APPROXIMATION PROCESSES OF FRACTIONAL INTERPOLATION TYPE OPERATORS

Ryukyu Math. J., 20 (2007), 25-42

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ABSTRACT. We consider approximation processes of fractional interpolation type operators on spaces of functions taking values in normed linear spaces. Consequently, we obtain a far-reaching generalization of the Bernstein type rational functions on $[0, \infty)$ to the multidimensional case for vector-valued functions, and approximation theorems by them with the rate of convergence in terms of the modulus of continuity of functions to be approximated.

1. Introduction

Let $\mathbb{N}$ denote the set of all natural numbers. Let g be a real-valued continuous function on the closed unit interval $\mathbb{I} = [0, 1]$ of the real line $\mathbb{R}$ and let $n \in \mathbb{N}$. Then nth Bernstein polynomial of g is defined by

$$(1) \qquad B_n(g)(t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} g\Big(\frac{k}{n}\Big) \qquad (t \in \mathbb{I}).$$

It is well known that the sequence $\{B_n(g)\}_{n\in\mathbb{N}}$ converges uniformly to g on $\mathbb{I}$, and the Bernstein polynomials and their generalizations play an important role in approximation theory (see, e,g., [1], [2], [8], [13]).

In view of these concernments, Balázs [3] introduced and studied several approximation properties of the Bernstein type rational functions defined as follows:

Received November 30, 2007.

Let f be a real-valued function on $[0,\infty)$ and let $n \in \mathbb{N}$, and define

$$(2) \quad R_n(f;x) = \frac{1}{(1+a_nx)^n} \sum_{k=0}^{n} \binom{n}{k} (a_nx)^k f\Big(\frac{k}{b_n}\Big) \qquad (x \in [0,\infty)),$$

where $a = \{a_n\}_{n\in\mathbb{N}}$ and $b = \{b_n\}_{n\in\mathbb{N}}$ are suitably chosen sequences of positive real numbers. To compare (1) and (2), setting

$$q_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k} \qquad (t \in \mathbb{I},\ k = 0,1,\dots,n)$$

and

$$r_{n,k}(x) = \binom{n}{k} \frac{(a_nx)^k}{(1+a_nx)^n} \qquad (x \in [0,\infty),\ k = 0,1,\dots,n),$$

we have

$$r_{n,k}(x) = q_{n,k}\Big(\frac{a_nx}{1+a_nx}\Big) \qquad (x \in [0,\infty),\ k = 0,1,\dots,n),$$

and so

$$R_n(f;x) = B_n(f|_{\mathbb{I}})\Big(\frac{a_nx}{1+a_nx}\Big),$$

where $f|_{\mathbb{I}}$ denotes the restriction of f to $\mathbb{I}$.

In [4], the estimate of the rate of convergence of $R_n(f;x)$ to $f(x)$ given in [3] is improved by an appropriate choice of a and b when f satisfies some more restrictive conditions. Furthermore, in [20] the saturation problem is discussed for $\{R_n\}_{n\in\mathbb{N}}$ and the uniform approximation problem is considered for R_n-like rational functions defined by

$$(3) \qquad R_n(B;a;f;x) = \frac{1}{(1+a_nx)^n} \sum_{k=0}^{n} \binom{n}{k} (a_nx)^k f(b_{n,k})$$

$$= \sum_{k=0}^{n} r_{n,k}(x) f(b_{n,k}) \qquad (x \in [0,\infty)),$$

where $B = (b_{n,k})_{0\le k\le n\,(n=1,2,\dots)}$ is a matrix whose entries satisfy

$$0 \le b_{n,0} < b_{n,1} < b_{n,2} < \cdots < b_{n,n},$$

and f is a real-valued continuous function on $[0,\infty)$ for which $\lim_{x\to\infty} f(x)$ exists. Note that if

$$a_n = 1 \qquad (n \in \mathbb{N}),$$

and if

$$b_{n,k} = \frac{k}{n-k+1} \qquad (0 \le k \le n,\ n \in \mathbb{N}),$$

then (3) reduces to

$$(4) \qquad L_n(f)(x) := \frac{1}{(1+x)^n} \sum_{k=0}^{n} \binom{n}{k} x^k f\Big(\frac{k}{n-k+1}\Big),$$

which was introduced by Bleimann, Butzer and Hahn [5]. In [21], the satutration properties of the sequence $\{L_n\}_{n \in \mathbb{N}}$ is established and it is showed that these operators satisfy an asymptotic relation of the Voronovskaja type, i.e.,

$$\lim_{n\to\infty} n(L_n(x_0) - f(x_0)) = f''(x_0)x_0(1+x_0)^2$$

if f is a real-valued continuous function on $[0,\infty)$ for which $\lim_{x\to\infty} f(x)$ exists and the second derivative $f''(x_0)$ exists at a point x_0.

Let $1 \le p \le \infty$ be fixed and let $\mathbb{R}^r$ denote the metric linear space of all r-tuples of real numbers, equipped with the usual metric

$$d_p(x,y) := \begin{cases} (\sum_{i=1}^r |x_i - y_i|^p)^{1/p} & (1 \le p < \infty) \\ \max\{|x_i - y_i| : 1 \le i \le r\} & (p = \infty), \end{cases}$$

$$(x = (x_1, x_2, \dots, x_r),\ y = (y_1, y_2, \dots, y_r) \in \mathbb{R}^r).$$

The purpose of this paper is to generalize (2) for vector-valued functions on the r-dimensional first hyperquadrant

$$[0,\infty)^r := \{x = (x_1, x_2, \dots, x_r) \in \mathbb{R}^r : x_i \ge 0,\ i = 1, 2, \dots, r\}$$

and to consider their uniform convergence with rates in terms of the modulus of continuity of functions to be approximated.

2. Convergence theorems

Let (X,d) be a locally compact metric space and let $(E, \|\cdot\|)$ be a normed linear space. Let $B(X,E)$ denote the normed linear space of all E-valued bounded functions on X with the supremum norm $\|\cdot\|_X$. Also, we denote by $C(X,E)$ the linear space consisting of all E-valued continuous functions on X and set $BC(X,E) = B(X,E) \cap C(X,E)$. Let $\{Y_\alpha : \alpha \in D\}$ be a family of finite sets, where D is a directed set and let $\{w_\alpha\}_{\alpha \in D}$ be a net of strictly positive real-valued functions on X. For each $\alpha \in D$, let ξ_α be a mapping of Y_α into X and let χ_α be a real-valued function on $X \times Y_\alpha$. Then we define an interpolation type operator with the weight function w_α by the form

$$U_\alpha(f)(x) = U_\alpha(E; f; x) = w_\alpha(x) \sum_{k\in Y_\alpha} \chi_\alpha(x,k) f(\xi_\alpha(k))$$

$$(\alpha \in D,\ f \in C(X,E),\ x \in X)$$

(cf. [14] - [19]), and set

$$W_\alpha(g)(x) = U_\alpha(\mathbb{R}; g; x) = w_\alpha(x) \sum_{k\in Y_\alpha} \chi_\alpha(x,k) g(\xi_\alpha(k))$$

$$(\alpha \in D,\ g \in C(X,\mathbb{R}),\ x \in X).$$

For each $x \in X$ and $q > 0$, we define

$$\mu_\alpha(x;q) = w_\alpha(x) \sum_{k\in Y_\alpha} |\chi_\alpha(x,k)| d^q(x, \xi_\alpha(k)) \qquad (\alpha \in D),$$

which is called the qth absolute moment of χ_α at x with respect to the weight function w_α. Also, 1_X stands for the unit function defined by $1_X(x) = 1$ for all $x \in X$.

Let X_0 be a compact subset of X and suppose that

$$\chi_\alpha(x,k) \geq 0 \qquad (\alpha \in D)$$

for all $(x,k) \in X_0 \times Y_\alpha$.

Theorem 1. *If*

(5) $$\lim_\alpha \|W_\alpha(1_X) - 1_X\|_{X_0} = 0$$

and if

(6) $$\lim_\alpha \|\mu_\alpha(\cdot;q)\|_{X_0} = 0$$

for some $q > 0$, *then*

(7) $$\lim_\alpha \|U_\alpha(f) - f\|_{X_0} = 0$$

for all $f \in BC(X,E)$.

Proof. This follows from [15, Corollary 3] (cf. [14, Theorem 2 (b)], [15, Theorem 8], [16, Theorem 1]), which remains true without the completeness of E for interpolation type operators (cf. [17]).

In the rest of this section, let $1 \leq p \leq \infty$ be fixed and let

$$(X,d) = ([0,\infty)^r, d_p).$$

For $i = 1, 2, \ldots, r$, p_i denotes the ith coordinate function on $\mathbb{R}^r$ defined by

$$p_i(x) = x_i \qquad (x = (x_1, x_2, \ldots, x_r) \in \mathbb{R}^r).$$

Tnen we have

$$\mu_\alpha(x) := \mu_\alpha(x;2) = w_\alpha(x) \sum_{i=1}^{r} \sum_{k\in Y_\alpha} |\chi_\alpha(x,k)|(p_i(x) - p_i(\xi_\alpha(k)))^2$$

$$(\alpha \in D,\ x \in X).$$

Note that

$$\mu_\alpha(x) = \sum_{i=1}^{r} W_\alpha\big((p_i(x)1_X - p_i)^2\big)(x) \qquad (\alpha \in D,\ x \in X_0),$$

and so we obtain the following Korovkin type result, which can be more convenient for applications to the concrete weighted interpolation type operators (cf. [14, Theorem 4], [15, Theorems 5 and 6]). For the background of Korovkin-type approximation theory, we refer to the book of Altomare and Campiti [1], in which an excellent source and vast literatures of this theory can be found (cf. [9], [10], [12]).

Corollary 1. *If (5) holds and if*

$$\lim_\alpha \mu_\alpha(x) = 0 \quad \textit{uniformly in } x \in X_0,$$

then (7) holds for all $f \in BC(X,E)$. In particular, if (5) holds and if

(8) $$\lim_\alpha \|W_\alpha(p_i) - p_i\|_{X_0} = 0 \qquad (i = 1,2,\cdots,r)$$

and

(9) $$\lim_\alpha \|W_\alpha(p_i^2) - p_i^2\|_{X_0} = 0 \qquad (i = 1,2,\dots,r),$$

then (7) holds for every $f \in BC(X,E)$.

Remark 1. Conditions (5), (8) and (9) imply that

$$\lim_\alpha \|W_\alpha(g) - g\|_{X_0} = 0$$

for all $g \in BC(X,\mathbb{R})$ (cf. [15, Theorem 6], [14, Theorem 4 (b)]).

Let $n \in \mathbb{N}$ and define

$$p_{n,k}(s,t) = \binom{n}{k} s^k t^{n-k} \qquad (s,t \in \mathbb{R},\ k = 0,1,\dots,n).$$

Lemma 1. *Let $s,t,u \in \mathbb{R}$. Then the following equalities hold :*

(10) $$\sum_{k=1}^{n} k p_{n,k}(s,t) = ns(s+t)^{n-1}.$$

$$\sum_{k=2}^{n} k(k-1)p_{n,k}(s,t) = n(n-1)s^2(s+t)^{n-2}. \tag{11}$$

$$\sum_{k=1}^{n} k^2 p_{n,k}(s,t) = ns(ns+t)(s+t)^{n-2}. \tag{12}$$

$$\sum_{k=0}^{n}(u-k)^2 p_{n,k}(s,t) \tag{13}$$

$$= (s+t)^{n-2}\big(s^2u^2 + nst + 2su(tu-ns) + (tu-ns)^2\big).$$

In particular, if $tu = ns$, then (13) reduces to

$$\sum_{k=0}^{n}(u-k)^2 p_{n,k}(s,t) = u(s+t)^{n-2}(s^2u+t^2).$$

Proof. (10) and (11) immediately follow from the binomial theorem. (12) follows from (10) and (11). Also, (13) follows from (10) and (12).

Lemma 2. *Let $\{m_\alpha\}_{\alpha\in D}$ be a net of positive integers and let $\{\beta_\alpha\}_{\alpha\in D}$ be a net of positive real numbers. Let $\{u_\alpha\}_{\alpha\in D}$ and $\{v_\alpha\}_{\alpha\in D}$ be nets of real-valued functions on $[0,\infty)$ such that $u_\alpha(t)+v_\alpha(t)\neq 0$ for all $\alpha\in D$ and all $t\in[0,\infty)$. We define*

$$\varphi_{\alpha,k}(t) = \binom{m_\alpha}{k} u_\alpha^k(t) v_\alpha^{m_\alpha-k}(t) \qquad (t\in[0,\infty),\ k=0,1,\dots,m_\alpha).$$

Then the following equalities hold for all $\alpha\in D$ and all $t\in[0,\infty)$:

$$\frac{1}{(u_\alpha(t)+v_\alpha(t))^{m_\alpha}} \sum_{k=0}^{m_\alpha} \varphi_{\alpha,k}(t) = 1. \tag{14}$$

$$\frac{1}{(u_\alpha(t)+v_\alpha(t))^{m_\alpha}} \sum_{k=1}^{m_\alpha} \frac{k}{\beta_\alpha}\varphi_{\alpha,k}(t) = \frac{m_\alpha u_\alpha(t)}{\beta_\alpha(u_\alpha(t)+v_\alpha(t))}. \tag{15}$$

$$\frac{1}{(u_\alpha(t)+v_\alpha(t))^{m_\alpha}} \sum_{k=1}^{m_\alpha} \left(\frac{k}{\beta_\alpha}\right)^2 \varphi_{\alpha,k}(t) \tag{16}$$

$$= \frac{m_\alpha u_\alpha(t)(m_\alpha u_\alpha(t)+v_\alpha(t))}{\beta_\alpha^2(u_\alpha(t)+v_\alpha(t))^2}.$$

Proof. (14) immediately follows from the binomial theorem. For (15) and (16), we take $n = m_\alpha, s = u_\alpha(t)$ and $v_\alpha(t)$ instead of t in Lemma 1 (10) and (12), respectively.

Lemma 3. *Let $\{m_\alpha\}_{\alpha\in D}$ be a net of positive integers and let $\{\beta_\alpha\}_{\alpha\in D}$ be a net of positive real numbers with $\lim_\alpha \beta_\alpha = +\infty$. Let $\{u_\alpha\}_{\alpha\in D}$ and $\{v_\alpha\}_{\alpha\in D}$ be nets of nonnegative functions in $C[0,\infty) := C([0,\infty),\mathbb{R})$ such that*

$$\inf\{u_\alpha(t) + v_\alpha(t) : t \in [0,\infty)\} > 0$$

for each $\alpha \in D$. We define

$$I_\alpha(t) = I_\alpha(m_\alpha;\beta_\alpha;u_\alpha;v_\alpha;t) = \frac{m_\alpha u_\alpha(t)}{\beta_\alpha(u_\alpha(t) + v_\alpha(t))}$$
$$(\alpha \in D,\ t \in [0,\infty))$$

and

$$J_\alpha(t) = J_\alpha(m_\alpha;\beta_\alpha;u_\alpha;v_\alpha;t) = \frac{m_\alpha u_\alpha(t)(m_\alpha u_\alpha(t) + v_\alpha(t))}{\beta_\alpha^2(u_\alpha(t) + v_\alpha(t))^2}$$
$$(\alpha \in D,\ t \in [0,\infty)).$$

Let K be a compact subset of $[0,\infty)$. If

$$\lim_\alpha I_\alpha(t) = t \quad \textit{uniformly in } t \in K,$$

then

$$\lim_\alpha J_\alpha(t) = t^2 \quad \textit{uniformly in } t \in K.$$

Proof. We have

$$J_\alpha(t) = I_\alpha^2(t) + \frac{I_\alpha(t)v_\alpha(t)}{\beta_\alpha(u_\alpha(t) + v_\alpha(t))} \qquad (\alpha \in D,\ t \in [0,\infty)),$$

from which the desired result follows, immediately.

Lemma 4. *Let $\{m_\alpha\}_{\alpha\in D}$, $\{\beta_\alpha\}_{\alpha\in D}$, $\{u_\alpha\}_{\alpha\in D}$ and $\{v_\alpha\}_{\alpha\in D}$ be as in Lemma 3 such that $\beta_\alpha = o(m_\alpha)$, $v_\alpha(t) > 0$ and $m_\alpha u_\alpha(t) = \beta_\alpha t v_\alpha(t)$ for all $\alpha \in D$ and all $t \in [0,\infty)$. Let K be a compact subset of $[0,\infty)$. Then there hold*

$$\lim_\alpha I_\alpha(t) = t \quad \textit{uniformly in } t \in K \tag{17}$$

and

$$\lim_\alpha J_\alpha(t) = t^2 \quad \textit{uniformly in } t \in K. \tag{18}$$

Proof. Since

$$u_\alpha(t) = \frac{\beta_\alpha t}{m_\alpha} v_\alpha(t) \qquad (\alpha \in D,\ t \in [0,\infty)),$$

we have

$$I_\alpha(t) = \frac{m_\alpha t}{\beta_\alpha t + m_\alpha} \qquad (\alpha \in D,\ t \in [0,\infty)),$$

which establishes (17). Therefore, (18) holds by Lemma 3.

Let $\{n_{\alpha,i}\}_{\alpha\in D}, i = 1,2,\dots,r$, be nets of positive integers and let $\{b_{n_{\alpha,i}}\}_{\alpha\in D}, i = 1,2,\dots,r$, be nets of positive real numbers such that

$$\lim_\alpha b_{n_{\alpha,i}} = +\infty \qquad (i = 1,2,\dots,r).$$

Let $\{g_{n_{\alpha,i}}\}_{\alpha\in D}$ and $\{h_{n_{\alpha,i}}\}_{\alpha\in D}, i = 1,2,\dots,r$, be nets of nonnegative functions in $C[0,\infty)$ such that

$$\inf\{g_{n_{\alpha,i}}(t) + h_{n_{\alpha,i}}(t) : t \in [0,\infty)\} > 0$$

for all $\alpha \in D$ and for $i = 1,2,\dots,r$. Then we define

$$\text{(19)} \qquad F_\alpha(f)(x) = F_\alpha(E;f;x) = \prod_{i=1}^{r} \frac{1}{(g_{n_{\alpha,i}}(x_i) + h_{n_{\alpha,i}}(x_i))^{n_{\alpha,i}}}$$

$$\times \sum_{k_1=0}^{n_{\alpha,1}} \sum_{k_2=0}^{n_{\alpha,2}} \cdots \sum_{k_r=0}^{n_{\alpha,r}} \prod_{i=1}^{r} \rho_{n_{\alpha,i},k_i}(x_i) f\Big(\frac{k_1}{b_{n_{\alpha,1}}}, \frac{k_2}{b_{n_{\alpha,2}}}, \dots, \frac{k_r}{b_{n_{\alpha,r}}}\Big)$$

$$(\alpha \in D,\ f \in C(X,E),\ x = (x_1,x_2,\dots,x_r) \in X),$$

where

$$\rho_{n_{\alpha,i},k_i}(x_i) = \binom{n_{\alpha,i}}{k_i} g_{n_{\alpha,i}}{}^{k_i}(x_i) h_{n_{\alpha,i}}{}^{n_{\alpha,i}-k_i}(x_i)$$

$$(\alpha \in D,\ i = 1,2,\dots,r).$$

We call $F_\alpha, \alpha \in D$, the fractinonal interpolation type operators on $C(X,E)$.

From now on let $K_i, i = 1,2,\dots,r$, be compact subsets of $[0,\infty)$ and we set

$$X_0 = \prod_{i=1}^{r} K_i.$$

Theorem 2. *If*

$$\text{(20)} \qquad \lim_\alpha I_\alpha(n_{\alpha,i}; b_{n_{\alpha,i}}; g_{n_{\alpha,i}}; h_{n_{\alpha,i}}; t) = t \quad \textit{uniformly in } t \in K_i$$

for $i = 1,2,\dots,r$, *then*

$$\text{(21)} \qquad \lim_\alpha \|F_\alpha(f) - f\|_{X_0} = 0$$

for all $f \in BC(X, E)$.

Proof. We define

$$G_\alpha(g)(x) = F_\alpha(\mathbb{R}; g; x) \qquad (\alpha \in D,\ g \in C(X, \mathbb{R}),\ x \in X).$$

Then by Lemma 2 (14) and (15), we have

$$G_\alpha(1_X)(x) = 1 \qquad (\alpha \in D,\ x \in X)$$

and

$$G_\alpha(p_i)(x) = I_\alpha(n_{\alpha,i}; b_{n_{\alpha,i}}; g_{n_{\alpha,i}}; h_{n_{\alpha,i}}; x_i)$$
$$(\alpha \in D,\ x = (x_1, x_2, \dots, x_r) \in X,\ i = 1, 2, \dots, r).$$

Therefore, (20), Lemma 2 (16) and Lemma 3 yield that

$$\lim_\alpha \|G_\alpha(p_i) - p_i\|_{X_0} = 0 \qquad (i = 1, 2, \dots, r)$$

and

$$\lim_\alpha \|G_\alpha(p_i^2) - p_i^2\|_{X_0} = 0 \qquad (i = 1, 2, \dots, r).$$

Thus, the desired result (21) follows by Corollary 1.

Let

$$a_{n_{\alpha,i}} := \frac{b_{n_{\alpha,i}}}{n_{\alpha,i}} \qquad (\alpha \in D,\ i = 1, 2, \dots, r) \tag{22}$$

and we define

$$T_\alpha(f)(x) = T_\alpha(E; f; x) = \prod_{i=1}^{r} \frac{1}{(1 + a_{n_{\alpha,i}} x_i)^{n_{\alpha,i}}} \tag{23}$$
$$\times \sum_{k_1=0}^{n_{\alpha,1}} \sum_{k_2=0}^{n_{\alpha,2}} \cdots \sum_{k_r=0}^{n_{\alpha,r}} \prod_{i=1}^{r} \rho_{n_{\alpha,i},k_i}(x_i) f\Big(\frac{k_1}{b_{n_{\alpha,1}}}, \frac{k_2}{b_{n_{\alpha,2}}}, \dots, \frac{k_r}{b_{n_{\alpha,r}}}\Big),$$
$$(\alpha \in D,\ f \in C(X, E),\ x = (x_1, x_2, \dots, x_r) \in X),$$

where

$$\rho_{n_{\alpha,i},k_i}(x_i) = \binom{n_{\alpha,i}}{k_i} \left(a_{n_{\alpha,i}} x_i\right)^{k_i}$$
$$(\alpha \in D,\ i = 1, 2, \dots, r).$$

Theorem 3. *If*

$$a_{n_{\alpha,i}} = o(1)$$

for $i = 1, 2, \dots, r$, *then*

$$\lim_\alpha \|T_\alpha(f) - f\|_{X_0} = 0$$

for all $f \in BC(X, E)$.

Proof. Let

$$h_{n_{\alpha,i}} = 1_{[0,\infty)} \qquad (\alpha \in D,\ i = 1,2,\dots,r)$$

and

$$g_{n_{\alpha,i}}(t) = a_{n_{\alpha,i}} t h_{n_{\alpha,i}}(t) \qquad (\alpha \in D,\ i = 1,2,\dots,r)$$

for all $t \in [0,\infty)$. Then (19) reduces to (23). Since

$$b_{n_{\alpha,i}} = o(n_{\alpha,i}) \qquad (i = 1,2.\dots,r)$$

and

$$n_{\alpha,i} g_{n_{\alpha,i}}(t) = b_{n_{\alpha,i}} t h_{n_{\alpha,i}}(t) \qquad (\alpha \in D,\ i = 1,2,\dots,r)$$

for all $t \in [0,\infty)$, we use Lemma 4 and apply Theorem 2 to $F_\alpha = T_\alpha$.

Remark 2. (23) generalizes (2) to the r-dimensional Bernstein type rational vector-valued functions. Also, (3) can be extended by the following form to the r-dimensional case for vector-valued functions:

$$(24) \qquad R_\alpha(f)(x) = R_\alpha(E;\mathcal{B};\mathcal{A};f;x) = \prod_{i=1}^{r} \frac{1}{(1 + a_{n_{\alpha,i}} x_i)^{n_{\alpha,i}}}$$

$$\times \sum_{k_1=0}^{n_{\alpha,1}} \sum_{k_2=0}^{n_{\alpha,2}} \cdots \sum_{k_r=0}^{n_{\alpha,r}} \prod_{i=1}^{r} \binom{n_{\alpha,i}}{k_i} \left(a_{n_{\alpha,i}} x_i\right)^{k_i} f\left(b_{n_{\alpha,1},k_1}, b_{n_{\alpha,2},k_2}, \dots, b_{n_{\alpha,r},k_r}\right)$$

$$(\alpha \in D,\ f \in C(X,E),\ x = (x_1,x_2,\dots,x_r) \in X),$$

where

$$\mathcal{A} = \{a_{n_{\alpha,i}} : \alpha \in D,\ i = 1,2,\dots,r\}$$

and

$$\mathcal{B} = \{b_{n_{\alpha,i},k_i} : 0 \le k_i \le n_{\alpha,i},\ \alpha \in D,\ i = 1,2,\dots,r\}$$

are families of positive real numbers with

$$0 \le b_{n_{\alpha,i},0} < b_{n_{\alpha,i},1} < b_{n_{\alpha,i},2} < \cdots < b_{n_{\alpha,i},n_{\alpha,i}}$$

$$(\alpha \in D,\ i = 1,2,\dots,r).$$

In particular, the operator $L_n(f)(x)$ defined by (4) is generalized to the r-dimensional case for vector-valued functions defined as follows:

$$(25) \qquad L_\alpha(f)(x) = L_\alpha(E;f;x) = \prod_{i=1}^{r} \frac{1}{(1+x_i)^{n_{\alpha,i}}} \sum_{k_1=0}^{n_{\alpha,1}} \sum_{k_2=0}^{n_{\alpha,2}} \cdots \sum_{k_r=0}^{n_{\alpha,r}}$$

$$\prod_{i=1}^{r} \binom{n_{\alpha,i}}{k_i} x_i^{k_i} f\left(\frac{k_1}{n_{\alpha,1} - k_1 + 1}, \frac{k_2}{n_{\alpha,2} - k_2 + 1}, \dots, \frac{k_r}{n_{\alpha,r} - k_r + 1}\right)$$

$$(\alpha \in D,\ f \in C(X,E),\ x = (x_1,x_2,\dots,x_r) \in X).$$

3. Rates of convergence

Le $f \in B(X, E)$ and let $\delta \geq 0$. Then we define

$$\omega(f, \delta) = \omega(d; f, \delta) = \sup\{\|f(x) - f(y)\| : x, y \in X, d(x, y) \leq \delta\},$$

which is called the modulus of continuity of f. Obviously, $\omega(f, \cdot)$ is a monotone increasing function on $[0, \infty)$ and

$$\omega(f, 0) = 0, \quad \omega(f, \delta) \leq 2\|f\|_X.$$

Also, f is uniformly continuous on X if and only if

$$\lim_{\delta \to +0} \omega(f, \delta) = 0.$$

We give here a quantitative form of Theorem 1, in which we estimate the rate of convergence in terms of the modulus of continuity of f. For this we suppose that for each $\alpha \in D$,

$$\chi_\alpha(x, k) \geq 0$$

for all $(x, k) \in X \times Y_\alpha$ and that there exist constants $C \geq 1$ and $K > 0$ such that

$$\omega(f, \xi\delta) \leq (C + K\xi)\omega(f, \delta) \tag{26}$$

for all $\xi, \delta \geq 0$ and all $f \in B(X, E)$.

For the sufficient condition such that (26) holds with $C = K = 1$, see [17, Lemma 1] (cf. [18, Lemma 2.4]).

Let $\{\epsilon_\alpha\}_{\alpha \in D}$ be a net of positive real numbers.

Theorem 4. *Let $q \geq 1$. Then for all $f \in BC(X, E), x \in X$ and all $\alpha \in D$,*

$$\begin{aligned} \|U_\alpha(f)(x) - f(x)\| \leq{} & |W_\alpha(1_X)(x) - 1|\|f(x)\| \\ & + (CW_\alpha(1_X)(x) + Kc_\alpha(x; q))\omega(f, \epsilon_\alpha), \end{aligned} \tag{27}$$

where

$$c_\alpha(x; q) = \min\{\epsilon_\alpha^{-q}\mu_\alpha(x; q),\ \epsilon_\alpha^{-1}(W_\alpha(1_X)(x))^{1-1/q}\mu_\alpha(x; q)^{1/q}\}.$$

In particular, if $W_\alpha(1_X) = 1_X$ for all $\alpha \in D$, then (27) reduces to

$$\|U_\alpha(f)(x) - f(x)\| \leq (C + Kc_\alpha(x; q))\omega(f, \epsilon_\alpha)$$

and

$$c_\alpha(x; q) = \min\{\epsilon_\alpha^{-q}\mu_\alpha(x; q),\ \epsilon_\alpha^{-1}\mu_\alpha(x; q)^{1/q}\}.$$

Proof. Let $f \in BC(X,E)$ and $x \in X$. Then for all $\alpha \in D$ we have

$$
\begin{aligned}
(28) \quad \|U_\alpha(f)(x) - f(x)\| &\le \Big\| w_\alpha(x) \sum_{k \in Y_\alpha} \chi_\alpha(x,k)(f(\xi_\alpha(k)) - f(x)) \Big\| \\
&\quad + \Big\| \Big(w_\alpha(x) \sum_{k \in Y_\alpha} \chi_\alpha(x,k) - 1 \Big) f(x) \Big\| \\
&= J_\alpha^{(1)}(x) + J_\alpha^{(2)}(x),
\end{aligned}
$$

say. We have

$$J_\alpha^{(2)}(x) = |W_\alpha(1_X)(x) - 1| \|f(x)\|.$$

Also, in view of [18, Lemma 2.7], we obtain

$$(29) \qquad J_\alpha^{(1)}(x) \le (CW_\alpha(1_X)(x) + Kc_\alpha(x;q,\delta))\omega(f,\delta) \qquad (\delta > 0),$$

where

$$c_\alpha(x;q,\delta) = \min\{\delta^{-q}\mu_\alpha(x;q),\ \delta^{-1}(W_\alpha(1_X)(x))^{1-1/q}\mu_\alpha(x;q)^{1/q}\}.$$

Therefore, putting $\delta = \epsilon_\alpha$ in (29), (28) yields the desired inequality (27).

Let Φ be a nonnegative real-valued function on $X^2 := X \times X$ and we define

$$\mu_\alpha(\Phi;x) = \omega_\alpha(x) \sum_{k \in Y_\alpha} \chi_\alpha(x,k)\Phi(x,\xi_\alpha(k)) \qquad (\alpha \in D,\ x \in X),$$

which is called the Φ-moment of χ_α at x with respect to the weight function w_α (cf. [14] - [19]).

Next we suppose that there exist constants $q \ge 1$ and $\kappa > 0$ such that

$$(30) \qquad d^q(x,y) \le \kappa\Phi(x,y)$$

for all $(x,y) \in X^2$.

Remark 3. If

$$(31) \qquad \lim_\alpha \mu_\alpha(\Phi;x) = 0 \quad \text{uniformly in } x \in X_0,$$

then (30) gives (6). Therefore, by Theorem 1, (5) and (31) establish that (7) holds for every $f \in BC(X,E)$.

Theorem 5. *For all $f \in BC(X,E), x \in X$ and all $\alpha \in D$,*

$$
\begin{aligned}
(32) \qquad \|U_\alpha(f)(x) - f(x)\| &\le |W(1_X)(x) - 1| \|f(x)\| \\
&\quad + (CW_\alpha(1_X)(x) + K\zeta_\alpha(\Phi;x;q))\omega(f,\epsilon_\alpha),
\end{aligned}
$$

where

$$\zeta_\alpha(\Phi;x;q)$$

$$= \min\{\kappa\epsilon_\alpha^{-q}\mu_\alpha(\Phi; x),\ \kappa^{1/q}\epsilon_\alpha^{-1}(W_\alpha(1_X)(x))^{1-1/q}\mu_\alpha(\Phi; x)^{1/q}\}.$$

In particular, if $W_\alpha(1_X) = 1_X$ *for all* $\alpha \in D$*, then (32) reduces to*

$$\|U_\alpha(f)(x) - f(x)\| \le (C + K\zeta_\alpha(\Phi; x; q))\omega(f, \epsilon_\alpha)$$

and

$$\zeta_\alpha(\Phi; x; q) = \min\{\kappa\epsilon_\alpha^{-q}\mu_\alpha(\Phi; x),\ \kappa^{1/q}\epsilon_\alpha^{-1}\mu_\alpha(\Phi; x)^{1/q}\}.$$

Proof. By (30), we have

$$\mu_\alpha(x; q) \le \kappa\mu_\alpha(\Phi; x) \qquad (\alpha \in D,\ x \in X),$$

which implies $c_\alpha(x; q) \le \zeta_\alpha(\Phi; x; q)$. Therefore, the desired estimate (32) follows from (27).

In the rest of this section, let $1 \le p \le \infty$ be fixed and let

$$(X, d) = ([0, \infty)^r, d_p).$$

For each $q \ge 1$, we set

$$c(p, q, r) = \begin{cases} r^{q/p} & (1 \le p < \infty,\ p \ne q) \\ 1 & (1 \le p < \infty,\ p = q) \\ 1 & (p = \infty). \end{cases}$$

Theorem 6. *For all* $f \in BC(X, E), x \in X$ *and all* $\alpha \in D$*,*

$$\|U_\alpha(f)(x) - f(x)\| \le |W_\alpha(1_X)(x) - 1|\|f(x)\| + (W_\alpha(1_X)(x) + \tau_\alpha(x; q))\omega(f, \epsilon_\alpha), \tag{33}$$

where

$$\tau_\alpha(x; q) = \min\Big\{c(p, q, r)\epsilon_\alpha^{-q}\sum_{i=1}^{r} W_\alpha(|p_i(x)1_X - p_i|^q)(x),\ c(p, q, r)^{1/q}\epsilon_\alpha^{-1}\Big(\sum_{i=1}^{r} W_\alpha(|p_i(x)1_X - p_i|^q)(x)\Big)^{1/q}\Big\}.$$

In particular, if $W_\alpha(1_X) = 1_X$ *for all* $\alpha \in D$*, then (33) reduces to*

$$\|U_\alpha(f)(x) - f(x)\| \le (1 + \tau_\alpha(x; q))\omega(f, \epsilon_\alpha).$$

Proof. By [17, Lemma 1 (b)] (cf. [18, Lemma 2.4 (b)]), (26) holds with $C = K = 1$. Also, we have

$$d_p^q(x, y) \le c(p, q, r)\sum_{i=1}^{r} |p_i(x) - p_i(y)|^q \qquad (x, y \in \mathbb{R}^r).$$

Therefore for all $(x,y)\in X^2$, (30) holds with

$$\kappa = c(p,q,r), \quad \Phi(x,y)=\sum_{i=1}^{r}|p_i(x)-p_i(y)|^q,$$

and so by Theorem 5, we obtain the desired result.

Corollary 2. *For all $f\in BC(X,E), x\in X$ and all $\alpha\in D$,*

$$\|U_\alpha(f)(x)-f(x)\| \le |W_\alpha(1_X)(x)-1|\|f(x)\| + (W_\alpha(1_X)(x)+\tau_\alpha(x))\omega(f,\epsilon_\alpha), \tag{34}$$

where

$$\tau_\alpha(x)=\min\{c(p,r)\epsilon_\alpha^{-2}\mu_\alpha(x),\ \sqrt{c(p,r)}\epsilon_\alpha^{-1}\sqrt{\mu_\alpha(x)}\}$$

and

$$c(p,r)=\begin{cases} r^{2/p} & (1\le p<\infty,\ p\ne 2)\\ 1 & (p=2,\ \infty).\end{cases}$$

In particular, if $W_\alpha(1_X)=1_X$ for all $\alpha\in D$, then (34) reduces to

$$\|U_\alpha(f)(x)-f(x)\|\le(1+\tau_\alpha(x))\omega(f,\epsilon_\alpha).$$

Concerning the rate of convergence of the net $\{F_\alpha\}_{\alpha\in D}$ of the fractional interpolation type operators defined by (19), we get the following result.

Theorem 7. *For all $f\in BC(X,E), x=(x_1,x_2,\dots,x_r)\in X$ and all $\alpha\in D$,*

$$\|F_\alpha(f)(x)-f(x)\|\le(1+\eta_\alpha(x))\omega(f,\epsilon_\alpha), \tag{35}$$

where

$$\eta_\alpha(x)=\min\{c(p,r)\epsilon_\alpha^{-2}\theta_\alpha(x),\ \sqrt{c(p,r)}\epsilon_\alpha^{-1}\sqrt{\theta_\alpha(x)}\} \tag{36}$$

and

$$\begin{aligned}\theta_\alpha(x)=\sum_{i=1}^{r}&\frac{1}{b_{n_{\alpha,i}}^2(g_{n_{\alpha,i}}(x_i)+h_{n_{\alpha,i}}(x_i))^2}\\ &\times\Big(\big(b_{n_{\alpha,i}}x_i g_{n_{\alpha,i}}(x_i)\big)^2+n_{\alpha,i}g_{n_{\alpha,i}}(x_i)h_{n_{\alpha,i}}(x_i)\\ &+2b_{n_{\alpha,i}}x_i g_{n_{\alpha,i}}(x_i)\big(b_{n_{\alpha,i}}x_i h_{n_{\alpha,i}}(x_i)-n_{\alpha,i}g_{n_{\alpha,i}}(x_i)\big)\\ &+\big(b_{n_{\alpha,i}}x_i h_{n_{\alpha,i}}(x_i)-n_{\alpha,i}g_{n_{\alpha,i}}(x_i)\big)^2\Big).\end{aligned}$$

Proof. We use Lemma 1 (13) and apply Corollary 2 to $U_\alpha=F_\alpha$.

Corollary 3. *Let* $a_{n_{\alpha,i}}$ $(\alpha \in D,\ i = 1, 2, \ldots, r)$ *be as in (22). Then for all* $f \in BC(X,E), x = (x_1, x_2, \ldots, x_r) \in X$ *and all* $\alpha \in D$,

$$\|T_\alpha(f)(x) - f(x)\| \le (1 + \eta_\alpha(x))\omega(f, \epsilon_\alpha),$$

where $\eta_\alpha(x)$ *is given by (36) and*

$$\theta_\alpha(x) = \sum_{i=1}^{r} \frac{a_{n_{\alpha,i}}^2 x_i^4 + x_i / b_{n_{\alpha,i}}}{(1 + a_{n_{\alpha,i}} x_i)^2}. \tag{37}$$

Remark 4. Corollary 3 sharply extends and improves [4, Theorem 1] to the very general settings.

Concerning the rate of convergence of the net $\{R_\alpha\}_{\alpha \in D}$ of operators definded by (24), we obtain the following result.

Theorem 8. *For all* $f \in BC(X,E), x = (x_1, x_2, \ldots, x_r) \in X$ *and all* $\alpha \in D$,

$$\|R_\alpha(f)(x) - f(x)\| \le (1 + \gamma_\alpha(x))\omega(f, \epsilon_\alpha),$$

where

$$\gamma_\alpha(x) = \min\{c(p,r)\epsilon_\alpha^{-2}\nu_\alpha(x),\ \sqrt{c(p,r)}\epsilon_\alpha^{-1}\sqrt{\nu_\alpha(x)}\}$$

and

$$\nu_\alpha(x) = \sum_{i=1}^{r} \sum_{k_i=0}^{n_{\alpha,i}} \binom{n_{\alpha,i}}{k_i} \frac{(a_{n_{\alpha,i}} x_i)^{k_i}}{(1 + a_{n_{\alpha,i}} x_i)^{n_{\alpha,i}}} \left(x_i - b_{n_{\alpha,i},k_i}\right)^2.$$

Proof. Apply Corollary 2 to $U_\alpha = R_\alpha$.

Theorem 9. *For all* $f \in BC(X,E), x = (x_1, x_2, \ldots, x_r) \in X$ *and all* $\alpha \in D$,

$$\|L_\alpha(f)(x) - f(x)\| \le (1 + \zeta_\alpha(x))\omega(f, \epsilon_\alpha),$$

where

$$\zeta_\alpha(x) = \min\{c(p,r)\epsilon_\alpha^{-2}\psi_\alpha(x),\ \sqrt{c(p,r)}\epsilon_\alpha^{-1}\sqrt{\psi_\alpha(x)}\}$$

and

$$\psi_\alpha(x) = \sum_{i=1}^{r} \sum_{k_i=0}^{n_{\alpha,i}} \binom{n_{\alpha,i}}{k_i} \frac{x_i^{k_i}}{(1 + x_i)^{n_{\alpha,i}}} \left(x_i - \frac{k_i}{n_{\alpha,i} - k_i + 1}\right)^2. \tag{38}$$

Proof. Apply Theorem 8 to $R_\alpha = L_\alpha$.

Remark 5. By [7, Remark 3] (cf. [11, (6)]), we have the the following more explicit expression for the second (absolute) moment (38) of L_α:

$$\psi_\alpha(x) = \sum_{i=1}^{r} \frac{(x_i - n_{\alpha,i}) x_i^{n_{\alpha,i}+1}}{(1 + x_i)^{n_{\alpha,i}}} + \frac{x_i^{n_{\alpha,i}+1}}{(1 + x_i)^{n_{\alpha,i}}} \sum_{k_i=2}^{n_{\alpha,i}+1} \binom{n_{\alpha,i}+1}{k_i} \frac{x_i^{1-k_i}}{k_i - 1}.$$

Theorem 10. *For all $f \in BC(X), x = (x_1, x_2, \dots, x_r) \in X$ and all $\alpha \in D$,*

$$\|L_\alpha(f)(x) - f(x)\| \le (1 + \kappa_\alpha(x))\omega(f, \epsilon_\alpha), \tag{39}$$

where

$$\kappa_\alpha(x) = \min\{c(p,r)\epsilon_\alpha^{-2}\sigma_\alpha(x),\ \sqrt{c(p,r)}\epsilon_\alpha^{-1}\sqrt{\sigma_\alpha(x)}\}$$

and

$$\sigma_\alpha(x) = 4\sum_{i=1}^{r} \frac{x_i(1+x_i)^2}{n_{\alpha,i}}.$$

Proof. By [11, (6)], we have

$$\psi_\alpha(x) \le \sum_{i=1}^{r}\Big(\frac{x_i}{n_{\alpha,i}} + \frac{x_i^2(1+x_i)}{n_{\alpha,i}} + \frac{3x_i(1+x_i)^2}{n_{\alpha,i}}\Big) \le \sigma_\alpha(x).$$

Therefore, the desired result follows from Theorem 9.

Remark 6. By putting $\epsilon_\alpha\sqrt{\sigma_\alpha(x)}$ instead of ϵ_α in (39), we get the following inequality for all $f \in BC(X, E), x \in X$ and all $\alpha \in D$:

$$\|L_\alpha(f)(x) - f(x)\| \le (1 + \min\{c(p,r)\epsilon_\alpha^{-2}, \sqrt{c(p,r)}\epsilon_\alpha^{-1}\}) \times \omega\Big(f, 2\epsilon_\alpha\sqrt{\sum_{i=1}^{r}\frac{x_i(x_i+1)^2}{n_{\alpha,i}}}\Big). \tag{40}$$

In particular, if $p = 2, \infty$, then (40) reduces to

$$\|L_\alpha(f)(x) - f(x)\| \le (1 + \min\{\epsilon_\alpha^{-1},\ \epsilon_\alpha^{-2}\}) \times \omega\Big(f, 2\epsilon_\alpha\sqrt{\sum_{i=1}^{r}\frac{x_i(x_i+1)^2}{n_{\alpha,i}}}\Big),$$

which generalizes the estimate given by Khan [11, Theorem 1].

Remark 7. We set

$$M(x) = \max\{p_i(x)(1+p_i(x))^2 : i = 1, 2, \dots, r\} \qquad (x \in X).$$

Then (40) yields the following estimate for all $f \in BC(X, E), x \in X$ and all $\alpha \in D$:

$$\|L_\alpha(f)(x) - f(x)\| \le \Big(1 + \min\Big\{\frac{4c(p,r)M(x)}{\epsilon_\alpha^2},\ \frac{2\sqrt{c(p,r)}\sqrt{M(x)}}{\epsilon_\alpha}\Big\}\Big)\omega\Big(f, \epsilon_\alpha\sqrt{\sum_{i=1}^{r}\frac{1}{n_{\alpha,i}}}\Big), \tag{41}$$

which particularly reduces to

$$\|L_\alpha(f)(x) - f(x)\|$$
$$\leq \Big(1 + \min\Big\{\frac{4M(x)}{\epsilon_\alpha^2},\ \frac{2\sqrt{M(x)}}{\epsilon_\alpha}\Big\}\Big)\omega\Big(f, \epsilon_\alpha\sqrt{\sum_{i=1}^{r}\frac{1}{n_{\alpha,i}}}\Big)$$

if $p = 2, \infty$.

Remark 8. If

$$n_{\alpha,i} = n_\alpha \qquad (\alpha \in D,\ i = 1, 2, \dots, r),$$

where $\{n_\alpha\}_{\alpha \in D}$ is a net of natural numbers, then by (41) we obtain the following estimate for all $f \in BC(X, E), x \in X$ and all $\alpha \in D$:

(42)
$$\|L_\alpha(f)(x) - f(x)\|$$
$$\leq \Big(1 + \min\Big\{4rc(p,r)M(x),\ 2\sqrt{rc(p,r)}\sqrt{M(x)}\Big\}\Big)\omega\Big(f, \sqrt{\frac{1}{n_\alpha}}\Big).$$

In particular, if $p = 2, \infty$, then (42) reduces to

$$\|L_\alpha(f)(x) - f(x)\| \leq \Big(1 + \min\Big\{4rM(x),\ 2\sqrt{r}\sqrt{M(x)}\Big\}\Big)\omega\Big(f, \sqrt{\frac{1}{n_\alpha}}\Big).$$

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