The flat coordinates of universal unfoldings of E^^~_6 and E^^~_7.

Bulletin of the College of Science
University of the Ryukyus No.42:5-10 (1986)

The flat coordinates of universal unfoldings of $\widetilde{E}_6$ and $\widetilde{E}_7$.

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(Received 21 May 1986)

Abstract

Let $F(x, y, z, t) = t_1 - F_1$ be a universal unfolding of the simple elliptic singularities $\widetilde{E}_6, \widetilde{E}_7$, given as follows.

$$\widetilde{E}_6 : F_1 = \frac{1}{3}(x^3 + y^3 + z^3) - t_0 xyz - t_{10} yz - t_{11} zx - t_{12} xy - t_{20} x - t_{21} y - t_{22} z,$$

$$\widetilde{E}_7 : F_1 = \frac{1}{4}(x^4 + y^4) - \frac{1}{2} t_0 x^2 y^2 - t_{10} x^2 y - t_{11} xy^2 - t_{20} x^2 - t_{21} xy - t_{22} y^2 - t_{30} x - t_{31} y.$$

These are weighted homogeneous by giving x, y, z, t_{ij}, t_0 the weights 1, 1, 1, i, 0.

The primitive formes $\zeta^{(0)}$ can be expressed as $\zeta^{(0)} = [\frac{1}{u(t_0)} dxdydz]$, where u is any solution of the differential equation

$$(*) \quad \begin{array}{lll} \text{(case of } \widetilde{E}_6) & (Du')' - t_0 u = 0, & D = 1 - t_0^3, \\ \text{(case of } \widetilde{E}_7) & (Du')' - \frac{1}{4} u = 0, & D = 1 - t_0^2, \end{array}$$

The flat coordinates are given as follows:

$$\widetilde{E}_6 : \tau_0 = \int dt_0/(u^2 D)$$

$$\tau_{1i} = \frac{1}{u} D^{-2/3} t_{1i} \quad (i = 0,1,2)$$

$$\tau_{2i} = \frac{1}{u} D^{-1/3} (t_{2i} + \frac{1}{D}(\frac{1}{2} t_0 t_{1i}^2 + t_0^2 t_{1j} t_{1k})) \quad (\{i,j,k\} = \{0,1,2\})$$

$$\tau_3 = t_3 + \frac{u'}{u} \Sigma t_{1i} t_{2i} + \frac{1}{D}(3 t_0 \frac{u'}{u} + 1)(t_0 t_{10} t_{11} t_{12} + \frac{1}{6} \Sigma t_{1i}^3)$$

where t_3 stands for t_1 because the weight of t_1 is 3.

$$\widetilde{E}_7 : \tau_0 = \int dt_0/(u^2 D)$$

$$\tau_{1i} = \frac{1}{u} D^{-3/4} t_{1i} \quad (i = 0,1)$$

$$\tau_{2\pm} = \frac{1}{2u} D^{-1/2}((1 \pm t_0)^{1/2}(t_{22} \pm t_{20}) + \frac{1}{D}(1 \pm t_0)^{3/2}(\frac{1}{2} t_{10}^2 \pm \frac{1}{2} t_{11}^2))$$

$$\tau_{21} = \frac{1}{u} D^{-1/2}(t_{21} + \frac{2}{D} t_0 t_{10} t_{11})$$

$$\tau_{30} = \frac{1}{u} D^{-1/4}(t_{30} + \frac{1}{D}(t_{11} t_{22} + t_0 t_{10} t_{21} + t_0 t_{11} t_{20}) + \frac{1}{D^2}(\frac{5}{6} t_0 t_{11}^3 + (2 + 3 t_0^2) \frac{1}{2} t_{10}^2 t_{11}))$$

$$\tau_{31}=\frac{1}{u}D^{-1/4}(t_{31}+\frac{1}{D}(t_{10}t_{20}+t_0t_{11}t_{21}+t_0t_{10}t_{22})+\frac{1}{D^2}(\frac{5}{6}t_0t_{10}^2+(2+3t_0^2)\frac{1}{2}t_{11}^2t_{10}))$$

$$\begin{aligned}\tau_4 &= t_4+2\frac{u'}{u}(t_{30}t_{11}+t_{31}t_{10}+t_{20}t_{22}+\frac{1}{2}t_{21}^2)+(1+2t_0\frac{u'}{u})(\frac{1}{2}t_{20}^2+\frac{1}{2}t_{22}^2)\\ &+\frac{1}{D}(1+8t_0\frac{u'}{u})(\frac{1}{2}t_{11}^2t_{20}+\frac{1}{2}t_{10}^2t_{22}+t_{10}t_{11}t_{21})\\ &+\frac{1}{D}(t_0+2(3+t_0^2)\frac{u'}{u})(\frac{1}{2}t_{10}^2t_{20}+\frac{1}{2}t_{11}^2t_{22})\\ &+\frac{1}{D^2}(8t_0+2(9+23t_0^2)\frac{u'}{u})\frac{1}{4}t_{10}^2t_{11}^2\\ &+\frac{1}{D^2}(5+3t_0^2+2(29+3t_0^2)t_0\frac{u'}{u})(\frac{1}{4!}t_{10}^4+\frac{1}{4!}t_{11}^4)\end{aligned}$$

where t_4 stands for t_1.

For a special solution u of (*) M. Noumi has already got the same results (including $\tilde{E}_8$).

The theory and the methods of calculation of the primitive forms and the flat coordinates are wholly depends on K. Saito [1]. The author expresses his thanks to K. Saito.

1. Notations

1.1 Let F, F_1 be as in Abstract, S be the parameter space of the universal unfolding F, T be the guotient space of S such that $\mathcal{O}_T=\{g\in\mathcal{O}_S:\partial g/\partial t_1=0\}$. We put $X=\mathbf{C}^3{}_{XYZ}\times T$. The coordinates of T are simply denoted by t', that of S is (t_1, t'). $\partial/\partial t_1$ called the primitive vector field is always fixed.

$\phi: X\to S: \phi(x, y, z, t')=(F_1(x,y,z,t'), t')$, $\pi: S\to T$ the projection.

$$\mathscr{H}_F^{(0)}=\phi_*\Omega^3_{X/T}/dF_1\wedge d\phi_*\Omega^1_{X/T}$$

$$\mathscr{H}_F^{(-1)}=\phi_*\Omega^2_{X/T}/dF_1\wedge\phi_*\Omega^1_{X/T}+d(\phi_*\Omega^1_{X/T})$$

$\mathscr{H}_F^{(0)}$, $\mathscr{H}_F^{(-1)}$ are $\mathcal{O}_S$-free modules of rank $\mu(=8$ for $\tilde{E}_6$, 9 for $\tilde{E}_7)$.
$\mathscr{H}_F^{(-1)}$ may be identified with the subset $\mathscr{H}_F^{(-1)}\wedge dF_1$ of $\mathscr{H}_F^{(0)}$.
An element ω of $\mathscr{H}_F^{(0)}$ is represented by $\omega=[\psi(x, y, z, t')\underline{dx}]$ $(\underline{dx}=dxdydz)$.
Put $\partial_i=\partial/\partial t_i$, and $\mathscr{G}=\Sigma\,\mathcal{O}_T\partial_i$.

1.2 The Gauss-Manin connection ∇

Let $\delta_1=\partial/\partial t_1$, $\omega=[a\partial F_1/\partial x_i dx]$ then $\nabla: Der_S\times H_F^{(-1)}\ \mathscr{H}_F^{(0)}$ is defined by

(1.2.1) $\nabla_{\delta_1}\ \omega$(simply denoted by $\delta_1\omega) = [\frac{\partial a}{\partial X_i}dx]$,

(1.2.2) $\nabla_{\partial_j}\omega$(simply $\partial_j\omega) = [(\frac{\partial a}{\partial t_j}\frac{\partial F_1}{\partial x_i} - \frac{\partial F_1}{\partial t_i}\frac{\partial a}{\partial x_i})dx]$.

Then we have

(1.2.3) $\nabla_{\partial_j}\nabla_{\delta_1}^{-1}[adx] = \nabla_{\delta_1}^{-1}[\partial_j adx] - [a\partial_j F_1 dx]$.

1.3 Higher residues $K^{(k)}$

$\mathcal{O}_T$ – bilinear forms $K^{(k)}: \pi_* \mathscr{H}_F^{(0\leq)} \times \pi_* \mathscr{H}_F^{(0)} \to \mathcal{O}_T$ $k = 0, 1, 2, \cdots$ satisfies

(1.3.1) $K^{(k)}$ is symmetric (skew-symmetric) if k is even (odd),

(1.3.2) $K^{(0)}([adx],[bdx]) = Res_{X/T}\begin{bmatrix} ad\ dx \\ \frac{\partial F_1}{\partial x}\frac{\partial F_1}{\partial y}\frac{\partial F_1}{\partial z}\end{bmatrix}$

(1.3.3) $K^{(K)}(\nabla_{\delta_1}^{-1}\omega_1, \omega_2) = K^{(k-1)}(\omega_1, \omega_2)$, $K^{(k)}(\omega_1, \nabla_{\delta_1}^{-1}\omega_2) = -K^{(k-1)}(\omega_1, \omega_2)$,

(1,3.4) $\delta K^{(k)}(\omega_1, \omega_2) = K^{(k+1)}(\nabla_\delta\nabla_{\delta_1}^{-1}\omega_1, \omega_2) - K^{(k+1)}(\omega_1, \nabla\hat{\delta}\nabla_{\delta_1}^{-1}\omega_2)$.

1.4. Primitive forms

For $\tilde{E}_6$, $\tilde{E}_7$, $\zeta^{(0)} = [\frac{1}{u(t_0)}dx]$ is a primitive form if

(1.4.1) $K^{(2)}(\partial_0^2\zeta^{(-2)}, \partial_0\zeta^{(-1)}) = 0$, where $\zeta^{(-k)} = \nabla_{\delta_1}^{-k}\zeta^{(0)}$.

1.5 ∇

A torsion free and integrable connection $\nabla: Der_T \times \mathscr{G} \to \mathscr{G}$ is defined by

(1.5.1) $\nabla_\delta\delta_i = \delta_j$ *if* $K^{(1)}(\delta\delta_i\zeta^{(-2)}, \delta'\zeta^{(-1)}) = K^{(0)}(\delta_j\zeta^{(-1)}, \delta'\zeta^{(-1)})$

for all δ' in $\mathscr{G}$

1.6 Flat coordinate system

There is a coordinate system $\tau_1, \tau_2, \cdots, \tau_\mu$ of S (fixing δ_1), called a flat coordinate system, such that $ker\nabla = \Sigma C\frac{\partial}{\partial\tau_i}$.

2 $\tilde{E}_6$

2.1 Diffrential equation

By (1.2.2), we get

(2.1.1) $\partial_0\zeta^{(-1)} = \delta_1^{-1}[-\frac{u'}{u^2}dx] + [\frac{xyz}{u}dx]$,

(2.1.2) $\partial_0^2\zeta^{(-2)} = \delta_1^{-2}[(-\frac{u''}{u^2} + 2\frac{(u')^2}{u^3})dx] + 2\delta_1^{-1}[-\frac{u'}{u^2}xyzdx] + [\frac{(xyz)^2}{u}dx]$.

Then from (1.4.1) we get

(2.1.3) $u'K'^{(0)}(xyzdx, dx) - u'K^{(1)}((xyz)^2dx, dx) - uK^{(2)}((xyz)^2dx, xyzdx)$.

Since

(2.1.4) $x^2y=\frac{1}{D}(yF_{1x}+t_0zF_{1y}+t_0^2xF_{1z}),\ x^3=xF_{1x}-t_0xyz,$

(1.2.1) implies

(2.1.5) $[x^2ydx]=0,\ [x_i^2x_jdx]=0$ if $i\neq j$,

(2.1.6) $[x^2yzdx]=\frac{1}{D}t_0^2\delta_1^{-1}[xdx].$

(2.1.7) $[(xyz)^2dx]=\frac{1}{D}(3t_0^2\delta_1^{-1}[xyzdx]+t_0\delta_1^{-2}[dx]).$

Substituting these to (2.1.3) and deviding by $K^{(0)}(xyzdx, dx)=\frac{1}{D}$ we get the equation (*).

2.2 $ker\nabla$

Since the weight of $K^{(-1)}(\delta\zeta^{(-1)}, \delta'\zeta^{(-1)})$ is non-positive,

(2.2.1) $\delta_1(=\partial/\partial t_3)\in ker\nabla$.

Put $\nabla_{\partial 0}\partial_{20}=\Sigma\alpha_i\partial_{2i}+\beta\delta_1$.

Let $m_2=xy$ or yz or zx. Then by the definition of ∇ (1.5.1), $K^{(1)}(\partial_0\partial_{20}\zeta^{(-2)}, m_2dx)=K^{(0)}(\frac{1}{u}(\alpha_0x+\alpha_1y+\alpha_2z)dx, m_2dx)$.

By (1.2.3), (1.2.1), (2.1.6), $\partial_0\partial_{20}\zeta^{(-2)}=\delta_1^{-1}[-\frac{u'}{u^2}xdx]+[\frac{1}{u}x^2yzdx]$
$=(-\frac{u'}{u^2}+\frac{1}{uD}t_0^2)\delta_1^{-1}[xdx]$.

By (2.1.5), we get $\alpha_0=-\frac{u'}{u}+\frac{1}{D}t_0^2,\ \alpha_1=\alpha_2=0.$

(2.2.2) $\nabla_{\partial 0}\partial_{2i}=(-\frac{u'}{u}+\frac{1}{D}t_0^2)\partial_{2i}.$

Assume that $\delta=\Sigma a_i\partial_{2i}+b\delta_1$ is in $ker\nabla$. Then by $\nabla_{\partial 0}\ \delta=0$ and by (2.2.2), we get $a_i'=a_i(\frac{u'}{u}+\frac{1}{D}t_0^2)$. Hence

(2.2.3) $\delta_{2i}\equiv uD^{1/3}\partial_{2i}$ (mod δ_1) is in $ker\nabla$.

By the same way, we get the followings:

(2.2.4) $\nabla_{\partial 1i}\partial_{1i}=\frac{1}{D}t_0\partial_{2i},\ \nabla_{\partial 1i}\partial_{1j}=\frac{1}{D}t_0^2\partial_{2k},$

(2.2.5) $\nabla_{\partial 0}\partial_{1i}=(-\frac{u'}{u}+\frac{2}{D}t_0^2)\partial_{1i}+\frac{1}{D^2}(1+t_0^3)t_{1i}\partial_{2i}+\frac{2}{D^2}t_0(t_{1j}\partial_{2k}+t_{1k}\partial_{2j}).$

(here we use the equality

$K^{(1)}(x^2y^2zdx, xydx)=\frac{1}{D^2}(1+5t_0^3)t_{12}K^{(0)}(xydx, dx),$

obtained by

$\partial_{12}K^{(1)}(x^2y^2zdx, xydx)=K^{(2)}(x^3y^3zdx, xydx)-K^{(2)}(x^2y^2zdx, x^2y^2dx),$

$[x^3y^3zdx]=\frac{1}{D^2}\nabla_{\delta_1}^{-2}[z+3t_0^3z],[x^2y^2zdx]=\frac{1}{D}\nabla_{\delta_1}^{-1}[2t_0^2xydx],[x^2y^2dx]=\frac{1}{D}\nabla_{\delta_1}^{-1}[t_0zdx],$

and so on).

(2.2.6) $\delta_{1i}\equiv uD^{2/3}(\partial_{1i}-\frac{1}{D}t_0t_{1i}\partial_{2i}-\frac{1}{D}t_0^2(t_{1j}\partial_{2k}+t_{1k}\partial_{2j}))\ (mod\ \delta_1)$

are in $ker\nabla$.

(2.2.7) $\delta_0\equiv u^2D(\partial_0+\Sigma(\frac{u'}{u}-\frac{2}{D}t_0^3)t_{1i}\partial_{1i}$

$+\Sigma((\frac{u'}{u}-\frac{1}{D}t_0^3)t_{2i}-\frac{1}{D}(\frac{u'}{u}t_0+1)\frac{1}{2}t_{1i}^2-\frac{1}{D}(\frac{u'}{u}t_0^2+2t_0)t_{1j}t_{1k})\partial_{2i})$

(mod δ_1)
is in $ker\nabla$.

2.3 Orthonormality of $\{\delta_i\}$

From the equalities

$\delta K^{(0)}(\delta_i\zeta^{(-1)},\delta_j\zeta^{(-1)})$

$=K^{(0)}(\nabla_\delta\delta_i\zeta^{(-1)},\delta_j\zeta^{(-1)})+K^{(0)}(\delta_i\zeta^{(-1)},\nabla_\delta\delta_j\zeta^{(-1)})=0,$

$K^{(0)}(\delta_i\zeta^{(-1)},\delta_j\zeta^{(-1)})$ are constant. From (2.1.5) we get

(2.3.1) $K^{(0)}(\delta_0\zeta^{(-1)},\delta_1\zeta^{(-1)})=1,\ K^{(0)}(\delta_{1i}\zeta^{(-1)},\delta_{2i}\zeta^{(-1)})=1,$

and other $K^{(0)}(\delta_i\zeta^{(-1)},\delta_j\zeta^{(-1)})$ are 0.

2.4 Flat coordinate system

For each δ in $ker\nabla$, $\omega_\delta=\Sigma K^{(0)}(\delta\zeta^{(-1)},\partial_i\zeta^{(-1)})dt_i$ is (independent of the coordinate system of S fixing δ_1) closed by the torsion freeness of ∇. Hence we obtain a coordinate system τ such that

(2.4.1) $d\tau_0=\omega_{\delta_1},\ d\tau_{1i}=\omega_{\delta_{2i}},\ d\tau_{2i}=\omega_{\delta_{1i}},\ d\tau_3=\omega_{\delta_0},$

integrating these, we get τ listed in the Abstract.
Since $\omega_\delta=\Sigma K^{(0)}(\delta\zeta^{(-1)},\partial/\partial\tau_i\zeta^{(-1)})d\tau_i$, we know $\delta_i=\partial/\partial\tau_i$.

3 $\tilde{E}_7$

For $\tilde{E}_7$, the following results are obtained similarly as in the case of $\tilde{E}_6$.

3.1

(1.4.1) implies the differential equation (*).

3.2 $ker\nabla$

δ_1 and the following vector fields are in $ker\nabla$.

$\delta_{3i}\equiv uD^{1/4}\partial_{3i}$ (mod δ_1) $i=0,1,$

$$\delta_{2\pm} \equiv uD^{1/2}((1 \pm t_0)^{-1/2}(\partial_{20} \pm \partial_{22}) - \frac{1}{D}(1 \pm t_0)^{1/2}(t_{10}\partial_{31} \pm t_{11}\partial_{30})) \pmod{\delta_1},$$

$$\delta_{21} \equiv uD^{1/2}(\partial_{21} - \frac{1}{D} t_0 (t_{10}\partial_{30} + t_{11}\partial_{31})) \pmod{\delta_1},$$

$$\delta_{10} \equiv uD^{3/4}(\partial_{10} - \frac{1}{D} t_0 t_{10}\partial_{20} - \frac{2}{D} t_0 t_{11}\partial_{21} - \frac{1}{D} t_{10}\partial_{22} - (\frac{1}{D} t_0 t_{21} + \frac{1}{D^2} t_{10} t_{11})\partial_{30}$$
$$- (\frac{1}{D} t_{20} + \frac{1}{D} t_0 t_{22} + \frac{1}{D^2} t_0 \frac{1}{2} t_{10}^2 + \frac{1}{D^2}(1 - t_0^2)\frac{1}{2} t_{11}^2)\partial_{31}) \pmod{\delta_1},$$

$$\delta_{11} \equiv uD^{3/4}(\partial_{11} - \frac{1}{D} t_0 t_{11}\partial_{22} - \frac{2}{D} t_0 t_{10}\partial_{21} - \frac{1}{D} t_{11}\partial_{20} - (\frac{1}{D} t_0 t_{21} + \frac{1}{D^2} t_{11} t_{10})\partial_{31}$$
$$- (\frac{1}{D} t_{22} + \frac{1}{D} t_0 t_{20} + \frac{1}{D^2} t_0 \frac{1}{2} t_{11}^2 + \frac{1}{D^2}(1 - t_0^2)\frac{1}{2} t_{10}^2)\partial_{30}) \pmod{\delta_1},$$

$$\delta_0 \equiv u^2 D(\partial_0 + (\frac{u'}{u} - \frac{6}{D} t_0)(t_{10}\partial_{10} + t_{11}\partial_{11})$$
$$+ ((\frac{u'}{u} - \frac{2}{D} t_0) t_{20} - \frac{1}{D} t_{22} - \frac{1}{D}(3 + 2\frac{u'}{u} t_0)\frac{1}{2} t_{10}^2 - \frac{u'}{2uD} t_{11}^2))\partial_{20}$$
$$+ ((\frac{u'}{u} - \frac{4}{D} t_0) t_{21} - \frac{4}{D}(1 + \frac{u'}{u} t_0) t_{10} t_{11})\partial_{21}$$
$$+ ((\frac{u'}{u} - \frac{2}{D} t_0) t_{22} - \frac{1}{D} t_{20} - \frac{1}{D}(3 + 2\frac{u'}{u} t_0)\frac{1}{2} t_{11}^2 - \frac{u'}{2uD} t_{10}^2))\partial_{22}$$
$$+ ((\frac{u'}{u} - \frac{2}{D} t_0) t_{30} - \frac{1}{D}(1 + 2 t_0 \frac{u'}{u}) t_{20} t_{11} - \frac{2}{D}(1 + t_0 \frac{u'}{u}) t_{21} t_{10} - \frac{u'}{uD} t_{22} t_{11}$$
$$- \frac{1}{D^2}(2 t_0 + (3 + 4 t_0^2)\frac{u'}{u})\frac{1}{2} t_{10}^2 t_{11} - \frac{1}{D^2}(1 + 8 t_0 \frac{u'}{u})\frac{1}{6} t_{11}^3)\partial_{30}$$
$$+ ((\frac{u'}{u} - \frac{2}{D} t_0) t_{31} - \frac{1}{D}(1 + 2 t_0 \frac{u'}{u}) t_{22} t_{10} - \frac{2}{D}(1 + t_0 \frac{u'}{u}) t_{21} t_{11} - \frac{u'}{uD} t_{20} t_{10}$$
$$- \frac{1}{D^2}(2 t_0 + (3 + 4 t_0^2)\frac{u'}{u})\frac{1}{2} t_{11}^2 t_{10} - \frac{1}{D^2}(1 + 8 t_0 \frac{u'}{u})\frac{1}{6} t_{10}^3)\partial_{31}) \pmod{\delta_1}.$$

3.3 Orthogonality of $\{\delta_i\}$

$K^{(0)}(\delta_1 \zeta^{(-1)}, \delta_0 \zeta^{(-1)}) = 1$, $K^{(0)}(\delta_{3i} \zeta^{(-1)}, \delta_{1j} \zeta^{(-1)}) = 1$ $\{i, j\} = \{0, 1\}$,
$K^{(0)}(\delta_{2\pm} \zeta^{(-1)}, \delta_{2\pm} \zeta^{(-1)}) = \pm 2$, $K^{(0)}(\delta_{21} \zeta^{(-1)}, \delta_{21} \zeta^{(-1)}) = 1$,
and other $K^{(0)}(\delta_i \zeta^{(-1)}, \delta_j \zeta^{(-1)})$ are 0.

3.4 Flat coordinate system

Let ω_δ be as in 2.4. Then we obtain a coordinate system τ such that $d\tau_0 = \omega_{\delta_1}$, $d\tau_{1i} = \omega_{\delta_{3j}}$, $d\tau_{2\pm} = \frac{1}{2}\omega_{\delta_{2\pm}}$, $d\tau_{21} = \omega_{\delta_{21}}$, $d\tau_{3i} = \omega_{\delta_{1j}}$,

where $\{i, j\} = \{0, 1\}$. Then $\partial/\partial\tau_i = \delta_i$, and we get τ_i listed in the Abstract.

Reference

1) Saito, K.,: Primitive forms for a universal unfolding of a function with an isolated critical point, J. Fac. Sci. Univ. Tokyo, Sec. IA, 28(3), 775-792(1982).