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ABSTRACT. For a nilpotent linear transformation $f : V \to V$ of type λ let $S(V,T)$ be the set of f-stable subspaces W associated to an LR (Littlewood-Richardson)-tableau T, i.e. W's such that $\dim f^{r-1}V \cap f^{t-1}W/\langle f^r V \cap f^{t-1}W, f^{r-1} \cap f^t W\rangle$ is equal to the number of cells (squares) of T filled with the letter t in the rth row for all t and r. Let $G(\lambda)$ be the subgroup of $\mathrm{GL}(V)$ consisting of elements commuting with f. It is given an example of $S(V,T)$ that does not have a dense $G(\lambda)$-orbit.

1. Introduction. Let $f : V \to V$ be a nilpotent linear transformation of a vector space V over $\mathbb{C}$. The purpose of this note is to give an example of the variety of subspaces stable under f associated to an LR(=Littlewood-Richardson) tableau that does not contain a dense orbit under the action of the subgroup of $\mathrm{GL}(V)$ commuting with f. The type of f is denoted by $\lambda = (\lambda_1, \cdots, \lambda_l)$, i.e. $\lambda_j = \dim \mathrm{Ker} f^j/\mathrm{Ker} f^{j-1} = \dim f^{j-1}V/f^jV$ where λ is a partition of the integer $\dim V$ because the induced maps $f : f^{j-1}V/f^jV \to f^jV/f^{j+1}V$ (and $f : \mathrm{Ker} f^j/\mathrm{Ker} f^{j-1} \to \mathrm{Ker} f^{j+1}/\mathrm{Ker} f^j$) are surjective. This means that the sizes of the Jordan blocks are the conjugate λ' of λ ; λ'_j is the number of i such that $\lambda_i \geq j$. A partition $\lambda = (\lambda_1, \cdots, \lambda_l)$ is identified with the diagram of row length $\lambda_1, \cdots, \lambda_l$ arranged like matrix entries, i.e. the cell (i,j) with the row index i increasing downwards and the column index j increasing to the right. For an f-stable subspace W of V, i.e. $fW \subset W$, the types of W and of V/W are those of the maps $f|_W : W \to W$ and of $f_{V/W} : V/W \to V/W$ induced by f, respectively.

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For an integer $0 < d < n = \dim V$ let $X(\lambda, d)$ be the set of f-stable subspaces of V of dimension d ;

$$X(\lambda, d) = \{ \ W \subset V; \ \dim \ W = d, \ fW \subset W \ \},$$

which is a closed set in the Grassmaniann $G(V, d)$ of d-dimensional subspaces of V. For an element $W \in X(\lambda, d)$ let $\mu^{(t)} = $ type V/f^tW, i.e. the rth row length of the diagram $\mu^{(t)}$ is equal to

$$\mu_r^{(t)} = \dim \frac{f^{r-1}(V/f^tW)}{f^r(V/f^tW)} = \dim \frac{\langle f^{r-1}V, f^tW\rangle}{\langle f^rV, f^tW\rangle}.$$

The canonical maps

$$\frac{\langle f^{r-1}V, f^{t+1}W\rangle}{\langle f^rV, f^{t+1}W\rangle} \to \frac{\langle f^{r-1}V, f^tW\rangle}{\langle f^rV, f^tW\rangle}, \quad f : \frac{f^{t-1}W}{f^tW} \to \frac{f^tW}{f^{t+1}W}$$

are surjective imply that $\mu \subset \mu^{(1)} \subset \cdots \subset \mu^{(l)} = \lambda$ where $f^{l-1}W \neq 0$ and $f^lW = 0$, and $(|\mu^{(1)}/\mu|, |\mu^{(2)}/\mu^{(1)}|, \cdots, |\lambda/\mu^{(l-1)}|)$ is a partition so we can define a skew tableau T of shape λ/μ and content ν by filling the letter t in the horizontal strip $\mu^{(t)}/\mu^{(t-1)}$, and $\nu_t = |\mu^{(t)}/\mu^{(t-1)}|$ for $1 \leq t \leq l$. Then type $W = \nu$ and T is an LR (Littlewood-Richardson)-tableau ([1],Lemma 1.1), which we call the LR-tableau associated to W. Thus $X(\lambda, d)$ is partitioned as $X(\lambda, d) = \bigcup_{\mu,\nu} S(\lambda/\mu, \nu) = \bigcup_{T \in \mathrm{LR}(\lambda,d)} S(V, T)$, where

$$\begin{aligned} S(\lambda/\mu, \nu) &= \{ \ W \in X(\lambda, d) \ ; \ \text{type } V/W = \mu, \ \text{type } W = \nu \ \}, \\ \mathrm{LR}(\lambda, d) &= \{ \ \text{LR-tableaux of shapes } \lambda/\mu\text{'s such that } |\lambda/\mu| = d \ \}, \\ S(V, T) &= \{ \ W \in X(\lambda, d) \ ; \ \text{type } V/f^tW = \text{shape } (\mu \cup T^{\mathrm{let} \leq t}) \\ & \qquad\qquad \text{for all } t \geq 0 \ \} \quad \text{if shape } T = \lambda/\mu. \end{aligned}$$

Here $T^{\mathrm{let} \leq t}$ is the subtableau of T consisting of cells (squares) filled with the letters less than or equal to t. The set $S(V, T)$ is a locally closed set in the Grassmaniann $G(V, d)$ and denote the closure of $S(V, T)$ by $X(V, T)$. The irreducible components of $S(\lambda/\mu, \nu)$ consist of $S(V, T)$ for the LR-tableaux $T \in \mathrm{LR}(\lambda, d)$ of shape λ/μ and content ν, and the dimension of $S(V, T)$ is equal to $n(\lambda) - n(\mu) - n(\nu)$ where $n(\lambda) =$

$\sum_j (j-1)\lambda'_j$ [1,Theorem A(1)]. For instance, if $\nu = (\nu_1)$, i.e. $fW = 0$ then $S(\lambda/\mu, \nu)$ is irreducible and a union of Schubert cells.

Let $G(\lambda) = \{\sigma \in \mathrm{GL}(V);\ \sigma f = f\sigma\}$ be the subgroup of $\mathrm{GL}(V)$ consisting of elements commuting with f. We consider two questions ;

(i) $S(V,T)$ is $G(\lambda)$-homogeneous ?
(ii) $S(V,T)$ has a dense $G(\lambda)$-orbit ?

We see easily that (i), hence (ii), is affirmative in the case $\nu = (\nu_1)$ above. Contrary to this case we remark in this short note that (i) (resp. (ii)) does not hold in general if $\nu'_1 \geq 2$ (resp. $\nu'_1 \geq 3$). Theoretical results like a condition that $S(V,T)$ has a dense $G(\lambda)$-orbit, are not contained in this note.

2. Notations and known results. Let $\{x_{ij};\ (i,j) \in \lambda\}$ be the Jordan basis of $f : V \to V$, i.e. $f(x_{ij}) = x_{i+1,j}$ with $x_{\lambda'_j+1,j} = 0$ where λ'_j is the number of i such that $\lambda_i \geq j$, and let $g : V \to V$ be the adjoint of f, i.e. $g(x_{ij}) = x_{i-1,j}$ with $x_{0,j} = 0$. We note two facts ; (i) an element $\sigma \in G(\lambda)$ is determined by the λ_1 vectors σx_{1j} for $1 \leq j \leq \lambda_1$ because $\sigma x_{ij} = \sigma f^{i-1} x_{1j} = f^{i-1}(\sigma x_{1j})$, (ii) $\sigma x_{1j} \in \mathrm{Ker} f^{\lambda'_j}$ because $f^{\lambda'_j}\sigma x_{1j} = \sigma(f^{\lambda'_j} x_{1j}) = 0$. Thus $G(\lambda)$ is an openset of $\prod_{1\leq j\leq\lambda_1} \mathrm{Ker}\ f^{\lambda'_j}$.

Definition(in [1]). For a tableau T,

(1) The generic vector v_a of $a \in T$ is defined by descending induction on letters to be $v_a = gv_{a\downarrow} + \sum_{b\in P(a)} \alpha_{ba} g v_b$, where $gv_b = x_{b\uparrow}$ if $b \in T^\infty$, and the coefficients α_{ba} are algebraically independent over $\mathbb{C}$. Here $P(a)$ is the set of cells $b \in T^\infty$ such that column $b <$ column a and letter$(b\uparrow) <$ letter $a <$ letter b, where $b\uparrow$ is the cell directly above b and T^∞ is the tableau added to T λ_1 cells filled with the letter ∞ in the positions $(\lambda'_j + 1, j)$ for $1 \leq j \leq \lambda_1$.

(2) The rational function field F_T is the purely transcendental extension field over $\mathbb{C}$ with the transcendental basis consisting of the parameters α_{ba} in the generic vectors v_a ; $F_T = \mathbb{C}(\alpha_{ba};\ b \in P(a), a \in T)$.

(3) The generic subspace W_T is the $f\otimes F_T$-stable subspace of $V \otimes F_T$ spanned by the generic vectors ; $W_T = \langle v_a;\ a \in T\rangle \subset V \otimes F_T$.

Example. The generic vectors of the LR-tableau $T = \begin{ytableau} \, & \, & \, & 1 \\ \, & \, & 2 \\ \, & 1 \\ 3 \end{ytableau}$ are

$$v_{41} = x_{41},$$
$$v_{32} = x_{32} + \alpha_{41,32} x_{31},$$
$$v_{23} = x_{23} + \alpha_{41,23} x_{31} + \alpha_{42,23} x_{32},$$
$$v_{14} = x_{14} + \alpha_{41,14} x_{31} + \alpha_{41,23}\alpha_{23,14} x_{21} + \alpha_{23,14} x_{13} + \alpha_{42,23}\alpha_{23,14} x_{22}.$$

In order to write the generic vectors more concisely we make use the diagram filled with the coefficients of the cell vectors in the corresponding entries, e.g.

$$v_{32} = \begin{ytableau} \, & \, & \, & \, \\ \, & \, & \, \\ \alpha & 1 \\ \, \end{ytableau}, \qquad v_{23} = \begin{ytableau} \, & \, & \, & \, \\ \, & \, & 1 \\ \beta & \gamma \\ \, \end{ytableau},$$

where $\alpha = \alpha_{41,32}$, $\beta = \alpha_{41,23}$ and $\gamma = \alpha_{42,23}$ for the above example. What we use in Section 3 and 4 is the following fact, the proof of which is contained in [1,Section 2].

Fact. For an LR-tableau T the generic subspace W_T is a generic point of $S(V,T)$, i.e. the field generated by the ratios of the Plücker coordinates of W_T, is isomorphic to the function field of the variety $X(V,T)$, the closure of $S(V,T)$ in $G(V,d)$. $\square$

In general, for a tableau T (not necessary LR-tabelau) the generic subspace W_T is a generic point of $S(V,\hat{T})$ where $\hat{T}$ is the LR-tableau obtained by applying the successions of shifts (coplactic operations) starting with T (Theorem B in [1]).

3. An example that $S(V,T)$ is not $G(\lambda)$-homogeneous (Example after Lemma 2.5 in [1]). For the LR-tableau $T = \begin{ytableau} \, & \, & 1 \\ \, & 1 \\ 2 \end{ytableau}$ let $W_{p,q} = \langle\, x_{31},\ px_{21}+x_{22},\ qx_{21}+x_{13} \,\rangle$, which is an element of $S(V,T)$ for all $(p,q) \neq (0,0)$. Consider the $G(\lambda)$-orbit of $W_{1,0}$. For any $\sigma \in G(\lambda)$ we see

$$\begin{aligned} \sigma x_{31} &= \alpha x_{31}, \\ \sigma(x_{21}+x_{22}) &= (\alpha x_{21} + *x_{31} + *x_{22}) + (\beta x_{22} + *x_{31}), \\ \sigma x_{13} &= \gamma x_{13} + *x_{22} + *x_{31}, \end{aligned}$$

i.e.

$$\begin{pmatrix} \sigma x_{31} \\ \sigma(x_{21}+x_{13}) \\ \sigma x_{13} \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ * & \alpha & *+\beta & 0 \\ * & 0 & * & \gamma \end{pmatrix} \cdot {}^t(x_{31}, x_{21}, x_{22}, x_{13})$$

where $\alpha\beta\gamma \neq 0$ and $*$ stands for some elements of $\mathbb{C}$. From this we see that the Plücker coordinates $p(\sigma W_{1,0}, (x_{31}, x_{21}, x_{13}))$ is non zero. On the other hand, σW_{01} is given by

$$\begin{pmatrix} \sigma x_{31} \\ \sigma x_{22} \\ \sigma(x_{21}+x_{13}) \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ * & 0 & \beta & 0 \\ * & \alpha & * & \gamma \end{pmatrix} \cdot {}^t(x_{31}, x_{21}, x_{22}, x_{13})$$

so the Plücker coodinats $p(W_{0,1}, (x_{31}, x_{21}, x_{13})) = 0$. The $G(\lambda)$-orbit decomposition is given by $S(V,T) = \mathcal{O}(W_{1,0}) \cup \mathcal{O}(W_{0,1})$ where $\mathcal{O}(W_{1,0}) \cong \mathbb{A}^2$, $\mathcal{O}(W_{0,1}) = S(V,T) \cap X(V,T') \cong \mathbb{P}^1 - \{2 \text{ points}\}$ with the LR-tableau $T' = \ytableaushort{{}{}1,{}2,1}$. On the other hand, we see $S(V,T') = \mathcal{O}(W') \cong \mathbb{A}^2$ is $G(\lambda)$-homogeneous.

4. An example of $S(V,T)$ that does not have a dense $G(\lambda)$-orbit. Consider two LR-tableaux

$$T_1 = \ytableaushort{{}{}1,{}12,{}3,2,4}\,, \qquad T_2 = \ytableaushort{{}{}1,{}{}2,{}1,{}3,2,4}$$

of shapes $\lambda/\mu = (3^2.2.1^2)/(2.1^2)$ and $(3^2.2^2.1^2)/(2^2.1^2)$, respectively, with $n(T_1) = n(T_2) = 1 + 3 = 4$. The generic vectors of $T_1^{\mathrm{let}\ 1}$ (resp. $T_2^{\mathrm{let}\ 1}$) are

$$\ytableaushort{{}x1,y{}{},z{},{},{}}\,, \quad \ytableaushort{{}{}{},{}1{},u{},{},{}} \quad \text{(resp.} \ \ytableaushort{{}{}1,{}x{},y{},z{},{},{}}\,, \quad \ytableaushort{{}{}{},{}{}{},{}1,u{},{},{}} \ \text{)}.$$

We see easily that any element of $S(V,T_1)$ (resp. $S(V,T_2)$) contains the 3-dimen- sional subspace $\langle x_{41}, x_{51}, x_{32}\rangle = \begin{ytableau} \ & \ & \ \\ \ & \ & \ \\ \ & * \\ * \\ * \end{ytableau}$ (resp. $\langle x_{51}, x_{61}, x_{42}\rangle = \begin{ytableau} \ & \ & \ \\ \ & \ & \ \\ \ & \ \\ \ & * \\ * \\ * \end{ytableau}$).

Theorem. $S(V,T_1)$ has a dense $G(\lambda)$-orbit while $S(V,T_2)$ does not.

The same proof below shows that $S(V,T_3)$ has a dense $G(\lambda)$-orbit for the LR-tableau $T_3 = \begin{ytableau} \ & \ & 1 \\ \ & \ & 2 \\ \ & \ \\ \ & 1 \\ 2 & 3 \\ 4 \end{ytableau}$ of shape $\lambda/\mu = (3^2.2^3.1)/(2^3.1)$. This means that T_2 is, in a sense, one of the simplest example of an LR-tableau for which $S(V,T)$ does not have a dense $G(\lambda)$-orbit.

Proof. Case T_1. By the remark before Theorem we consider $\sigma \in G(\lambda)$ such that

$$\sigma x_{21} = \begin{ytableau} \ & \ & \ \\ a & * & * \\ * & \ \\ \ \\ \ \end{ytableau}, \quad \sigma x_{12} = \begin{ytableau} \ & \beta & \gamma \\ \ & * & * \\ * & \ \\ \ \\ \ \end{ytableau}, \quad \sigma x_{13} = \begin{ytableau} \ & \ & \delta \\ \ & * & * \\ \ & \ \\ \ \\ \ \end{ytableau}.$$

Then

$$v_1 = \begin{ytableau} \ & 1 & 1 \\ 1 & \ & \ \\ \ & \ \\ \ \\ \ \end{ytableau} \to \sigma v_1 = \begin{ytableau} \ & \beta & \epsilon \\ a & * & * \\ * & \ \\ \ \\ \ \end{ytableau}, \quad v_2 = \begin{ytableau} \ & \ & \ \\ \ & 1 & \ \\ 2 & \ \\ \ \\ \ \end{ytableau} \to \sigma v_2 = \begin{ytableau} \ & \ & \ \\ \ & \beta & \gamma \\ \eta & \ \\ * \\ \ \end{ytableau},$$

where $\epsilon = \gamma + \delta$ and $\eta = 2a$. Hence $(\sigma v_1,\ f\sigma v_1,\ \sigma v_2)$ is equal to

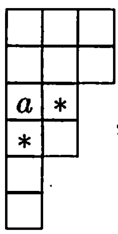
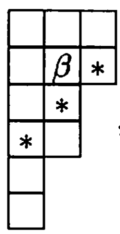
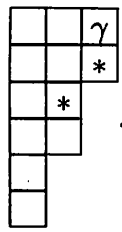
$$\begin{pmatrix} \alpha & \beta & \gamma+\delta & * & * & * \\ 0 & 0 & 0 & \alpha & \beta & \gamma+\delta \\ 0 & 0 & 0 & 2\alpha & \beta & \gamma \end{pmatrix} \cdot {}^t(x_{21}, x_{12}, x_{13}, x_{31}, x_{22}, x_{23}).$$

The Plücker coordinates of σW where $W = \langle\ v_1, fv_1, v_2, x_{41}, x_{51}, x_{32}\ \rangle$ are reduced to the 3-minors of the above 3×6 matrix. If the columns are numbered as $1, \cdots, 6$ from left to right then the 3-minors ijk are given by

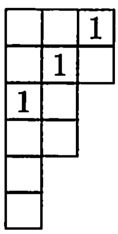
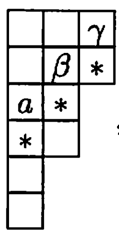
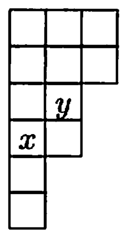
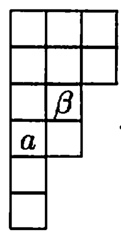
$$\begin{aligned}
&(145 : 156 : 245 : 246 : 456)\\
&= (\ -\alpha^2\beta : -\alpha\beta\delta : -\alpha\beta^2 : \alpha\beta(-\gamma-2\delta) : 456 \ )\\
&= (\ 1 : \frac{\delta}{\alpha} : \frac{\beta}{\alpha} : \frac{\gamma+2\delta}{\alpha} : \frac{456}{\alpha^2\beta} \ ).
\end{aligned} \tag{1}$$

The field generated by the four elements (1) over $\mathbb{C}$ is of 4-dimensional provided that the entries $*$'s are algebraically independent over $\mathbb{C}(\alpha, \beta, \gamma, \delta)$. This means that the $G(\lambda)$-orbit of the W is dense in $S(V, T_1)$.

Case T_2. By the remark before Theorem we consider $\sigma \in G(\lambda)$ such that

$\sigma x_{31} =$  , $\sigma x_{22} =$  , $\sigma x_{13} =$  .

Then

$v_1 =$  $\rightarrow \sigma v_1 =$  , $v_2 =$  $\rightarrow \sigma v_2 =$  .

Hence

$$\begin{pmatrix} \sigma v_1 \\ f\sigma v_1 \\ \sigma v_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & * & * & 0 \\ 0 & 0 & 0 & \alpha & \beta & \gamma \\ 0 & 0 & 0 & x\alpha & y\beta & 0 \end{pmatrix} \cdot {}^t(x_{31}, x_{22}, x_{13}, x_{41}, x_{32}, x_{22}).$$

As in the case in T_1 the Plücker coordinates of σW with $W = \langle \ v_1, fv_1, v_2, x_{41}, x_{51}, x_{32} \ \rangle$, are reduced to the 3-minors of the above 3×6 matrix, which are homogeneous of degree 3 with respect to α, β, γ together with

$$456 = \begin{vmatrix} * & * & 0 \\ \alpha & \beta & \gamma \\ x\alpha & y\beta & 0 \end{vmatrix}.$$

The field generated by their ratios over $\mathbb{C}$ are of 3-dimesnional even if α, β, γ and the two $*$'s are algebraically independent over $\mathbb{C}$. Similarly, we get a 3-dimensional $G(\lambda)$-orbit if we start with any specializations of the generic vectors of $T_2^{\mathrm{let}\,1}$, i.e.

$$v_1 = \begin{array}{|c|c|c|} \hline \ & \ & 1 \\ \hline \ & x & \ \\ \hline y & \ & \multicolumn{1}{c}{} \\ \cline{1-2} z & \ & \multicolumn{1}{c}{} \\ \cline{1-2} \ & \multicolumn{2}{c}{} \\ \cline{1-1} \ & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}, \qquad v_2 = \begin{array}{|c|c|c|} \hline \ & \ & \ \\ \hline \ & \ & \ \\ \hline \ & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} u & \ & \multicolumn{1}{c}{} \\ \cline{1-2} \ & \multicolumn{2}{c}{} \\ \cline{1-1} \ & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

for any constants $x, y, z, u \in \mathbb{C}$. This implies that $S(V, T_2)$ does not have a dense $G(\lambda)$-orbit. $\square$

References

1. T. Maeda, *A partial order on Littlewood-Richardson tableaux*, J. Alg., submitted.

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