The Simplest Equivalent Circuit of a Multi-Terminal Network

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Abstract

The Helmholtz-Thévenin theorem together with its dual equivalent Mayer-Norton's one is generalized to that of n-terminal networks. The new theorem asserts that any n-terminal network can be reduced to a set of the equivalent circuits which consist of $(n-1)(n-2)/2$ impedances and $(n-1)$ active 2-terminal elements(Helmholtz-Thévenin's or Mayer-Norton's equivalent circuits). Their graph is the complete one, and the active elements are connected to each other so that they make a tree. The number of the possible equivalent circuits is n^{n-2} for an n-terminal network, if we do not distinguish a Helmholtz-Thévenin's circuit from Mayer-Norton's one.

Introduction

It is well known that any two-terminal network has a simple equivalent circuit given by the Helmholtz-Thévenin theorem. Such a circuit has a special significance in the successive analysis of a cascade connection of two ports[1]. The circuit as well as its dual equivalent one given by Mayer-Norton's theorem consists of two components, an impedance and an electric source. Since neither component can be omitted to simulate the network completely, these theorems are regarded as to give the simplest equivalent circuits to a two-terminal network. Such consideration naturally leads to a question upon the simplest equivalent circuit for a network which has more terminals than two. Of course, this classical problem must have been long since investigated deeply, but the results seem to be scattered in the literature[2,3]. Then a systematic up-to-date treatment is given here.

Definitions

An n-terminal network is defined as a circuit which has n terminals and contains some linear elements (R, C and L) and some independent electric sources. As in Fig. 1, we can choose one terminal as the ground and measure all the voltages and currents from this node. So there are $(n-1)$ voltages and $(n-1)$ currents which determine the state of the circuit. Our purpose is to get a circuit which is equivalent to such an n-terminal network as to the voltages and currents at the terminals and is as simple as possible.

The choice of components of the equivalent circuit needs careful consideration. Impedances and electric sources may seem natural and inevitable components at first sight. However such a choice brings the following inconvenience. If we apply it to a two-terminal network, Helmholtz-Thévenin's theorem gives an equivalent circuit with three nodes while Mayer-Norton's theorem gives one with two nodes as in Fig. 2. Then Mayer-Norton's circuit is regarded as simpler than Helmholtz-Thévenin's one as to the nodes. Therefore the above choice violates the equivalence of the two circuits which should be

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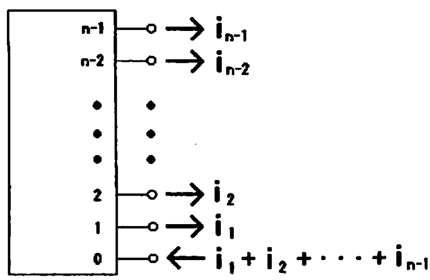

Fig. 1 An n-terminal network.

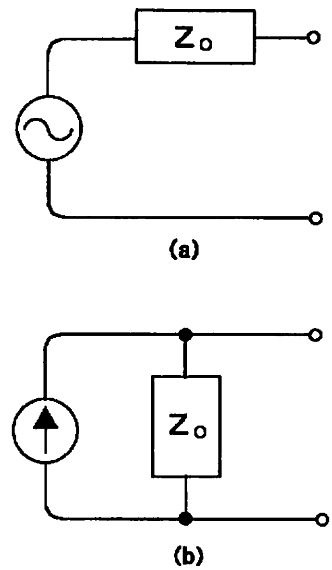

Fig. 2 Source elements.
(a) Helmholtz-Thevnin's element.
(b) Mayer-Norton's element.

treated alternatively to each other. To avoid this disadvantage, let us adopt these two circuits themselves as a constituent element and neglect their inner structures. This choice is not artificial but based on the nature of the actual source. If there were any source without accompanying an impedance, it could supply an infinite energy to a load. Therefore a sole source is an ideal element which can never exist physically. A set of a source and an impedance should be always treated as an inseparable component. Since our definition does not take the inner node in the Helmholtz-Thévenin's circuit into account, both circuits have two nodes at their ends. We make no distinction between the two circuits and call either circuit a source element. Since the other kind of components is just an impedance, source elements and impedances are the components of our equivalent circuit hereafter.

The use of physically realizable sources instead of ideal ones should be also applied to the given circuit, because a solitary source often brings an inconvenient but not essential restriction to a systemmatic analysis of the circuit. So we assume that in the given circuit every voltage or current source has an impedance in series or in parallel respectively between its two nearest terminals.

Reduction of Nodes and Branches

In order to get a simplified equivalent circuit we should reduce the nodes and branches as many as possible. Each step of the reduction can be performed with either one of the following procedures.

(1) Unifying a series connection, which decreases a node and a branch.
(2) Unifying a parallel connection, which decreases a branch.
(3) Applying a generalized Y-Δ (star-mesh) transformation, which decreases a node at a cost of increasing some branches unless the original branches are less than 4.
(4) Applying a generalized Δ-Y(mesh-star)transformation, which decreases some branches at a cost of increasing a node unless the original branches are less than 4.

Since (1) and (2) have no counteraction, they should be done by all means. The procedure (3) must have priority over (4), because (3) is always applicable to any star, while (4) is only possible to the mesh which satisfies a certain special condition[4,5]. Let us apply the procedures (1), (2) and (3) as far as there remains a node or a branch parallel to another branch within the circuit. Then we will finally get a graph with exactly one branch connecting each pair of terminals and no inner nodes. Such a graph is called a complete graph and is obviously the simplest with regard to the nodes, because there are no nodes other than the given terminals. We can not give assurance that the circuit is also the simplest with regard to the branches, because they may be reduced further at a cost of adding inner nodes. However since a generalized Δ-Y transformation is impossible in general, such a reduction of branches is possible only in the case when the special relation holds among the elements given. Thus the complete graph is usually the simlest one as to the branches too. So we regard the complete graph as the most favorable one and always choose it as our final object.

Reduction of Sources

Even after we have obtained the complete graph, there remains another possibility of reduction. Some electric sources may be eliminated.

According to our assumption, a source does not connect the terminals solely. We can expect that this condition is kept after all the equivalent transformations. Then any voltage source can be transformed into a current source and vice versa. Let all the sources be current sources for the convenience of discussion.

If any current sources make a loop, at least one can be eliminated with the following process. We may add a set of the identical current sources along a loop, with each source parallel to each branch, which gives no influences on the other part of the circuit[5]. It means that we can eliminate any one source by putting the sources with the same value inversely to all the branches in parallel as in Fig. 3. Therefore we can always cut every loop open, if any. If we trim the loops completely, we wll reach the final circuit that has no loop of the sources. In the graph theory, a set of branches with no loop is said to make a tree.

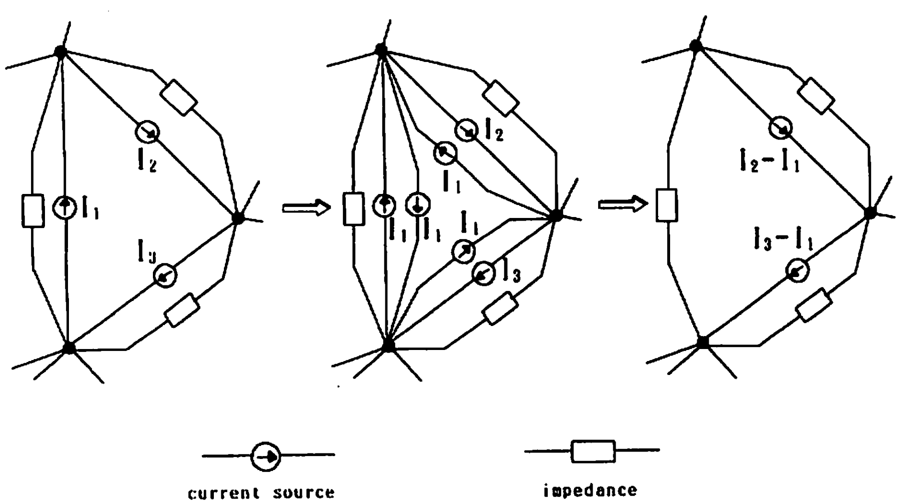

Fig. 3 An example of reducible current source.

A tree of a complete n-terminal graph is composed of $(n-1)$ branches. Hence our final equivalent circuit contains only $(n-1)$ sources that make a tree. Such sources are contained in source elements according to our convention. Since each source element has two constants, the total number of the constants in source elements is $2(n-1)$. All the other branches may be made of impedances and their number is $n(n-1)/2-(n-1)=(n-2)(n-1)/2$. The total number of the characteristic parameters is $2(n-1)+(n-2)(n-1)/2=(n+2)(n-1)/2$. Let us call such a circuit a standard equivalent circuit.

Thus we have reached the result which can be regarded as an expansion of Helmholtz-Thévenin's theorem and Mayer-Norton's one. It should be noted that the standard equivalent circuit is not unique in general but has an arbitrariness of choosing a tree of sources. The number of the standard equivalent circuits is n^{n-2} for the complete graph of n-nodes according to Cayley's theorem[6]. If we apply the result to a 2-terminal case, we get a unique circuit that contains a single branch composed of a source element, as is expected.

Characteristic Equations and their reduction

Let us consider how each component of a standard equivalent circuit can be obtained from those of the given one. Suppose that the original circuit has b branches and m nodes which include n terminals. The voltages and the currents of all the branches are independent variables which can specify a state of the circuit. Among these 2b variables, there hold $(m-1)$ equations by KCL and $(b-m+1)$ ones by KVL. In addition, the voltage and the current of each branch are connected by Ohm's law, or either value is given directly when the element is a source. They give other b equations. Thus the total number of the equations is $(m-1)+(b-m+1)+b=2b$ which is equal to that of the variables. We will call these equations the inner equations.

Let us assume that each terminal is newly connected to the ground through some arbitrary element. These $(n-1)$ outer elements are introduced only for the purpose of completing the set of the equations and their characters have no influences on the following procedure. When we add a new branch between some two nodes, a new loop equation is added. So we get $(n-1)$ new equations on the whole. We also must change $(n-1)$ node equations into those which contain the outer current. Thus the following two vector equations are obtained in general.

$$\begin{bmatrix} \quad A \quad \end{bmatrix} \begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} \quad B \quad \end{bmatrix} \begin{bmatrix} \\ x \\ \\ \end{bmatrix} + \begin{bmatrix} p \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} \quad C \quad \end{bmatrix} \begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} \quad D \quad \end{bmatrix} \begin{bmatrix} \\ x \\ \\ \end{bmatrix} + \begin{bmatrix} q \end{bmatrix} \tag{2}$$

Equations (1) and (2) correspond to the set of revised inner equations and that of new loop equations respectively. Here y is a vector whose components are 2b inner variables, and **x** is one whose components are (n−1) terminal variables, while the others are either a constant matrix or a constant vector. The size of **A**, **B**, **C**, **D**, **p** and **q** are $2b \times 2b$, $2b \times 2n$, $(n-1) \times 2b$, $(n-1) \times 2n$, $2b \times 1$ and $(n-1) \times 1$ respectively.

Let us multiply both ends of Eq. (1) by the inverse matrix of **A**, and solve the equation for **y**. Substituting the value of **y** into Eq. (2), we get the following equation for **x** only.

$$(\mathbf{C}\mathbf{A}^{-1}\mathbf{B} - \mathbf{D}) \cdot \mathbf{x} = \mathbf{q} - \mathbf{C}\mathbf{A}^{-1}\mathbf{p} \tag{3}$$

Let us rewrite it as follows.

$$\begin{bmatrix} & K & \end{bmatrix} \begin{bmatrix} \\ x \\ \\ \end{bmatrix} = \begin{bmatrix} r \end{bmatrix} \tag{4}$$

Here we have defined as $\boldsymbol{K} = \mathbf{C}\mathbf{A}^{-1}\mathbf{B} - \mathbf{D}$, and $\boldsymbol{r} = \boldsymbol{q} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{p}$, whose size are $(n-1) \times 2(n-1)$, and $(n-1) \times 1$ respectively. This equation should be called a terminal equation.

Further we can simplify the equation by solving it for any half of the variables. For the convenience of the discussion, we choose the following equation,

$$\begin{bmatrix} U & L \end{bmatrix} \begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} s \end{bmatrix} \tag{5}$$

where **U** is an identity matrix, and **I** and **V** are the terminal currents and voltages respectively. It can be rewritten as

$$\begin{bmatrix} I \end{bmatrix} + \begin{bmatrix} & L & \end{bmatrix} \begin{bmatrix} V \end{bmatrix} = \begin{bmatrix} s \end{bmatrix} \tag{6}$$

The matrix **L** is symmetric because of the reciprocity theorem. So among its $(n-1) \times (n-1)$ components only the $n(n-1)/2$ ones are independent. Since **s** has (n−1) components, the total number of the independent constants in Eq. (6) is $n(n-1)/2 + (n-1) = (n+2)(n-1)/2$ which agrees with that of a standard equivalent circuit.

Determination of the Constants of a Standard Equivalent Circuit

Now we can determine every value of the components of any standard equivalent circuit by comparing it with the corresponding constant in Eq. (6). For instance, we get the followng values when the active elements are made of Mayer-Norton's circuits.

$$Z_{ij} = -1/L_{ij} \quad (i \neq 0,\ i \neq j) \tag{7}$$

$$Z_{0i} = 1/\sum_{j} L_{ij} \tag{8}$$

$$\sum_{j=0\,(j \neq i)}^{n-1} I_{ij} = s_i \tag{9}$$

where Z_{ij} and I_{ij} are an impedance and a current source respectively between the terminal i and j. The value of I_{ij} is defined along the direction from j to i and therefore $I_{ij}=-I_{ji}$ holds.

When the active elements are composed of Helmholtz-Thevenin's circuits, we have only to replace Eq. (9) to the following.

$$\sum_{j=0(j\neq i)}^{n-1} E_{ij}/Z_{ij} = s_i \tag{10}$$

where E_{ij} is a voltage source between the terminal i and j defined along the direction from j to i.

Thus we got a standard equivalent circuit explicitly.

Examples

As was stated previously, the standard equivalent circuit is not unique but has an arbitrariness of choosing a tree of sources. The number of the standard equivalent circuits is n^{n-2} for the complete graph of n-nodes according to Cayley's theorem[6]. If we apply the conclusion to a 3-terminal network and a 4-terminal one, the types shown in Fig. 4 are obtained as their standard equivalent circuits.

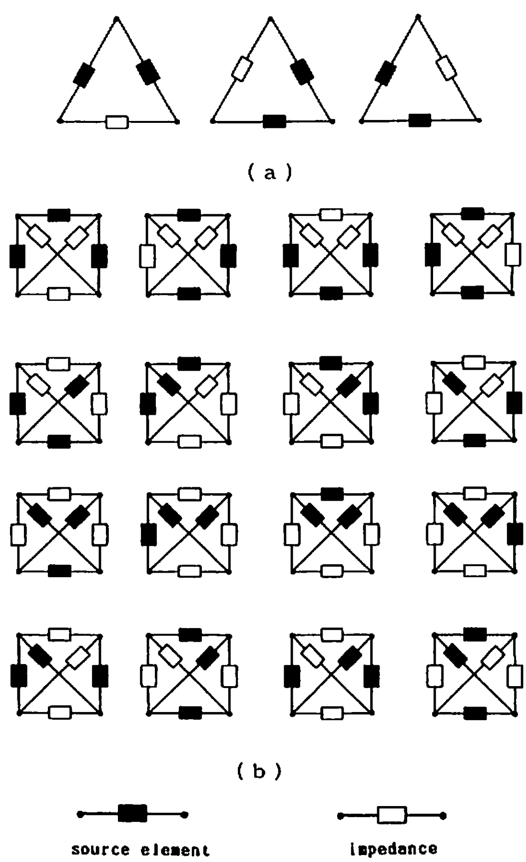

Fig. 4 Examples of standard equivalent circuits.
(a) 3-terminal circuits.
(b) 4-terminal circuits.

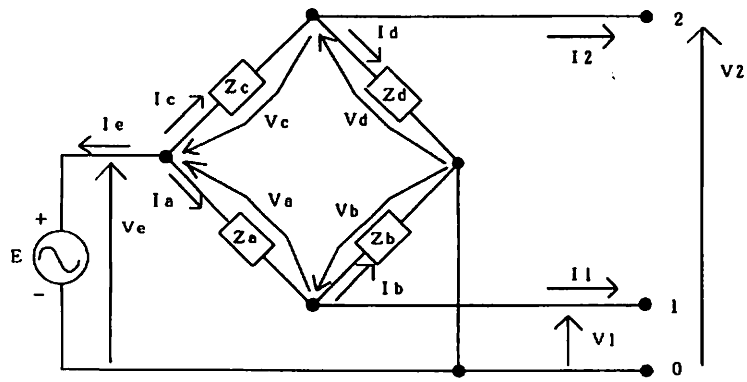

Fig. 5 An example of 3-terminal circuit, a Wheatstone bridge.

Let us apply our result to a familiar Wheatstone bridge in Fig. 5. The circuit is regarded as a 3-terminal one because either of the two output terminals is not grounded. The equations corresponding to (1) and (2) are as follows.

$$\begin{pmatrix} -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & Z_a & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & Z_b & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & Z_c & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & Z_d & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_a \\ V_b \\ V_c \\ V_d \\ V_e \\ I_a \\ I_b \\ I_c \\ I_d \\ I_e \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ E \end{pmatrix} \quad (11)$$

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_a \\ V_b \\ V_c \\ V_d \\ V_e \\ I_a \\ I_b \\ I_c \\ I_d \\ I_e \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (12)$$

If we eliminate all the inner variables in these equations, we get the following terminal equation which corresponds to (6).

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \begin{pmatrix} 1/Z_a + 1/Z_b & 0 \\ 0 & 1/Z_c + 1/Z_d \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} E/Z_a \\ E/Z_c \end{pmatrix} \tag{13}$$

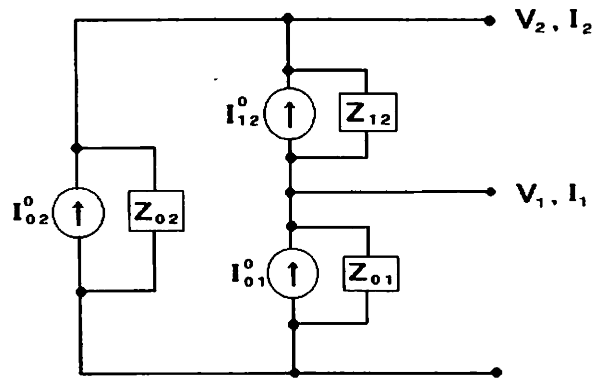

Fig. 6 The standard equivalent circuit of 3-terminal one.

On the other hand, the standard equivalent circuit of 3-terminal network has a form shown in Fig. 6, which corresponds to either of the three types in Fig. 4 (a), if we equate one of the sources I^o_{01}, I^o_{02}, or I^o_{12} to zero. Its terminal equation is given as follows using equations (7), (8), and (9).

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \begin{pmatrix} 1/Z_{10} + 1/Z_{12} & -1/Z_{12} \\ -1/Z_{12} & 1/Z_{20} + 1/Z_{12} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} I^o_{10} + I^o_{12} \\ I^o_{20} - I^o_{12} \end{pmatrix} \tag{14}$$

Comparison of (14) with (13) gives the following equations.

$$Z_{12} = \infty \tag{15}$$

$$Z_{10} = Z_a Z_b / (Z_a + Z_b) \tag{16}$$

$$Z_{20} = Z_c Z_d / (Z_c + Z_d) \tag{17}$$

$$I^o_{10} + I^o_{12} = E/Z_a \tag{18}$$

$$I^o_{20} - I^o_{12} = E/Z_c \tag{19}$$

Thus the impedances are determined uniquely with the first three equations, but the values of current sources in (18) and (19) have the following three possible solutions.

(a) In case of $I^{o}_{12} = 0$,

$$I^{o}_{10} = E/Z_a \tag{20}$$

$$I^{o}_{20} = E/Z_c \tag{21}$$

(b) In case of $I^{o}_{20} = 0$,

$$I^{o}_{10} = 1/Z_a + 1/Z_c \tag{22}$$

$$I^{o}_{12} = -\ E/Z_c \tag{23}$$

(c) In case of $I^{o}_{10} = 0$,

$$I^{o}_{12} = E/Z_a \tag{24}$$

$$I^{o}_{20} = 1/Z_a + 1/Z_c \tag{25}$$

These three solutions are shown in Fig. 7. Though the three types are equivalent, the first one is most favorable to give the wellknown equilibrium condition,

$$Z_b/Z_a = Z_d/Z_c \tag{26}$$

which is easily obtained from the comparison of V_1 with V_2.

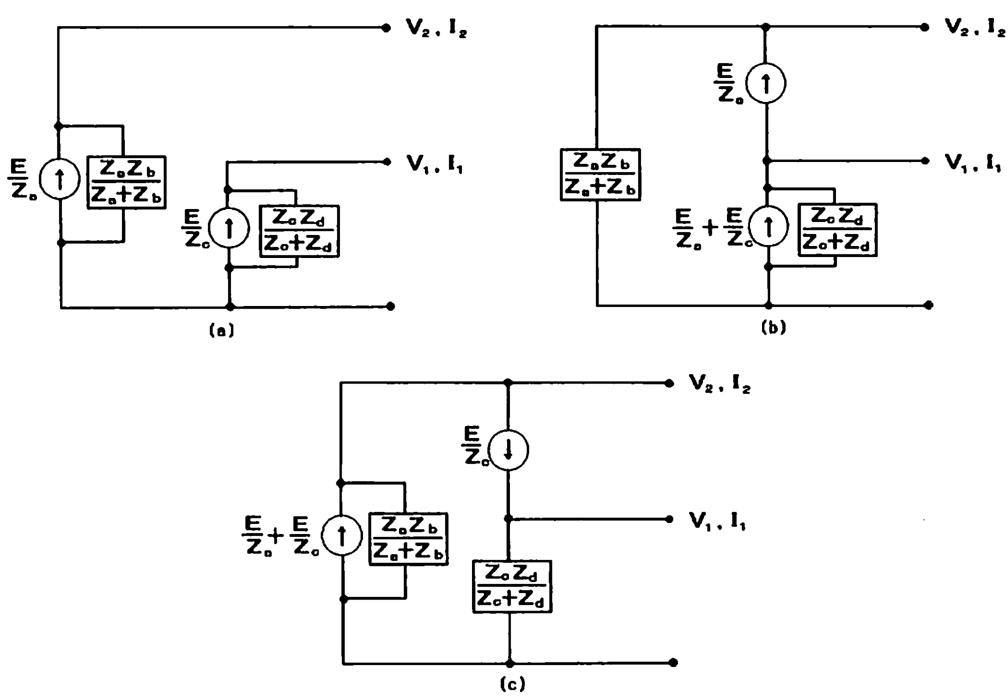

Fig. 7 The three types of the standard equivalent circuit of Fig. 5.

Conclusions

We conclude this paper as follows.

(1) Every n-terminal network can be reduced to a standard equivalent circuit, the graph of which is complete.

(2) In a standard equivalent circuit $(n-1)$ branches chosen to make a tree consist of source elements (Helmholtz-Thevenin's circuits or Mayer-Norton's ones), while the other branches are composed of impedances.

(3) A standard equivalent circuit is always the simplest as to the nodes, and also is the simplest as to the branches in usual cases where no special relation holds among the elements of the given network.

(4) A standard equivalent circuit has alternatives due to different choice of a tree of the sources. Their total number is n^{n-2} for an n-terminal network.

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