The Straightforward Expansion of Helmholtz-Thévenin Theorem to Multi-Terminal Networks

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The Straightforward Expansion of Helmholtz-Thévenin Theorem to Multi-Terminal Networks

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Abstract

The most convenient expansion of Helmholtz-Thévenin theorem is given to n-terminal networks. The new theorem provides a unique equivalent circuit to any n-terminal network whether its inner structure is known or not. The equivalent circuit consists of $n(n-1)/2$ impedances and $(n-1)$ voltage sources, the values of which can be easily given by the open-circuit voltages and the short-circuit currents.

1. Introduction

We treat the same object as in a previous paper[1]. An n-terminal network N_n is defined as a circuit which has n terminals and contains some linear elements (R, C and L) and some independent electric sources. As in Fig.1, we can choose one terminal as the ground and measure all the voltages and currents from this node. So there are $(n-1)$ voltages and $(n-1)$ currents which determine the state of the circuit.

In this paper, any n-terminal network is shown to be equivalent to the Helmholtz-Thévenin's circuit(HTC) and its dual equivalent Mayer-Norton's circuit(MNC) expanded to an n-terminal case, and the explicit contents of HTC and MNC are given.

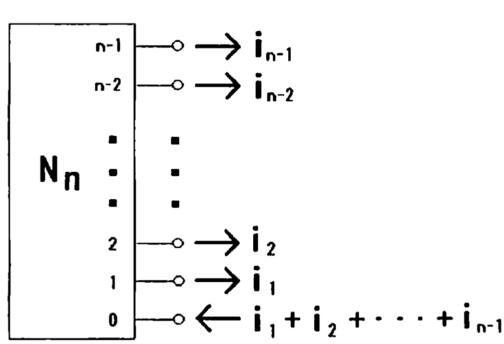

Fig.1 An n-terminal network Nn.

In the previous paper[1], the author gave the procedure how to get the simplest equivalent circuit of a multi-terminal network. The resultant circuit is made of a complete graph of impedances and sources involving an inner impedance and has been named a standard equivalent circuit(SEC). The whole process can be regarded as to generalize the Helmholtz-Thévenin's theorem and Mayer-Norton's one because the original theorems give the simplest equivalent circuits to any two-terminal networks. So the previous paper implies that a SEC is an HTC or MNC if all the sources contained are assumed to consist of voltages or currents, respectively. However such a treatment causes inconvenience in some aspects. The first default is that a SEC is not unique but has n^{n-2} versions. The

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second is that the values of its elements have no simple relations to those measured from the outside in general. Hence we will newly define an HTC and MNC below.

New Difinition

Let us give the new definition as follows. The Helmholtz-Thévenin's circuit(HTC) of an n-terminal network N_n is an n-terminal network N_{no} in series with (n−1) voltage sources $E_{o1},E_{o2},...,E_{o,n-1}$ at each terminal of N_{no},where N_{no} is a complete graph of n(n−1)/2 impedances, and each voltage $E_{o1},E_{o2},...,E_{o,n-1}$ is the open-circuit one of N_n at the corresponding terminal(Fig. 2). The corresponding Mayer-Norton's circuit(MNC) of an n-terminal network N_n is an n-terminal network N_{no} in parallel with (n−1) current sources $I_{o1},I_{o2},...,I_{o,n-1}$ between each terminal and the ground of N_{no},where N_{no} is a complete graph of n(n−1)/2 impedances, and each current $I_{o1},I_{o2},...,I_{o,n-1}$ is the short-circuit one of N_n between the corresponding terminal and the ground(Fig. 3).

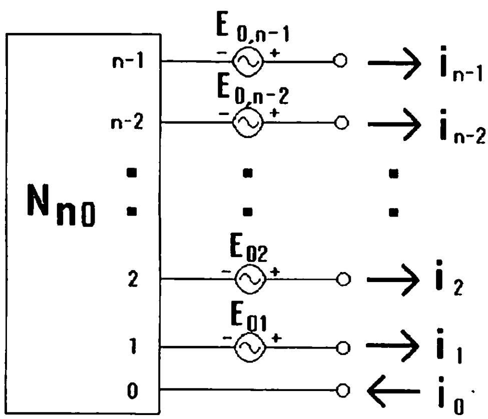

Fig.2 The Helmholtz-Thévenin's equivalent circuit of N_n. N_{no} consists of a complete graph of n(n−1)/2 impedances. E_{oi} is the open-circuit voltage at i th terminal of N_n.

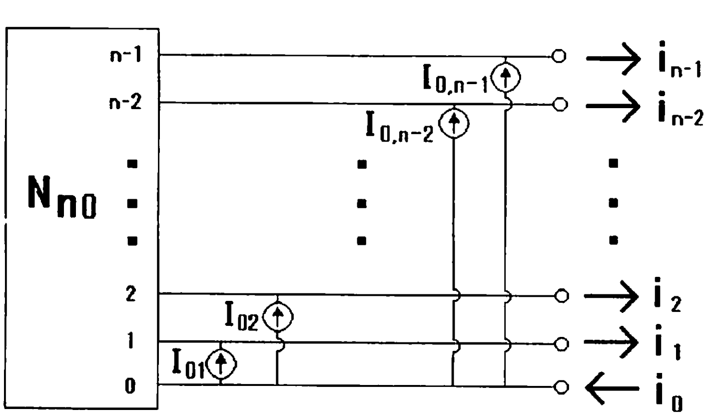

Fig.3 The Mayer-Norton's equivalent circuit of N_n. N_{no} is the same as in Fig.2, and I_{oi} is the short-circuit current between i th terminal and 0 th one of N_n.

Comparing the two circuits with SECs, we see that an MNC is a kind of SEC while an HTC is not because its nodes in N_{no} are not the terminals of N_n so that it contains (n−1) inner nodes forbidden in a SEC. At the cost of losing its simplicity, the HTC has gained the merit that all the values of sources coincide with the open-circuit ones directly as in the case of two terminals.

Oono has given an equivalent circuit to a 3-terminal network as an expanded example of the Helmholtz-Thévenin's theorem, which differs from our SEC and coincides with the present HTC[2]. The present paper has been hinted by Oono's result, but needed some additional consideration. He stated only that N_{no} is obtained from N_n by setting all sources to zero.

Main usefulness of the Helmholtz-Thévenin's theorem may lie in that N_n can be treated as a black box whose inner structure is not known. From this point of view, getting N_{no} from N_n by setting all sources to zero seems to be meaningless because it needs the knowledge of the inside of N_n. Although N_{no} is given easily as an impedance or a triangular loop of impedances in the case of 2 or 3 terminals respectively, its general structure should be specified explicitly. Thus we need the above definition.

Now the new Helmholtz-Thévenin's circuit and Mayer-Norton's one have the following appropriate properties which deserve their names. Firstly, they have the minimal set of elements, (n−1) sources and n(n−1)/2 impedances, and therefore each of them is one of the easiest circuits to handle among all the equivalents. Secondly, all the sources in both circuits are isolated from each other, which implies that an open-circuit voltage at any terminal in an HTC is not affected by the other sources when the other terminals are open, while a short-circuit current between any terminal and the ground in an MNC is not affected by the other sources when the other terminals are short-circuited.

We will give the proof that the new HTC and MNC exist for any n-terminal network below.

The proof

Since we have shown in detail that any circuit is equivalent to its standard equivalent one, the above two may also be equivalent to it. The Mayer-Norton's equivalent circuit is one of the standard equivalent circuits already obtained in Ref. (1), and therefore the proof is unnecessary. The Helmholtz-Thévenin's equivalent circuit can be reduced to a standard equivalent one as follows. As in Fig. 4, Blakesley's theorem[3] assures that any voltage source connected to an n-branch node can be moved to the other (n−1) branches with its direction kept. We will show the necessary procedure in Fig.5 of 5-terminal case. Fig.5 (a) is an HTC. If we apply the Blakesley transformation to every source, we get the circuit in Fig. 5 (b), and finally the diagram in Fig. 5 (c), for we can eliminate the sources on any branch among a loop of sources[1]. Since the sources in Fig. 5 (c) makes a tree of (n−1) branches on the complete graph of impedances, the circuit is an SEC. Thus we have

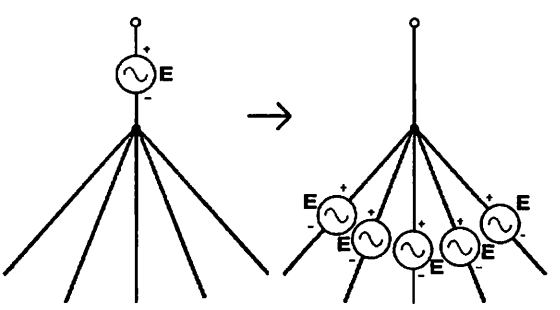

Fig.4 Blakesley Transformation of voltage sources.

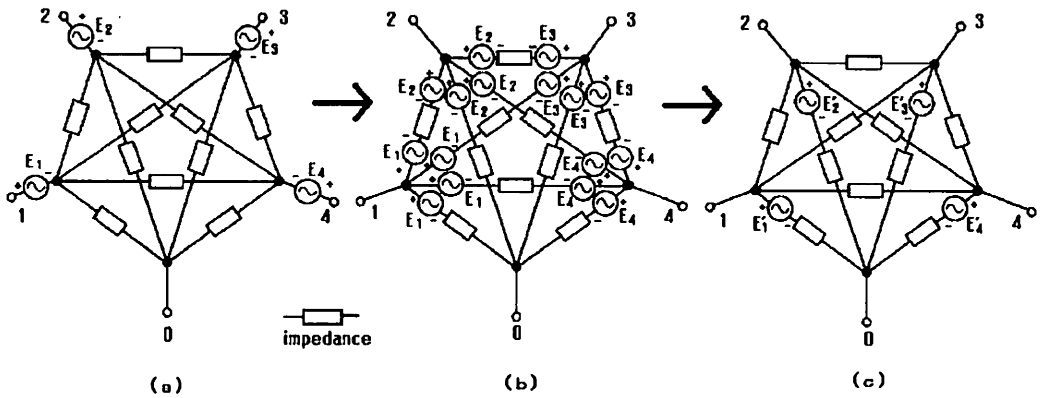

Fig.5 Transformations proving that a Helmholtz-Thévenin's equivalent circuit is reduced into a standard equivalent circuit. $E_1'=4E_1-E_2-E_3-E_4$, $E_2'=4E_2-E_1-E_3-E_4$, $E_3'=4E_3-E_1-E_2-E_4$, $E_4'=4E_4-E_1-E_2-E_3$.

accomplished the proof.

How to Get the Constituent Elements

It should be noticed that the transformations in Fig. 5 accompany no change of impedances. Therefore all the impedances in the Helmholtz-Thévenin's equivalent circuit are the same as those in the Mayer-Norton's one or in any standard equivalent circuit. We can get the following equation among the currents and voltages at the terminals[1]

$$\left[I \right] + \left[\; L \; \right] \left[V \right] = \left[s \right] \qquad (1)$$

where $\boldsymbol{I}$ and $\boldsymbol{V}$ are a vector whose components are $(I_1, I_2, \ldots, I_{n-1})$ and $(V_1, V_2, \ldots, V_{n-1})$ respectively and $\boldsymbol{L}$ and $\boldsymbol{s}$ are a constant matrix whose size is $(n-1) \times (n-1)$ and $(n-1) \times 1$ respectively.

If we apply Eq. (1) to the Mayer-Norton's equivalent circuit, each component of $\boldsymbol{s}$ is regarded as the short-circuit current of the corresponding terminal. That is,

$s_i = I_i$ when all the terminals are short-circuited to the ground.

On the other hand each component of $\boldsymbol{L}$ is given by the values when a certain terminal is open and the others are short-circuited as follows.

$L_{ij} = (s_j - I_j)/V_i$ when ith terminal is open and the others are short-circuited to the ground.

Thus we can determine the impedances Z_{ij} which connects ith terminal and jth one as follows[1].

$$Z_{ij} = -1/L_{ij} \quad (i \neq 0,\ i \neq j) \qquad (2)$$

$$Z_{0i} = 1/\sum_j L_{ij} \qquad (3)$$

Obviously the values of voltage source in the Helmholtz-Thévenin's equivalent circuit are given as those of open-circuit voltage. Hence all the values of the constituent elements in both circuits are obtained from the open- or short-circuited ones of the original circuit.

Examples

Let us apply our result to a familiar Wheatstone bridge in Fig. 6.
From the open-circuit voltages,

$$V_{01} = Z_bE/(Z_a+Z_b), \quad (4)$$

$$V_{02} = Z_dE/(Z_c+Z_d). \quad (5)$$

From the case when all the terminals are short-circuited,

$$s_1 = E/Z_a, \quad (6)$$

$$s_2 = E/Z_c. \quad (7)$$

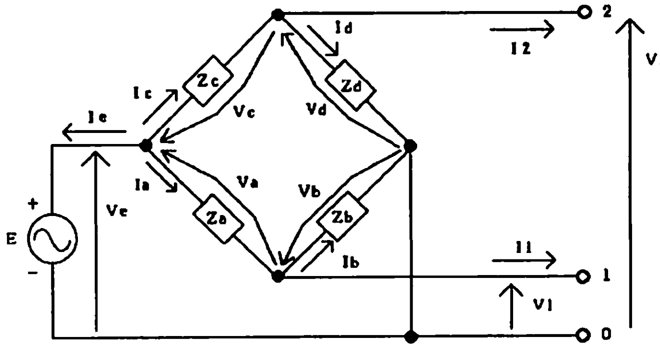

Fig.6 An example of 3-terminal network, a Wheatstone bridge.

If we open the first terminal and short-circuit the second, we get

$$L_{11} = (Z_a+Z_b)/Z_aZ_b, \quad (8)$$

$$L_{21} = 0. \quad (9)$$

When we try the opposite case, we obtain

$$L_{12} = 0, \quad (10)$$

$$L_{22} = (Z_c+Z_d)/Z_cZ_d. \quad (11)$$

Applying Eqs. (2) and (3), we get

$$Z_{12} = Z_{21} = \infty, \quad (12)$$

$$Z_{01} = Z_aZ_b/(Z_a+Z_b), \quad (13)$$

$$Z_{02} = Z_cZ_d/(Z_c+Z_d). \quad (14)$$

The results are shown in Fig. 7.

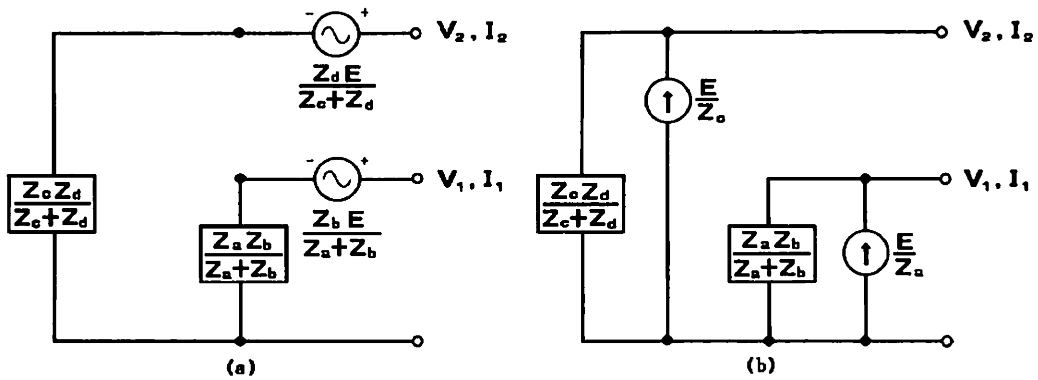

Fig.7 (a) The Helmholtz-Thévenin's equivalent circuit of Fig. 6.
(b) The Mayer-Norton's equivalent circuit of Fig. 6.

Discussions

Oono derived the two general equivalent circuits to a 3-terminal network, Δ- and Y-type[2] (Fig. 11·6 (b) and 11·6 (a) in Ref.(2) respectively). His Δ-type solution obviously coincides with those shown in Fig. 4(a) of Ref. (1) so that it is a standard equivalent circuit. (His solution includes three sources, but one of them can be arbitrarily omitted according to his own comment.) His Y-type is not contained in the results of Ref.(1) because it has the inner node other than the three terminals. It also differs from the present Helmholtz-Thévenin's equivalent circuit. In general it has no corresponding Y-type(star-type) equivalent circuit in the case when the number of terminals exceeds 3 because the Δ-Y transformation is impossible[4].

Errata

In Ref. (1), Equations (22) and (25) should be revised as follows.

$$I^0_{10} = E/Z_a + E/Z_c,$$

$$I^0_{20} = E/Z_a + E/Z_c.$$

Figures 6 and 7 also should be revised into the present Figures 8 and 9 respectively.

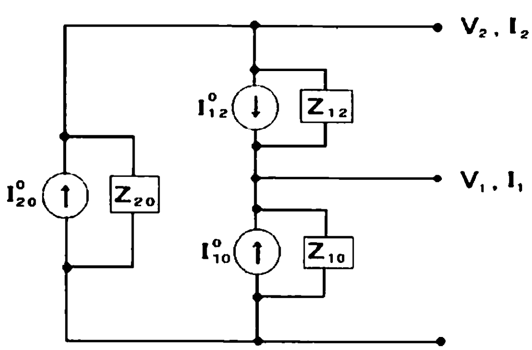

Fig.8 (Revised Fig. 6 in Ref. (1)) The standard equivalent circuit of 3-terminal one.

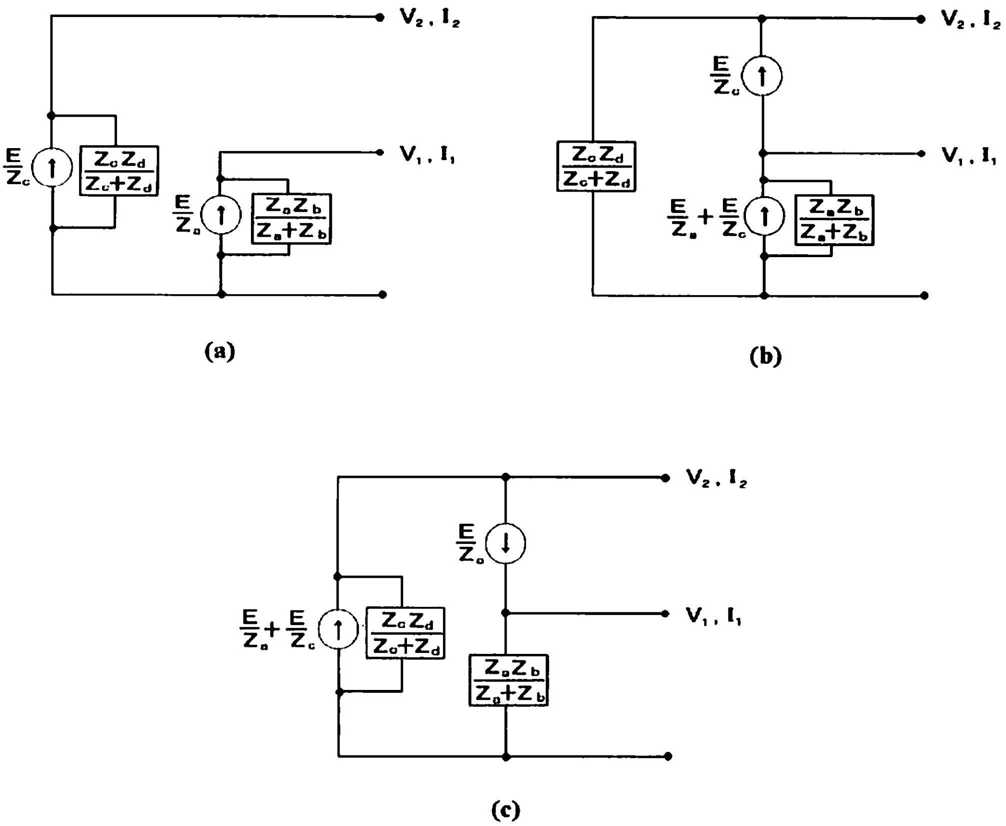

Fig.9 (Revised Fig. 7 in Ref. (1)) The three types of the standard equivalent circuit of Fig.5 in Ref. (1).

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