1/pa$  will appear in the numerator the same number of times as  $a'$

appears in the numerator, and also,  $1/pa$  will appear in the denominator the same number of times as  $a'$  appears in the denominator. Similar things will happen for  $1/pb, 1/pc, 1/pd, 1/pe$  and  $r^2$ . Since (\*) holds for this fraction, all  $1/pa, \dots, 1/pe$  and  $r^2$  will be cancelled out, and we get the equality (3).

To see the *only if* part, suppose that for some point-symbol, say,  $x$ , the number of times it appears in the numerator is not equal to the number of times it appears in the denominator. Then we cannot cancel out  $1/px$ , and the fraction would not be invariant under inversions.  $\square$

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