The b function of a μ-constant deformation of x^7+y^5.

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Introduction

The versal deformation of the plane curve singularity $x^7+y^5=0$ has 24-dimensional parameter space containing 4-dimensional μ-constant subspace. The μ-constant deformation is given by

$(1/7)x^7+(1/5)y^5-t_1x^3y^3-t_4x^5y^2-t_6x^4y^3-t_{11}x^5y^3=0.$

In this paper we stratify the 4-dimensional parameter space into 7 strata, by the condition that on each stratum the b function is constant. The 7 strata are $\{t_1\neq 0,\ t_4+6t_1^4\neq 0\}$, $\{t_1\neq 0,\ t_4+6t_1^4=0\}$, $\{t_1=0,\ t_4t_6\neq 0\}$, $\{t_1=0, t_4\neq 0, t_6=0\}$, $\{t_1=0,\ t_4=0,\ t_6\neq 0\}$, $\{t_1=t_4=t_6=0,\ t_{11}\neq 0\}$, *and* $\{t_1=t_4=t_6=t_{11}=0\}$.

The idea of calculation is essentially due to Yano [4].

Definitions

$\mathbf{N}$: the set of natural numbers, $\mathbf{N}_0=\mathbf{N}\cup\{0\}$.

$\mathbf{O}$: the set of germs of holomorphic functions in a neiborhood of 0 in $\mathbf{C}^{n+1}$.

$\mathbf{D}=O[\partial/\partial x_0, \ldots, \partial/\partial x_n]$.

$J_f(s)=\{P(s)\in D[s] : P(s)f^s=0\}$, where $f\in O$.

$B_{pt}=D\delta=\mathbf{C}[\partial/\partial x_0, \ldots, \partial/\partial x_n]\delta$, where δ is the *δ-function* whose support is $\{0\}$.

Let $f\in O$, the unitary generator of the ideal $\{b(s)\in \mathbf{C}[s] : P(s)f^{s+1}=b(s)f^s$ *for some* $P(s)\in D[s]\}$ is called the *b function* of f and denoted by $b_f(s)$.

If $f(0)=0$, $b_f(s)$ is devided by $s+1$, and we put $\tilde{b}_f(s)=b_f(s)/(s+1)$.

Let d be a degree on O, then we can extend d on $D[s]$, B_{pt} by the rules $d(\partial/\partial x_i)=-d(x_i)$, $d(\delta)=d(s)=0$.

For $f\in O$, $P(s)\in D[s]$, f^* *and* $P^*(s)$ denote the homogeneous parts of f and $P(s)$, of the lowest degrees.

$A_mO=\{f\in O : d(f)\geqq m\}$, $A_m^+O=\{f\in O : d(f)>m\}$

§1. Preliminary results

1.1 *(Theorem of Brieskorn - Pham)*

Put $f(x_0, \ldots, x_n)=x_0^{a_0}+\ldots+x_n^{a_n}$. *The characteristic polynomial of the local monodromy of* $(f^{-1}(0), 0)$ *is given by*

$$\triangle(t)=\Pi(t-\alpha_0\alpha_1\ldots\alpha_n),$$

where each α_j *ranges over all* a_j*-th roots of unity other than 1.*

For the proof, see Milnor [2]

1.2. *(Malgrange [1])*

Assume f *has an isolated singularity, then we have*

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$\{$*the eigenvalues of local monodromy*$\} = \{exp(2\pi\sqrt{-1}\alpha) : b_f(\alpha)=0\}$.
And the degree of $b_f(s)$ *is not greater than* μ, *the Milnor number of* $(f^{-1}(0), 0)$.

1.3. *(Kashiwara - Yano)*

Assume f *has an isolated singularity. Let* α *be a root of* $b_f(s)$, *then there exists* $\triangle \in B_{pt}$ *with the following properties :*

(1.3.1) $f\triangle = 0$; $(\partial f/\partial x_i)\triangle = 0$, $i=0, \ldots, n$.

(1.3.2) *For all* $P(s) \in J_f(s)$, $P(\alpha)\triangle = 0$.

The converse is also true.

1.4. *Let* $f(x, y) = (1/m)x^m + (1/n)y^n$, $(m,n)=1$, *and* F *be a versal deformation defined by* $F(x, y) = f(x, y) - \Sigma t_{i,j}x^i y^j$, *where the summation ranges all* (i, j) *satisfying* $0 \leqq i \leqq m-2$, $0 \leqq j \leqq n-2$ *and* $i/m + j/n \geqq 1$. *Then for each fixed* t, $F^{-1}(0)$ *has the same local monodromy as* $f^{-1}(0)$, *and by 1.1, the eigenvalues are* $exp(2\pi\sqrt{-1}(\ (i+1)/m + (j+1)/n\ ))$; $0 \leqq i \leqq m-2$, $0 \leqq j \leqq n-2$.

$b_f(s)$ *is known to be* $\Pi(s + (i+1)/m + (j+1)/n)$, *where the product ranges* $0 \leqq i \leqq m-2$, $0 \leqq j \leqq n-2$, *(Miwa [3]).*

1.5. Let

$F(x,y) = (1/m)x^m + (1/n)y^n - \Sigma_{i \geqq 0}\ t_i x^{a_i} y^{b_i}$ *where* $(m, n)=1$, $t_0 \neq 0$, $r = (a_0/m) + (b_0/n) - 1 > 0$, *and* $a_i \geqq a_0$, $b_i \geqq b_0$ *for* $i \geqq 0$.

We assume that (a_0, b_0) *satisfies the following condition :*

(＊) *If* (p, q) *satisfies*

(1.5.1) $0 \leqq p \leqq m-2$, $0 \leqq q \leqq n-2$,

(1.5.2) $r = (p/m) + (q/n) - 1 = kr$ *for some natural number* k,

then we have $p \geqq a_0$, $q \geqq b_0$.

Then

(1.5.3) $b_F(s) = \prod_{0 \leqq i \leqq m-2,\ 0 \leqq j \leqq n-2} (s + (i+1)/m + (j+1)/n - e_{i,j})$

where $e_{i,j} = \begin{cases} 1 & \text{if } i \geqq a_0,\ j \geqq b_0, \\ 0 & \text{otherwise.} \end{cases}$

We first prove the following lemma.

1.6. Lemma.

Let F *be as in 1.5. Assume* $(u, v) \in \mathbf{N}_0 \times \mathbf{N}_0$ *satisfies*

(1.6.1) $u/m + v/n - 2 = kr$ *for some* $k \in \mathbf{N}$.

Then we have

(1.6.2) $x^u y^v \in (x^{u1}y^{v1}, x^{u2}y^{v2}, x^{u3}y^{v3})O + (F_x, F_y)(F, F_x, F_y)O$, *for some* $(u_i, v_i) \in \mathbf{N}_0 \times \mathbf{N}_0$ *satisfying*

(1.6.3) $u_i/m + v_i/n - 2 = (k+1)r$.

Proof.

By(1.6.1), we may assume, without loss of generality, that $u \geqq m$.
Then

(1.6.4) $x^u y^u \in (x^{u-m+1}y^v F_x,\ x^{u-m+a0}y^{v+b0})O$.

Since $(u-m, v)$ satisfies (1.5.2) in 1.5, at least one of the following three conditions holds:

(1.6.5) $v \geqq n-1$,

(1.6.6) $u-m\geqq m-1$,

(1.6.7) $u-m\geqq a_0,\ v\geqq b_0$.

If (1.6.5) holds we have

(1.6.8) $x^uy^u\in (F_xF_y,\ x^{u+a0}y^{v-n+b0},\ x^{u-m+a0}y^{v+b0},\ x^{u-m+2a0}y^{v-n+2b0})O.$

If (1.6.6) holds we have

(1.6.9) $x^uy^v\in(F_x^2,\ x^{u-m+a0}y^{v+b0},\ x^{u-2m+2a0}y^{v+2b0})O.$

If (1.6.7) holds we have

(1.6.10) $x^uy^v\in(F_x\cdot(F-X_0F),\ x^{u-m+a0}y^{v+b0})O$, *where* $X_0=(1/m)xD_x+(1/n)yD_y$.

In either case, (1.6.2) holds.

1.7. Proof of 1.5.

Let F be as in 1.5, d be the degree defined by $d(x)=1/m$, $d(y)=1/n$, and X_0 be the Euler operator : $X_0=(1/m)xD_x+(1/n)yD_y$.

Since F has an isolated singularity, there exists $m\in N$ so that

(1.7.1) $A_{mr}O\subset(F_x,\ F_y)^2O.$

X_0 has the property:

(1.7.2) $F\text{-}X_0F=\Sigma_{i\geq 0}r_it_ix^{a_i}y^{b_i}$, where $r_i=a_i/m+b_i/n-1$, $r_0=r$.

Hence $(F\text{-}X_0F)^2\in(x^{2a0}y^{2b0})O$, and then by Lemma 1.6, we have

(1.7.3) $(F\text{-}X_0F)^2\in(F_x,\ F_y).\cdot(F,F_x,F_y)\ +\ A_{mr}O$

$\subset(F_x,F_y)\cdot(F,F_x,Fy).$

This means that there exists an operator $P_2'(s)\in D[s]$ of the form

(1.7.4) $P_2'(s)=(s+r-X_0)(s-X_0)+$ *(higher terms with respect to d)*,

so that

(1.7.5) $P_2'(s)F^s=g(x,y)sF^{s-1}$, with $g\in(F_{xx})O\cap A_r^+O+(F_{xy})O\cap A_r^+O$
$+(F_{yy})O\cap A_r^+O+(x^ay^b)\cap A_r^+O\subset(F_x)O\cap A_r^+O+(F_y)O\cap A_r^+O+(x^ay^b)\cap A_r^+O.$

Hence we have an operator $P_2(s)\in J_F(s)$ of the form

(1.7.6) $P_2(s)=P_2'(s)+g_0(x,y)(s-X_0)+g_1(x,y)D_x+g_2(x,y)D_y$,

with $d(g_0)>0$, $d(g_1)>r+d(x)$, $d(g_2)>r+d(y)$.

We also have operators of "order 1" in $J_F(s)$:

(1.7.7) $P_{11}(s)=x^{m-a0-1}(s-X_0)+$ *(higher terms)*,

(1.7.8) $P_{12}(s)=y^{n-b0-1}(s-X_0)+$ *(higher terms)*.

Now we determine $\tilde{b}_F(s)$ by 1.3.

Let α be a root of $\tilde{b}_F(s)$, and $P(D_x,D_y)\delta$ be the corresponding element of B_{pt} stated in 1.3. By (1.3.1), we have

(1.7.9) $F_xP\delta=F_yP\delta=(F-X_0F)P\delta=0.$

Taking the homogeneous parts of the lowest degree, we get

(1.7.9)* $x^{m-1}P^*\delta=y^{n-1}P^*\delta=x^ay^bP^*\delta=0.$

This means

(1.7.10) $P^*=D_x^pD_y^q$ with $0\leqq p\leqq m-2,\ 0\leqq q<b$ or $0\leqq p<a,\ 0\leqq q\leqq n-2.$

By (1.3.2), applied to P_{11}, P_{12}, and P_2, we get

(1.7.11) $(x^{m-a0-1}(\alpha-X_0)+\ .\ .\ .\ )P\delta=(y^{n-b0-1}(\alpha-X_0)+\ .\ .\ .)P\delta=0,$

(1.7.12) $((\alpha+r-X_0)(\alpha-X_0)+\ .\ .\ .)P\delta=0.$

Since $X_0D_x^pD_y^q\delta=-((p+1)/m+(q+1)/n)D_x^pD_y^q\delta$, these mean

(1.7.11)* $x^{m-a0-1}(\alpha+(p+1)/m+(q+1)/n)D_x^pD_y^q\delta$

$=y^{n-b0-1}(\alpha+(p+1)/m+(q+1)/n)D_x^pD_y^q\delta$

$=0,$

(1.7.12)* $(\alpha+r+(p+1)/m+(q+1)/n)(\alpha+(p+1)/m+(q+1)/n)D_x^pD_y^q\delta=0.$

If

(1.7.13) $p\geqq m-a_0-1$ *or* $q\geqq n-b_0-1$ then by (1.7.11)*,

(1.7.14) $\alpha=-((p+1)/m+(q+1)/n)$

If

(1.7.15) $p<m-a_0-1$ and $q<n-b_0-1$ then by (1.7.12)*,

(1.7.16) $\alpha=-((p+1)/m+(q+1)/n)$ or $-((p+1)/m+(q+1)/n+r)$

The latter is equal to $-((p+a_0+1)/m+(q+b_0+1)/n-1)$.

By 1.4. and 1.2. $\tilde{b}_F(s)$ has distinct $(m-1)(n-1)$ roots, hence (1.7.14) and (1.7.16) give all the roots. This proves 1.5.

§ 2. **Example.** $F=(1/7)x^7+(1/5)y^5-t_1x^3y^3-t_4x^5y^2-t_6x^4y^3-t_{11}x^5y^3.$

A

$t_1\neq 0$

2.1.

d is defined by $d(x)=1/7,\ d(y)=1/5$, and $X_0=(1/7)xD_x+(1/5)yD_y$.

It is easily seen that

(2.1.1) $A_{35}O/(F,F_x,F_y)\cap A_{35}O$ is generated by x^5y^2 over C.

$A_{70}O=(xF_x,yF_y)\cdot A_{35}O$, and

$x^5y^2\cdot x^7\equiv y^2F_x^2,\ x^5y^2\cdot y^5\equiv t_1^{-1}(F-X_0F)x^2F_y$ *(mod higher terms)*, hence we have

(2.1.2) $A_{70}O=((F_x,F_y)\cap A_{35}O)\cdot((F,F_x,F_y)\cap A_{35}O)$

This assures that there exists an equation :

(2.1.3) $(F-X_0F)^2-(1/35)^2(t_1^2y^2F_xF_y+3t_1^3x^2y\ F_y^2-3t_1^3x^3F_xF_y+18t_1^3x^2(F-X_0F)F_y$
$+9t_1^4y^2F_x^2+\ldots)=0$

This together with (2.1.1) imply the existence of an operator $P_2(s)\in D[s]$ of the form :

(2.1.4) $P_2(s)\ =\ (s+1/35-X_0)\ (s-X_0)-(1/35)^2\ (t_1^2y^2D_xD_y+3t_1^3x^2yD_y^2$
$+3t_1^3x^3D_xD_y+18t_1^3x^2(s-1-X_0)D_y\ +9t_1^4y^2D_x^2+501t_1^3x^2D_y+\ldots)$

so that we have

(2.1.5) $P_2(s)F^s\ =\ -(12/35^2)\ (t_4+6t_1^4)x^5y^2sF^{s-1}.$

2.2.

By (2.1.1) and (2.1.2) we have an operator $P_3(s)\in J_F(s)$ of the form :

(2.2.1) $P_3(s)\ =\ (s+4/35-X_0)\ (s+1/35-X_0)\ (s-X_0)\ +\ $*(higher terms)*.

We also have operators in $J_F(s)$ of "order 2 and 1" :

(2.2.2) $P_{21}(s)\ =xP_2(s)+\ldots,\ P_{22}(s)\ =yP_2(s)+\ldots,$

(2.2.3) $P_{11}(s)\ =\ x^3(s-X_0)-(1/35)(t_1y^3D_x+3t_1x^2y^2D_v+9t_1^3x^5D_y+\ldots)$,

$P_{12}(s)\ =\ y(s-X_0)-(1/35)(t_1x^3D_y+3t_1^2y^2D_x+9t_1^3x^2yD_y$
$+t_1^{-1}(27t_1^4+4t_4)x^2(s-X_0)+\ldots).$

2.3. Assume α be a root of $\tilde{b}_F(s)$, and $P(D_x, D_y)\delta$ be the corresponding element of B_{pt} with the properties (1.3.1), (1.3.2).
Then

(2.3.1) $P^* = D_x^p D_y^q$, for $0 \leqq p \leqq 5$, $0 \leqq q \leqq 3$.

2.4. If $p \geqq 3$ or $q \geqq 1$ then the corresponding root α is equal to $-((p+1)/7 + (q+1)/5)$.

Proof. Apply (1.3.1) to $P_{11}(s)$, $P_{12}(s)$.

2.5. (p,q) cannot be either $(3,3)$, $(4,3)$ or $(5,3)$

Proof. Use the equality $(F - X_0F)P\delta = 0$.

2.6. (p,q) cannot be $(5,2)$ unless $t_4 + 6t_1^4 = 0$.

Proof. Let $(p,q) = (5,2)$, $\alpha = -(6/7 + 3/5)$. Put

(2.6.1) $P = D_x^5 D_y^2 + a_1 D_x^2 D_y^4 + a_2 D_x^6 D_y + a_3 D_x^3 D_y^3 + \ldots$.

The coefficient of $D_x^2\delta$ in $F_yP\delta$ is $-(24/35)(a_1 + 15t_1)$.
The coefficient of δ in $(F - X_0F)P\delta$ is $-(1/35)(36a_3t_1 - 960t_4)$.
The coefficient of $D_y^3\delta$ in $P_{11}(\alpha)P\delta$ is $(120a_1t_1^2 - 18a_3 - 1080t_1^3)/35$.

Since all these coefficients must be zero, we get $t_4 + 6t_1^4 = 0$.

2.7. Assume $t_4 + 6t_1^4 \neq 0$

If $p = 1$ or 2 and $q = 0$, then by use of $P_{21}(s)$, we know that α is equal to

(2.7.1) $-((p+1)/7 + 1/5)$ or $-((p+1)/7 + 1/5 + 1/35) = -((p+3+1)/7 + (3+1)/5 - 1)$.

If $p = q = 0$, then by use of $P_3(s)$, α is equal to

(2.7.2) $1/7 + 1/5$ or $1/7 + 1/5 + 1/35 = 4/7 + 4/5 - 1$
or $1/7 + 1/5 + 4/35 = 6/7 + 3/5 - 1$.

Sine (p,q) cannot be $(3,3)$, $(5,2)$, $(4,3)$, $(5,3)$ we find that 2.4. and (2.7.1) and (2.7.2) give all the roots of $\tilde{b}_F(s)$.

By (2.7.1), (2.7.2) together with 2.4, 2.5, and 2.6, we have

(2.7.3) $\tilde{b}_F(s) = \Pi_{0 \leq p \leq 5,\ 0 \leq q \leq 3} (s + (p+1)/7 + (q+1)/5 - e_{p,q})$,

where $e_{p,q} = \begin{cases} 1 & \text{if } (p,q) = (3,3), (5,2), (4,3), (5,3). \\ 0 & \text{otherwise.} \end{cases}$

2.8. Assume $t_4 + 6t_1^4 = 0$.

In this case $P_2(s)$ at (2.1.4) belongs to $J_F(s)$. Applying 1.3. to $P_2(s)$, we know that for $0 \leqq p \leqq 2$, $q = 0$, α is equal to

(2.8.1) $-((p+1)/7 + 1/5)$ or $-((p+1)/7 + 1/5 + 1/35) = -((p+3+1)/7 + (3+1)/5 - 1)$.

We get $\tilde{b}_F(s)$:

(2.8.2) $\tilde{b}_F(s) = \Pi_{0 \leq p \leq 5,\ 0 \leq q \leq 3} (s + (p+1)/7 + (q+1)/5 - e_{p,q})$,

where $e_{p,q} = \begin{cases} 1 & \text{if } (p,q) = (3,3), (4,3), (5,3), \\ 0 & \text{otherwise.} \end{cases}$

B

$\mathbf{t_1 = 0}$

2.9. Assume $t_4 \neq 0$, $t_6 \neq 0$.

We have following operators in J_F(s):

(2.9.1) $Q_3(s)=(s+6/35-X_0)(s+4/35-X_0)(s-X_0)+$ *higher terms.*

(2.9.2) $Q_2(s)=y(s+4/35-X_0)(s-X_0)+$ *higher terms.*

(2.9.3) $Q_{11}(s)=y^2(s-X_0)+$ *higher terms,*

$Q_{12}(s)=(2t_4x-3t_6y)(s-X_0)+$ *higher terms.*

By these operators we can determine $\tilde{b}_F(s)$:

(2.9.4) $\tilde{b}_F(s)=\prod_{0\leq p\leq 5,\ 0\leq q\leq 3}(s+(p+1)/7+(q+1)/5-e_{p,q})$,

where $e_{p,q}=\begin{cases}1 \text{ if } (p,q)=(5,2),\ (4,3),\ (5,3),\\ 0 \text{ otherwise.}\end{cases}$

2.10. Assume $t_4t_6=0$.

We can apply 1.5.

(2.10.1) If $t_4\neq 0,\ t_6=0$ then $e_{p,q}=\begin{cases}1 \text{ if } (p,q)=(5,2),\ (5,3)\\ 0 \text{ otherwise.}\end{cases}$

(2.10.2) If $t_4=0,\ t_6\neq 0$ then $e_{p,q}=\begin{cases}1 \text{ if } (p,q)=(4,3),\ (5,3)\\ 0 \text{ otherwise.}\end{cases}$

(2.10.3) If $t_4=t_6=0,\ t_{11}\neq 0$ then $e_{p,q}=\begin{cases}1 \text{ if } (p,q)=(5,3)\\ 0 \text{ otherwise.}\end{cases}$

(2.10.4) If all t's are zero then by 1.4, all $e_{p,q}$ are zero.

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