有限モノドロミー群をもつ超幾何微分方程式のSchwarz map

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	作成者: 加藤, 満生, Kato, Mitsuo
	メールアドレス:
	所属:
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Connection Formulas for Appell's F_2 and Some Applications

Mitsuo KATO

Abstract. In this paper, we derive several connection formulas for Appell's F_2 from which we obtain a monodromy representation of it. We determine the irreducibility condition of F_2. We also obtain a monodromy representation of generalized hypergeometric function ${}_3F_2$.

1 Introduction.

Appell's hypergeometric function

$$F_2(a,b,b',c,c';x,y) = \sum_{m,n=0}^{\infty} \frac{(a,m+n)(b,m)(b',n)}{(c,m)(c',n)(1,m)(1,n)} x^m y^n,$$

where $(a,n) = \Gamma(a+n)/\Gamma(a)$, satisfies the following system of differential equations of rank four ([A-K]):

$$\begin{cases} x(1-x)z_{xx} - xyz_{xy} + (c-(a+b+1)x)z_x - byz_y - abz = 0 \\ y(1-y)z_{yy} - xyz_{xy} + (c'-(a+b'+1)x)z_y - b'xz_x - ab'z = 0 \end{cases}$$

which we denote by $E_2(a,b,b',c,c')$. This is an extension of Gauss' hypergeometric differential equation

$$x(1-x)z'' + (c-(a+b+1)x)z' - abz = 0$$

which we denote by $E(a,b,c)$.

If neither c nor c' is an integer then E_2 has the following four linearly independent solutions defined at the origin ([A-K], [Kmr])).

$$\begin{aligned} f_1 &= F_2(a,b,b',c,c';x,y), \\ f_2 &= x^{1-c}F_2(1+a-c,1+b-c,b',2-c,c';x,y), \\ f_3 &= y^{1-c'}F_2(1+a-c',b,1+b'-c',c,2-c';x,y), \\ f_4 &= x^{1-c}y^{1-c'}F_2(2+a-c-c',1+b-c,1+b'-c',2-c,2-c';x,y). \end{aligned} \tag{1.1}$$

In this paper, we always define the value of multivalued function of the form

$$A^\alpha$$

by the condition that

$$\arg(A) = 0 \quad \text{if} \quad A > 0.$$

In Section 3, we will construct other systems of fundamental solutions at several points on the singular locus of E_2. In Section 4, we derive connection formulas among theses systems of fundamental solutons. Connection formulas for Appell's F_2 have been obtained by several authors ([Skg], [Tky]) but for the sake of self-containedness we derive the formulas in this paper.

Theorem 5.1 gives a monodromy representation of E_2 with respect to a system $\omega_1, \omega_2, \omega_3, \omega_4$ (see (5.3)) of fundamental solutions of E_2. Other representations are also known (see [Sa-Ta]).

Theorem 6.1 and 7.1 give the irreducibility condition of E_2, Table 6.1 gives the complete list of the invariant subspaces of the solution space of reducible E_2.

Theorem 8.3 gives a monodromy representation of the generalized hypergeometric function ${}_3F_2$ with respect to a system of fundamental solutions defined at the origin. Other representation is already known (see [B-H] and [Ohr]).

2 Phaffian system.

Let A be a constant (n,n) matrix and λ an eigenvalue of A with

$$\mathrm{rank}(A-\lambda I)=n-m.$$

We suppose that any positive integer is not an eigenvalue of $A-\lambda I$. Let $B(x,y)$ and $C(x,y)$ be (n,n) matricies whose components are holomorphic on the closure of $D=\{(x,y)\in(C)^2 \,|\, |x|<r, |y|<r\}$. Put

$$\Omega=\left(\frac{A}{x}+B(x,y)\right)dx+C(x,y)dy$$

and we assume the integrabity condition

$$d\Omega=\Omega\wedge\Omega.$$

Put $Q(y)=C(0,y)$. Then we have the following theorem ([Kt1, Theorem 1.1]).

Theorem 2.1. *The differential equation*

$$du=\Omega u \tag{2.1}$$

has m linearly independent solutions of the form $u=x^\lambda v(x,y)$, where $v(x,y)$ is holomorphic in D. Then "the boundary value"

$$v_0(y)=v(0,y) \tag{2.2}$$

of v(x,y) satisfies

$$Av_0=\lambda v_0, \tag{2.3}$$

$$\frac{dv_0}{dy}=Qv_0. \tag{2.4}$$

Conversely, if $v_0(y)$ satisfies (2.3) and (2.4) then there exists a unique solution $u=x^\lambda v(x,y)$ of (2.1) satisfying (2.2).

It is well known that the differential equation $E_2(a,b,b',c,c')$ is equivalent to the following Phaffian system:

$$du=Pu, \tag{2.5}$$

where

$$u={}^t(z,xz_x,yz_y,xyz_{xy}) \tag{2.6}$$

and

$$P=P_1\frac{dx}{x}+P_2\frac{dy}{y}+P_3\frac{dx}{x-1}+P_4\frac{dy}{y-1}+P_5\frac{d(x+y)}{x+y-1}, \tag{2.7}$$

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1-c & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1-c \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1-c' & 0 \\ 0 & 0 & 0 & 1-c' \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -ab & c-a-b-1 & -b & -1 \\ 0 & 0 & 0 & 0 \\ abb' & b'(1+a+b-c) & bb' & b' \end{pmatrix},$$

$$P_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -ab' & -b' & c'-a-b'-1 & -1 \\ abb' & bb' & b(1+a+b'-c') & b \end{pmatrix},$$

$$P_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -abb' & b'(c-1-a-b) & b(c'-1-a-b') & \lambda_1 - 2 \end{pmatrix},$$

with $\lambda_1 = c + c' - a - b - b'$.

Remark. From (2.7) we know that the singular locus of E_2 is a subset of

$$\{(x,y) \in \mathbf{C}^2 | \; xy(x-1)(y-1)(x+y-1) = 0\}. \qquad (2.8)$$

In fact, it is known that the singular locus of E_2 with generic parameters α_j and β_j is (2.8).

3 Local solutions at (1,0).

We will have a blowing up at the point (1,0). That is we put

$$x = x_1, \quad y = (1 - x_1)y_1. \tag{3.1}$$

We will consider the local solutions along the exceptional divisor $\{x_1 = 1\}$. The matrix P at (2.7) now has the following form.

$$\begin{aligned} P = &\{(P_2 + P_3 + P_5)/(x_1 - 1) + P_1/x_1 - P_4 y_1/(y - 1)\}\, dx_1 \\ &+ \{P_2/y_1 + P_5/(y_1 - 1) - P_4(x_1 - 1)/(y - 1)\}\, dy_1. \end{aligned} \tag{3.2}$$

In order to use Theorem 2.1, we put $A = P_2 + P_3 + P_5$ and $Q = P_2/y_1 + P_5(y_1 - 1)$. Then

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -ab & c - a - b - 1 & -b & 0 \\ 0 & 0 & 1 - c' & 0 \\ 0 & 0 & b(c' - a - 1) & c - a - b - 1 \end{pmatrix} \tag{3.3}$$

and

$$\begin{aligned} Q = &\frac{1}{y_1}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 - c' & 0 \\ 0 & 0 & 0 & 1 - c' \end{pmatrix}. \\ &+ \frac{1}{y_1 - 1}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -abb' & b'(c - 1 - a - b) & b(c' - 1 - a - b')_2 & \lambda_1 - 2 \end{pmatrix} \end{aligned} \tag{3.4}$$

where $\lambda_1 = c + c' - a - b - b'$.

The eigenvalues of A are 0, $1 - c'$, $c - a - b - 1$, $c - a - b - 1$. We assume that

these eigenvalues do not differ by integers.

We will apply Theorem 2.1 for $\lambda = 0, 1 - c'$ and $c - a - b - 1$.

First let $\lambda = 0$. Then the solution u of (2.5) has the form

$$u = v_0(y_1) + (1 - x_1)v_1(y_1) + (1 - x_1)^2 v_2(y_1) + \cdots,$$

where $v_0(y_1)$ is a solution of

$$Av_0 = 0, \quad v_0'(y_1) = Qv_0(y_1). \tag{3.5}$$

This implies that v_0 is constant, that is,

$$v_0 = {}^t(c - a - b - 1, ab, 0, 0) \tag{3.6}$$

up to a contant multiplication.

Let $\lambda = 1 - c'$. Then, by the same reasoning as above, (2.5) has a solution u of the following form:

$$u = (1 - x_1)^{1-c'}(v_0(y_1) + (1 - x_1)v_1(y_1) + \cdots),$$
$$v_0(y_1) = y_1^{1-c'} \cdot {}^t(1, *, *, *)$$

for some constant numbers $*$. Then, by (2.5), we know that all $v_j(y_1)$ are multiples of $y_1^{1-c'}$ inductively. Hence we have a solution of the following form:

$$u = ((1 - x_1)y_1)^{1-c'}(v_0 + (1 - x_1)v_1(y_1) + \cdots), \tag{3.7}$$
$$v_0 = {}^t(1, *, *, *). \tag{3.8}$$

Let $\lambda = c - a - b - 1$, then the solution of (2.5) takes the form

$$u = (1 - x_1)^{c-a-b-1}(v_0(y_1) + (1 - x_1)v_1(y_1) + \cdots). \tag{3.9}$$

By (2.3), v_0 takes the form

$$v_0 = {}^t(0, f(y_1), 0, g(y_1)) \tag{3.10}$$

and (2.4) is equivalent to

$$\frac{d}{dy_1}\begin{pmatrix} f \\ g \end{pmatrix} = \left(\frac{1}{y_1}\begin{pmatrix} 0 & 1 \\ 0 & 1 - c' \end{pmatrix} + \frac{1}{y_1 - 1}\begin{pmatrix} 0 & 0 \\ b'(c - a - b - 1) & \lambda_1 - 2 \end{pmatrix}\right)\begin{pmatrix} f \\ g \end{pmatrix}, \tag{3.11}$$

where $\lambda_1 = c + c' - a - b - b'$.

Put

$$v_1(y_1) = {}^t(h(y_1), h_1(y_1), h_2(y_1), h_3(y_1)). \tag{3.12}$$

Then (2.5) implies

$$-(c - a - b)(c + c' - a - b - 1)h = (c + c' - a - b - 1 - b'y_1)f + (1 - y_1)g$$

and (3.11) implies that $h(y_1)$ is a solution of Gauss' hypergeometric equation $E(a + b - c, b', c')$. Thus we have the following theorem.

Theorem 3.1. *$E_2(a, b, b', c, c')$ has the solutions of the following forms along the exceptional divisor $X = \{x_1 = 1\}$.*

(1). $f(x_1, y_1)$, where f is holomorphic along X and $f(1, y_1)$ is constant.

(2). $((1-x_1)y_1)^{1-c'}f(x_1, y_1)$, where f is holomorphic along X and $f(1, y_1)$ is constant.

(3). $(1 - x_1)^{c-a-b}f(x_1, y_1)$, where f is holomorphic along X and $f(1, y_1)$ is a solution of $E(a + b - c, b', c')$ as a function of y_1.

Thanks to this theorem, we can define the following local solutions at three points on the exceptional divisor X.

At the point $x_1 = 1,\ y_1 = 0$, the following solutions exist:

$$\begin{aligned}
\varphi_1(x_1, y_1) &= \tilde{\varphi}_1(x_1, y_1), \\
\varphi_2(x_1, y_1) &= (1 - x_1)^{1-c'}y_1^{1-c'}\tilde{\varphi}_2(x_1, y_1), \\
\varphi_3(x_1, y_1) &= (1 - x_1)^{c-a-b}\tilde{\varphi}_3(x_1, y_1), \\
\varphi_4(x_1, y_1) &= (1 - x_1)^{c-a-b}y_1^{1-c'}\tilde{\varphi}_4(x_1, y_1),
\end{aligned} \tag{3.13}$$

where $\tilde{\varphi}_j$ are holomorphic at $x_1 = 1, y_1 = 0$ and

$$\tilde{\varphi}_j(1,0) = 1 \ (1 \le j \le 4), \quad \tilde{\varphi}_j(1,y_1) \equiv 1 \ (1 \le j \le 2).$$

At the point $x_1 = 1,\ y_1 = 1$, the following solutions exist:

$$\begin{aligned}
\psi_1(x_1,y_1) &= \tilde{\psi}_1(x_1,y_1), \\
\psi_2(x_1,y_1) &= (1-x_1)^{1-c'} y_1^{1-c'} \tilde{\psi}_2(x_1,y_1), \\
\psi_3(x_1,y_1) &= (1-x_1)^{c-a-b} \tilde{\psi}_3(x_1,y_1), \\
\psi_4(x_1,y_1) &= (1-x_1)^{c-a-b}(1-y_1)^{c+c'-a-b-b'} \tilde{\psi}_4(x_1,y_1),
\end{aligned} \tag{3.14}$$

where $\tilde{\psi}_j$ are holomorphic at $x_1 = 1, y_1 = 1$ and

$$\tilde{\psi}_j(1,1) = 1 \ (1 \le j \le 4), \quad \tilde{\psi}_j(1,y_1) \equiv 1 \ (1 \le j \le 2).$$

At the point $x_1 = 1,\ y_1 = \infty$, we can choose

$$x_2 = 1/y_1 \quad \text{and} \quad y_2 = y_1(1-x_1)$$

as the local coordinates there, and the following solutions exist:

$$\begin{aligned}
\kappa_1(x_2,y_2) &= \tilde{\kappa}_1(x_2,y_2), \\
\kappa_2(x_2,y_2) &= y_2^{1-c'} \tilde{\kappa}_2(x_2,y_2), \\
\kappa_3(x_2,y_2) &= e^{i\pi(a+b-c)} y_2^{c-a-b} \tilde{\kappa}_3(x_2,y_2), \\
\kappa_4(x_2,y_2) &= e^{i\pi b'} x_2^{c-a-b+b'} y_2^{c-a-b} \tilde{\kappa}_4(x_2,y_2),
\end{aligned} \tag{3.15}$$

where $\tilde{\kappa}_j$ are holomorphic at $x_2 = 0, y_2 = 0$ and

$$\tilde{\kappa}_j(0,0) = 1 \ (1 \le j \le 4), \quad \tilde{\kappa}_j(x_2,0) \equiv 1 \ (1 \le j \le 2).$$

We now have a blowing up

$$x = (1-y_1)x_1, \quad y = y_1 \tag{3.16}$$

at the point (0,1) and obtain the exceptional divisor

$$X' = \{y_1 = 1\}.$$

By the symmety

$$(a,b,b',c,c',x,y) \longrightarrow (a,b',b,c',c,y,x), \tag{3.17}$$

we obtain the following local solutions at three points on the exceptional divisor X'. At the point $x_1 = 0,\ y_1 = 1$, the following solutions exist:

$$\begin{aligned}
\varphi_1'(x_1,y_1) &= \tilde{\varphi}_1'(x_1,y_1), \\
\varphi_2'(x_1,y_1) &= (1-y_1)^{1-c} x_1^{1-c} \tilde{\varphi}_2'(x_1,y_1), \\
\varphi_3'(x_1,y_1) &= (1-y_1)^{c'-a-b'} \tilde{\varphi}_3'(x_1,y_1), \\
\varphi_4'(x_1,y_1) &= (1-y_1)^{c'-a-b'} x_1^{1-c} \tilde{\varphi}_4'(x_1,y_1),
\end{aligned} \tag{3.18}$$

where $\tilde{\varphi}'_j$ are holomorphic at $x_1 = 0, y_1 = 1$ and

$$\tilde{\varphi}'_j(0,1) = 1 \ (1 \le j \le 4), \quad \tilde{\varphi}'_j(x_1, 1) \equiv 1 \ (1 \le j \le 2).$$

At the point $x_1 = 1,\ y_1 = 1$, the following solutions exist:

$$\begin{aligned}
\psi'_1(x_1,y_1) &= \tilde{\psi}'_1(x_1,y_1),\\
\psi'_2(x_1,y_1) &= (1-y_1)^{1-c}x_1^{1-c}\tilde{\psi}'_2(x_1,y_1),\\
\psi'_3(x_1,y_1) &= (1-y_1)^{c'-a-b'}\tilde{\psi}'_3(x_1,y_1),\\
\psi'_4(x_1,y_1) &= (1-y_1)^{c'-a-b'}(1-x_1)^{c+c'-a-b-b'}\tilde{\psi}'_4(x_1,y_1),
\end{aligned} \tag{3.19}$$

where $\tilde{\psi}'_j$ are holomorphic at $x_1 = 1,\ y_1 = 1$ and

$$\tilde{\psi}'_j(1,1) = 1 \ (1 \le j \le 4), \quad \tilde{\psi}'_j(x_1, 1) \equiv 1 \ (1 \le j \le 2).$$

At the point $x_1 = \infty,\ y_1 = 1$, we can choose

$$y_2 = 1/x_1 \quad \text{and} \quad x_2 = x_1(1-y_1)$$

as the local coordinates there, and the following solutions exist:

$$\begin{aligned}
\kappa'_1(x_2,y_2) &= \tilde{\kappa}'_1(x_2,y_2),\\
\kappa'_2(x_2,y_2) &= x_2^{1-c}\tilde{\kappa}'_2(x_2,y_2),\\
\kappa'_3(x_2,y_2) &= e^{i\pi(a+b'-c')}x_2^{c'-a-b'}\tilde{\kappa}'_3(x_2,y_2),\\
\kappa'_4(x_2,y_2) &= e^{i\pi b}y_2^{c'-a-b'+b}x_2^{c'-a-b'}\tilde{\kappa}'_4(x_2,y_2),
\end{aligned} \tag{3.20}$$

where $\tilde{\kappa}'_j$ are holomorphic at $x_2 = 0, y_2 = 0$ and

$$\tilde{\kappa}'_j(0,0) = 1 \ (1 \le j \le 4), \quad \tilde{\kappa}'_j(0, y_2) \equiv 1 \ (1 \le j \le 2).$$

4 Connection formulas.

In this section we derive the connection formulas among the fundamental systems f_j (see Section 1), φ_j, ψ_j, κ_j φ'_j, ψ'_j, κ'_j (see Section 3) of solutions of $E_2(a,b,b',c,c')$. In order to determine the values of these multi-valued solutions, we take the argument of positive number to be zero.

Formula 4.1 (The connection formula along $\{0 \le x \le 1,\ y=0\}$**).**

$$f_1 = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\varphi_1 + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\varphi_3,$$

$$f_2 = \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)}\varphi_1 + \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(1+a-c)\Gamma(1+b-c)}\varphi_3,$$

$$f_3 = \frac{\Gamma(c)\Gamma(c+c'-a-b-1)}{\Gamma(c-b)\Gamma(c+c'-a-1)}\varphi_2 + \frac{\Gamma(c)\Gamma(1+a+b-c-c')}{\Gamma(b)\Gamma(1+a-c')}\varphi_4,$$

$$f_4 = \frac{\Gamma(2-c)\Gamma(c+c'-a-b-1)}{\Gamma(1-b)\Gamma(c'-a)}\varphi_2 + \frac{\Gamma(2-c)\Gamma(1+a+b-c-c')}{\Gamma(1+b-c)\Gamma(2+a-c-c')}\varphi_4,$$

$$\varphi_1 = \frac{\Gamma(1+a+b-c)\Gamma(1-c)}{\Gamma(1+a-c)\Gamma(1+b-c)}f_1 + \frac{\Gamma(1+a+b-c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)}f_2,$$

$$\varphi_2 = \frac{\Gamma(2+a+b-c-c')\Gamma(1-c)}{\Gamma(2+a-c-c')\Gamma(1+b-c)}f_3 + \frac{\Gamma(2+a+b-c-c')\Gamma(c-1)}{\Gamma(1+a-c')\Gamma(b)}f_4,$$

$$\varphi_3 = \frac{\Gamma(1+c-a-b)\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)}f_1 + \frac{\Gamma(1+c-a-b)\Gamma(c-1)}{\Gamma(c-a)\Gamma(c-b)}f_2,$$

$$\varphi_4 = \frac{\Gamma(c+c'-a-b)\Gamma(1-c)}{\Gamma(c'-a)\Gamma(1-b)}f_3 + \frac{\Gamma(c+c'-a-b)\Gamma(c-1)}{\Gamma(c+c'-a-1)\Gamma(c-b)}f_4.$$

Formula 4.2 (The connection formula along $\{x_1 = 1,\ 0 \le y_1 \le 1\}$).

$$\varphi_1 = \psi_1,$$

$$\varphi_2 = \psi_2,$$

$$\varphi_3 = \frac{\Gamma(c')\Gamma(c+c'-a-b-b')}{\Gamma(c+c'-a-b)\Gamma(c'-b')}\psi_3 + \frac{\Gamma(c')\Gamma(a+b+b'-c-c')}{\Gamma(a+b-c)\Gamma(b')}\psi_4,$$

$$\varphi_4 = \frac{\Gamma(2-c')\Gamma(c+c'-a-b-b')}{\Gamma(1+c-a-b)\Gamma(1-b')}\psi_3 + \frac{\Gamma(2-c')\Gamma(a+b+b'-c-c')}{\Gamma(1+a+b-c-c')\Gamma(1+b'-c')}\psi_4,$$

$$\psi_3 = \frac{\Gamma(1+a+b+b'-c-c')\Gamma(1-c')}{\Gamma(1+a+b-c-c')\Gamma(1+b'-c')}\varphi_3 + \frac{\Gamma(1+a+b+b'-c-c')\Gamma(c'-1)}{\Gamma(a+b-c)\Gamma(b')}\varphi_4,$$

$$\psi_4 = \frac{\Gamma(1+c+c'-a-b-b')\Gamma(1-c')}{\Gamma(1+c-a-b)\Gamma(1-b')}\varphi_3 + \frac{\Gamma(1+c+c'-a-b-b')\Gamma(c'-1)}{\Gamma(c+c'-a-b)\Gamma(c'-b')}\varphi_4.$$

Formula 4.3 (The connection formula along $\{x_1 = 1,\ y_1 > 0\}$).

$$\varphi_1 = \kappa_1,$$

$$\varphi_2 = \kappa_2,$$

$$\varphi_3 = \frac{\Gamma(c')\Gamma(b'+c-a-b)}{\Gamma(c+c'-a-b)\Gamma(b')}\kappa_3 + \frac{\Gamma(c')\Gamma(a+b-b'-c)}{\Gamma(c'-b')\Gamma(a+b-c)}\kappa_4,$$

$$\varphi_4 = e^{i\pi(1-c')}\left\{\frac{\Gamma(2-c')\Gamma(b'+c-a-b)}{\Gamma(1+c-a-b)\Gamma(1+b'-c')}\kappa_3 + \frac{\Gamma(2-c')\Gamma(a+b-b'-c)}{\Gamma(1-b')\Gamma(1+a+b-c-c')}\kappa_4\right\},$$

$$\kappa_3 = \frac{\Gamma(1-c')\Gamma(1+a+b-b'-c)}{\Gamma(1-b')\Gamma(1+a+b-c-c')}\varphi_3 + \frac{\Gamma(c'-1)\Gamma(1+a+b-b'-c)}{\Gamma(c'-b')\Gamma(a+b-c)}e^{i\pi(c'-1)}\varphi_4,$$

$$\kappa_4 = \frac{\Gamma(1+b'+c-a-b)\Gamma(1-c')}{\Gamma(1+c-a-b)\Gamma(1+b'-c')}\varphi_3 + \frac{\Gamma(1+b'+c-a-b)\Gamma(c'-1)}{\Gamma(b')\Gamma(c+c'-a-b)}e^{i\pi(c'-1)}\varphi_4.$$

Formula 4.4 (The connection formula among f and ψ).

$$f_1 = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\psi_1 + \frac{\Gamma(c)\Gamma(c')\Gamma(a+b-c)\Gamma(c+c'-a-b-b')}{\Gamma(a)\Gamma(b)\Gamma(c+c'-a-b)\Gamma(c'-b')}\psi_3$$
$$+ \frac{\Gamma(c)\Gamma(c')\Gamma(a+b+b'-c-c')}{\Gamma(a)\Gamma(b)\Gamma(b')}\psi_4,$$

$$f_2 = \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)}\psi_1$$
$$+ \frac{\Gamma(2-c)\Gamma(c')\Gamma(a+b-c)\Gamma(c+c'-a-b-b')}{\Gamma(1+a-c)\Gamma(1+b-c)\Gamma(c+c'-a-b)\Gamma(c'-b')}\psi_3$$
$$+ \frac{\Gamma(2-c)\Gamma(c')\Gamma(a+b+b'-c-c')}{\Gamma(1+a-c)\Gamma(1+b-c)\Gamma(b')}\psi_4,$$

$$f_3 = \frac{\Gamma(c)\Gamma(c+c'-a-b-1)}{\Gamma(c+c'-a-1)\Gamma(c-b)}\psi_2$$
$$+ \frac{\Gamma(c)\Gamma(2-c')\Gamma(1+a+b-c-c')\Gamma(c+c'-a-b-b')}{\Gamma(1+a-c')\Gamma(b)\Gamma(1-b')\Gamma(1+c-a-b)}\psi_3$$
$$+ \frac{\Gamma(c)\Gamma(2-c')\Gamma(a+b+b'-c-c')}{\Gamma(1+a-c')\Gamma(b)\Gamma(1+b'-c')}\psi_4,$$

$$f_4 = \frac{\Gamma(2-c)\Gamma(c+c'-a-b-1)}{\Gamma(c'-a)\Gamma(1-b)}\psi_2$$
$$+ \frac{\Gamma(2-c)\Gamma(2-c')\Gamma(1+a+b-c-c')\Gamma(c+c'-a-b-b')}{\Gamma(2+a-c-c')\Gamma(1+b-c)\Gamma(1-b')\Gamma(1+c-a-b)}\psi_3$$
$$+ \frac{\Gamma(2-c)\Gamma(2-c')\Gamma(a+b+b'-c-c')}{\Gamma(2+a-c-c')\Gamma(1+b-c)\Gamma(1+b'-c')}\psi_4,$$

$$\psi_1 = \frac{\Gamma(1+a+b-c)\Gamma(1-c)}{\Gamma(1+a-c)\Gamma(1+b-c)} f_1 + \frac{\Gamma(1+a+b-c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)} f_2,$$

$$\psi_2 = \frac{\Gamma(2+a+b-c-c')\Gamma(1-c)}{\Gamma(2+a-c-c')\Gamma(1+b-c)} f_3 + \frac{\Gamma(2+a+b-c-c')\Gamma(c-1)}{\Gamma(1+a-c')\Gamma(b)} f_4,$$

$$\psi_3 = \frac{\Gamma(1+a+b+b'-c-c')\Gamma(1-c)\Gamma(1-c')\Gamma(1+c-a-b)}{\Gamma(1+a+b-c-c')\Gamma(1+b'-c')\Gamma(1-a)\Gamma(1-b)} f_1$$
$$+ \frac{\Gamma(1+a+b+b'-c-c')\Gamma(c-1)\Gamma(1-c')\Gamma(1+c-a-b)}{\Gamma(1+a+b-c-c')\Gamma(1+b'-c')\Gamma(c-a)\Gamma(c-b)} f_2$$
$$+ \frac{\Gamma(1+a+b+b'-c-c')\Gamma(1-c)\Gamma(c'-1)\Gamma(c+c'-a-b)}{\Gamma(a+b-c)\Gamma(b')\Gamma(c'-a)\Gamma(1-b)} f_3$$
$$+ \frac{\Gamma(1+a+b+b'-c-c')\Gamma(c-1)\Gamma(c'-1)\Gamma(c+c'-a-b)}{\Gamma(a+b-c)\Gamma(b')\Gamma(c+c'-a-1)\Gamma(c-b)} f_4,$$

$$
\begin{aligned}
\psi_4 = {} & \frac{\Gamma(1+c+c'-a-b-b')\Gamma(1-c)\Gamma(1-c')}{\Gamma(1-a)\Gamma(1-b)\Gamma(1-b')} f_1 \\
& + \frac{\Gamma(1+c+c'-a-b-b')\Gamma(c-1)\Gamma(1-c')}{\Gamma(c-a)\Gamma(c-b)\Gamma(1-b')} f_2 \\
& + \frac{\Gamma(1+c+c'-a-b-b')\Gamma(1-c)\Gamma(c'-1)}{\Gamma(c'-a)\Gamma(1-b)\Gamma(c'-b')} f_3 \\
& + \frac{\Gamma(1+c+c'-a-b-b')\Gamma(c-1)\Gamma(c'-1)}{\Gamma(c+c'-a-1)\Gamma(c-b)\Gamma(c'-b')} f_4.
\end{aligned}
$$

Formula 4.5 (The connection formula among f and κ).

$$f_1 = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\kappa_1 + \frac{\Gamma(c)\Gamma(c')\Gamma(a+b-c)\Gamma(b'+c-a-b)}{\Gamma(a)\Gamma(b)\Gamma(b')\Gamma(c+c'-a-b)}\kappa_3 + \frac{\Gamma(c)\Gamma(c')\Gamma(a+b-b'-c)}{\Gamma(a)\Gamma(b)\Gamma(c'-b')}\kappa_4,$$

$$f_2 = \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)}\kappa_1 + \frac{\Gamma(2-c)\Gamma(c')\Gamma(a+b-c)\Gamma(b'+c-a-b)}{\Gamma(1+a-c)\Gamma(1+b-c)\Gamma(b')\Gamma(c+c'-a-b)}\kappa_3 + \frac{\Gamma(2-c)\Gamma(c')\Gamma(a+b-b'-c)}{\Gamma(1+a-c)\Gamma(1+b-c)\Gamma(c'-b')}\kappa_4,$$

$$f_3 = \frac{\Gamma(c)\Gamma(c+c'-a-b-1)}{\Gamma(c-b)\Gamma(c+c'-a-1)}\kappa_2 + e^{i\pi(1-c')}\left\{\frac{\Gamma(c)\Gamma(2-c')\Gamma(1+a+b-c-c')\Gamma(b'+c-a-b)}{\Gamma(1+a-c')\Gamma(b)\Gamma(1+b'-c')\Gamma(1+c-a-b)}\kappa_3 + \frac{\Gamma(c)\Gamma(2-c')\Gamma(a+b-b'-c)}{\Gamma(1+a-c')\Gamma(b)\Gamma(1-b')}\kappa_4\right\},$$

$$f_4 = \frac{\Gamma(2-c)\Gamma(c+c'-a-b-1)}{\Gamma(c'-a)\Gamma(1-b)}\kappa_2 + e^{i\pi(1-c')}\left\{\frac{\Gamma(2-c)\Gamma(2-c')\Gamma(1+a+b-c-c')\Gamma(b'+c-a-b)}{\Gamma(2+a-c-c')\Gamma(1+b-c)\Gamma(1+b'-c')\Gamma(1+c-a-b)}\kappa_3 + \frac{\Gamma(2-c)\Gamma(2-c')\Gamma(a+b-b'-c)}{\Gamma(2+a-c-c')\Gamma(1+b-c)\Gamma(1-b')}\kappa_4\right\},$$

$$\kappa_1 = \frac{\Gamma(1+a+b-c)\Gamma(1-c)}{\Gamma(1+a-c)\Gamma(1+b-c)}f_1 + \frac{\Gamma(1+a+b-c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)}f_2,$$

$$\kappa_2 = \frac{\Gamma(2+a+b-c-c')\Gamma(1-c)}{\Gamma(2+a-c-c')\Gamma(1+b-c)}f_3 + \frac{\Gamma(2+a+b-c-c')\Gamma(c-1)}{\Gamma(1+a-c')\Gamma(b)}f_4,$$

$$\kappa_3 = \frac{\Gamma(1+a+b-b'-c)\Gamma(1-c)\Gamma(1-c')\Gamma(1+c-a-b)}{\Gamma(1-a)\Gamma(1-b)\Gamma(1-b')\Gamma(1+a+b-c-c')}f_1 + \frac{\Gamma(1+a+b-b'-c)\Gamma(c-1)\Gamma(1-c')\Gamma(1+c-a-b)}{\Gamma(c-a)\Gamma(c-b)\Gamma(1-b')\Gamma(1+a+b-c-c')}f_2 + \frac{\Gamma(1+a+b-b'-c)\Gamma(1-c)\Gamma(c'-1)\Gamma(c+c'-a-b)}{\Gamma(c'-a)\Gamma(1-b)\Gamma(c'-b')\Gamma(a+b-c)}e^{i\pi(c'-1)}f_3 + \frac{\Gamma(1+a+b-b'-c)\Gamma(c-1)\Gamma(c'-1)\Gamma(c+c'-a-b)}{\Gamma(c+c'-a-1)\Gamma(c-b)\Gamma(c'-b')\Gamma(a+b-c)}e^{i\pi(c'-1)}f_4,$$

$$
\begin{aligned}
\kappa_4 = & \frac{\Gamma(1+b'+c-a-b)\Gamma(1-c)\Gamma(1-c')}{\Gamma(1-a)\Gamma(1-b)\Gamma(1+b'-c')} f_1 \\
& + \frac{\Gamma(1+b'+c-a-b)\Gamma(c-1)\Gamma(1-c')}{\Gamma(c-a)\Gamma(c-b)\Gamma(1+b'-c')} f_2 \\
& + \frac{\Gamma(1+b'+c-a-b)\Gamma(1-c)\Gamma(c'-1)}{\Gamma(c'-a)\Gamma(1-b)\Gamma(b')} e^{i\pi(c'-1)} f_3 \\
& + \frac{\Gamma(1+b'+c-a-b)\Gamma(c-1)\Gamma(c'-1)}{\Gamma(c+c'-a-1)\Gamma(c-b)\Gamma(b')} e^{i\pi(c'-1)} f_4.
\end{aligned}
$$

By the symmetry (3.14), we have the following connection formulas. We remark that

$$f_2 \text{ and } f_3 \text{ are exchanged by (3.14).}$$

Formula 4.6 (The connection formula among f and ψ').

$$f_1 = \frac{\Gamma(c')\Gamma(c'-a-b')}{\Gamma(c'-a)\Gamma(c'-b')}\psi'_1 + \frac{\Gamma(c)\Gamma(c')\Gamma(a+b'-c')\Gamma(c+c'-a-b-b')}{\Gamma(a)\Gamma(b')\Gamma(c+c'-a-b')\Gamma(c-b)}\psi'_3 + \frac{\Gamma(c)\Gamma(c')\Gamma(a+b+b'-c-c')}{\Gamma(a)\Gamma(b)\Gamma(b')}\psi'_4,$$

$$f_3 = \frac{\Gamma(2-c')\Gamma(c'-a-b')}{\Gamma(1-a)\Gamma(1-b')}\psi'_1 + \frac{\Gamma(2-c')\Gamma(c)\Gamma(a+b'-c')\Gamma(c+c'-a-b-b')}{\Gamma(1+a-c')\Gamma(1+b'-c')\Gamma(c+c'-a-b')\Gamma(c-b)}\psi'_3 + \frac{\Gamma(2-c')\Gamma(c)\Gamma(a+b+b'-c-c')}{\Gamma(1+a-c')\Gamma(1+b'-c')\Gamma(b)}\psi'_4,$$

$$f_2 = \frac{\Gamma(c')\Gamma(c+c'-a-b'-1)}{\Gamma(c+c'-a-1)\Gamma(c'-b')}\psi'_2 + \frac{\Gamma(c')\Gamma(2-c)\Gamma(1+a+b'-c-c')\Gamma(c+c'-a-b-b')}{\Gamma(1+a-c)\Gamma(b')\Gamma(1-b)\Gamma(1+c'-a-b')}\psi'_3 + \frac{\Gamma(c')\Gamma(2-c)\Gamma(a+b+b'-c-c')}{\Gamma(1+a-c)\Gamma(b')\Gamma(1+b-c)}\psi'_4,$$

$$f_4 = \frac{\Gamma(2-c')\Gamma(c+c'-a-b'-1)}{\Gamma(c-a)\Gamma(1-b')}\psi'_2 + \frac{\Gamma(2-c)\Gamma(2-c')\Gamma(1+a+b'-c-c')\Gamma(c+c'-a-b-b')}{\Gamma(2+a-c-c')\Gamma(1+b'-c')\Gamma(1-b)\Gamma(1+c'-a-b')}\psi'_3 + \frac{\Gamma(2-c)\Gamma(2-c')\Gamma(a+b+b'-c-c')}{\Gamma(2+a-c-c')\Gamma(1+b-c)\Gamma(1+b'-c')}\psi'_4,$$

$$\psi'_1 = \frac{\Gamma(1+a+b'-c')\Gamma(1-c')}{\Gamma(1+a-c')\Gamma(1+b'-c')} f_1 + \frac{\Gamma(1+a+b'-c')\Gamma(c'-1)}{\Gamma(a)\Gamma(b')} f_3,$$

$$\psi'_2 = \frac{\Gamma(2+a+b'-c-c')\Gamma(1-c')}{\Gamma(2+a-c-c')\Gamma(1+b'-c')} f_2 + \frac{\Gamma(2+a+b'-c-c')\Gamma(c'-1)}{\Gamma(1+a-c)\Gamma(b')} f_4,$$

$$\psi'_3 = \frac{\Gamma(1+a+b+b'-c-c')\Gamma(1-c)\Gamma(1-c')\Gamma(1+c'-a-b')}{\Gamma(1+a+b'-c-c')\Gamma(1+b-c)\Gamma(1-a)\Gamma(1-b')} f_1 + \frac{\Gamma(1+a+b+b'-c-c')\Gamma(c'-1)\Gamma(1-c)\Gamma(1+c'-a-b')}{\Gamma(1+a+b'-c-c')\Gamma(1+b-c)\Gamma(c'-a)\Gamma(c'-b')} f_3 + \frac{\Gamma(1+a+b+b'-c-c')\Gamma(1-c')\Gamma(c-1)\Gamma(c+c'-a-b')}{\Gamma(a+b'-c')\Gamma(b)\Gamma(c-a)\Gamma(1-b')} f_2 + \frac{\Gamma(1+a+b+b'-c-c')\Gamma(c-1)\Gamma(c'-1)\Gamma(c+c'-a-b')}{\Gamma(a+b'-c')\Gamma(b)\Gamma(c+c'-a-1)\Gamma(c'-b')} f_4,$$

$$\begin{aligned}\psi_4' = &\frac{\Gamma(1+c+c'-a-b-b')\Gamma(1-c)\Gamma(1-c')}{\Gamma(1-a)\Gamma(1-b)\Gamma(1-b')} f_1 \\ &+ \frac{\Gamma(1+c+c'-a-b-b')\Gamma(c'-1)\Gamma(1-c)}{\Gamma(c'-a)\Gamma(c'-b')\Gamma(1-b)} f_3 \\ &+ \frac{\Gamma(1+c+c'-a-b-b')\Gamma(1-c')\Gamma(c-1)}{\Gamma(c-a)\Gamma(1-b')\Gamma(c-b)} f_2 \\ &+ \frac{\Gamma(1+c+c'-a-b-b')\Gamma(c-1)\Gamma(c'-1)}{\Gamma(c+c'-a-1)\Gamma(c-b)\Gamma(c'-b')} f_4.\end{aligned}$$

Formula 4.7 (The connection formula among f and κ').

$$f_1 = \frac{\Gamma(c')\Gamma(c'-a-b')}{\Gamma(c'-a)\Gamma(c'-b')}\kappa_1' + \frac{\Gamma(c)\Gamma(c')\Gamma(a+b'-c')\Gamma(b+c'-a-b')}{\Gamma(a)\Gamma(b)\Gamma(b')\Gamma(c+c'-a-b')}\kappa_3'$$
$$+ \frac{\Gamma(c)\Gamma(c')\Gamma(a+b'-b-c')}{\Gamma(a)\Gamma(b')\Gamma(c-b)}\kappa_4',$$

$$f_3 = \frac{\Gamma(2-c')\Gamma(c'-a-b')}{\Gamma(1-a)\Gamma(1-b')}\kappa_1'$$
$$+ \frac{\Gamma(2-c')\Gamma(c)\Gamma(a+b'-c')\Gamma(b+c'-a-b')}{\Gamma(1+a-c')\Gamma(1+b'-c')\Gamma(b)\Gamma(c+c'-a-b')}\kappa_3'$$
$$+ \frac{\Gamma(2-c')\Gamma(c)\Gamma(a+b'-b-c')}{\Gamma(1+a-c')\Gamma(1+b'-c')\Gamma(c-b)}\kappa_4',$$

$$f_2 = \frac{\Gamma(c')\Gamma(c+c'-a-b'-1)}{\Gamma(c'-b')\Gamma(c+c'-a-1)}\kappa_2'$$
$$+ e^{i\pi(1-c)}\left\{\frac{\Gamma(c')\Gamma(2-c)\Gamma(1+a+b'-c-c')\Gamma(b+c'-a-b')}{\Gamma(1+a-c)\Gamma(b')\Gamma(1+b-c)\Gamma(1+c'-a-b')}\kappa_3'\right.$$
$$\left. + \frac{\Gamma(c')\Gamma(2-c)\Gamma(a+b'-b-c')}{\Gamma(1+a-c)\Gamma(b')\Gamma(1-b)}\kappa_4'\right\},$$

$$f_4 = \frac{\Gamma(2-c')\Gamma(c+c'-a-b'-1)}{\Gamma(c-a)\Gamma(1-b')}\kappa_2'$$
$$+ e^{i\pi(1-c)}\Bigg\{$$
$$\frac{\Gamma(2-c)\Gamma(2-c')\Gamma(1+a+b'-c-c')\Gamma(b+c'-a-b')}{\Gamma(2+a-c-c')\Gamma(1+b-c)\Gamma(1+b'-c')\Gamma(1+c'-a-b')}\kappa_3'$$
$$\left. + \frac{\Gamma(2-c)\Gamma(2-c')\Gamma(a+b'-b-c')}{\Gamma(2+a-c-c')\Gamma(1+b'-c')\Gamma(1-b)}\kappa_4'\right\},$$

$$\kappa_1' = \frac{\Gamma(1+a+b'-c')\Gamma(1-c')}{\Gamma(1+a-c')\Gamma(1+b'-c')}f_1 + \frac{\Gamma(1+a+b'-c')\Gamma(c'-1)}{\Gamma(a)\Gamma(b')}f_3,$$

$$\kappa_2' = \frac{\Gamma(2+a+b-c-c')\Gamma(1-c')}{\Gamma(2+a-c-c')\Gamma(1+b'-c')}f_2 + \frac{\Gamma(2+a+b'-c-c')\Gamma(c'-1)}{\Gamma(1+a-c)\Gamma(b')}f_4,$$

$$\kappa_3' = \frac{\Gamma(1+a+b'-b-c')\Gamma(1-c)\Gamma(1-c')\Gamma(1+c'-a-b')}{\Gamma(1-a)\Gamma(1-b)\Gamma(1-b')\Gamma(1+a+b'-c-c')}f_1$$
$$+ \frac{\Gamma(1+a+b'-b-c')\Gamma(c'-1)\Gamma(1-c)\Gamma(1+c'-a-b')}{\Gamma(c'-a)\Gamma(c'-b')\Gamma(1-b)\Gamma(1+a+b'-c-c')}f_3$$
$$+ \frac{\Gamma(1+a+b'-b-c')\Gamma(1-c')\Gamma(c-1)\Gamma(c+c'-a-b')}{\Gamma(c-a)\Gamma(1-b')\Gamma(c-b)\Gamma(a+b'-c')}e^{i\pi(c-1)}f_2$$
$$+ \frac{\Gamma(1+a+b'-b-c')\Gamma(c-1)\Gamma(c'-1)\Gamma(c+c'-a-b')}{\Gamma(c+c'-a-1)\Gamma(c-b)\Gamma(c'-b')\Gamma(a+b'-c')}e^{i\pi(c-1)}f_4,$$

$$\begin{aligned}\kappa_4' &= \frac{\Gamma(1+b+c'-a-b')\Gamma(1-c)\Gamma(1-c')}{\Gamma(1-a)\Gamma(1-b')\Gamma(1+b-c)}\, f_1 \\ &+ \frac{\Gamma(1+b+c'-a-b')\Gamma(c'-1)\Gamma(1-c)}{\Gamma(c'-a)\Gamma(c'-b')\Gamma(1+b-c)}\, f_3 \\ &+ \frac{\Gamma(1+b+c'-a-b')\Gamma(1-c')\Gamma(c-1)}{\Gamma(c-a)\Gamma(1-b')\Gamma(b)}\, e^{i\pi(c-1)} f_2 \\ &+ \frac{\Gamma(1+b+c'-a-b')\Gamma(c-1)\Gamma(c'-1)}{\Gamma(c+c'-a-1)\Gamma(c'-b')\Gamma(b)}\, e^{i\pi(c-1)} f_4.\end{aligned}$$

Formula 4.8 (The connection formula among ψ and ψ').

$$\psi_1 = \frac{\Gamma(1+a+b-c)\Gamma(1-c)\Gamma(c')\Gamma(c'-a-b')}{\Gamma(1+a-c)\Gamma(1+b-c)\Gamma(c'-a)\Gamma(c'-b')}\psi_1'$$
$$+\frac{\Gamma(1+a+b-c)\Gamma(c-1)\Gamma(c')\Gamma(c+c'-a-b'-1)}{\Gamma(a)\Gamma(b)\Gamma(c+c'-a-1)\Gamma(c'-b')}\psi_2'$$
$$+\frac{\Gamma(1+a+b-c)\Gamma(c')\Gamma(a+b'-c')\Gamma(1+a+b'-c-c')}{\Gamma(a)\Gamma(b')\Gamma(1+a-c)\Gamma(1+a+b+b'-c-c')}\psi_3',$$

$$\psi_2 = \frac{\Gamma(2+a+b-c-c')\Gamma(1-c)\Gamma(2-c')\Gamma(c'-a-b')}{\Gamma(2+a-c-c')\Gamma(1+b-c)\Gamma(1-a)\Gamma(1-b')}\psi_1'$$
$$+\frac{\Gamma(2+a+b-c-c')\Gamma(c-1)\Gamma(2-c')\Gamma(c+c'-a-b'-1)}{\Gamma(1+a-c')\Gamma(b)\Gamma(c-a)\Gamma(1-b')}\psi_2'$$
$$+\frac{\Gamma(2+a+b-c-c')\Gamma(2-c')\Gamma(a+b'-c')\Gamma(1+a+b'-c-c')}{\Gamma(2+a-c-c')\Gamma(1+a-c')\Gamma(1+b'-c')\Gamma(1+a+b+b'-c-c')}\psi_3',$$

$$\psi_3 = \frac{\Gamma(c+c'-a-b)\Gamma(1-c)\Gamma(1+c-a-b)\Gamma(c'-a-b')}{\Gamma(1-a)\Gamma(1-b)\Gamma(c'-a)\Gamma(c+c'-a-b-b')}\psi_1'$$
$$+\frac{\Gamma(c+c'-a-b)\Gamma(c-1)\Gamma(1+c-a-b)\Gamma(c+c'-a-b'-1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(c+c'-a-1)\Gamma(c+c'-a-b-b')}\psi_2'$$
$$+\frac{\Gamma(c+c'-a-b)\Gamma(a+b'-c')\Gamma(1+c-a-b)\Gamma(1+a+b'-c-c')}{\Gamma(1-b)\Gamma(c-b)\Gamma(b')\Gamma(1+b'-c')}\psi_3',$$

$$\psi_4 = \psi_4',$$

$$\psi_1' = \frac{\Gamma(1+a+b'-c')\Gamma(1-c')\Gamma(c)\Gamma(c-a-b)}{\Gamma(1+a-c')\Gamma(1+b'-c')\Gamma(c'-a)\Gamma(c-b)}\psi_1$$
$$+\frac{\Gamma(1+a+b'-c')\Gamma(c'-1)\Gamma(c)\Gamma(c+c'-a-b-1)}{\Gamma(a)\Gamma(b')\Gamma(c+c'-a-1)\Gamma(c-b)}\psi_2$$
$$+\frac{\Gamma(1+a+b'-c')\Gamma(c)\Gamma(a+b-c)\Gamma(1+a+b-c-c')}{\Gamma(a)\Gamma(b)\Gamma(1+a-c')\Gamma(1+a+b+b'-c-c')}\psi_3,$$

$$\psi_2' = \frac{\Gamma(2+a+b'-c-c')\Gamma(1-c')\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(2+a-c-c')\Gamma(1+b'-c')\Gamma(1-a)\Gamma(1-b)}\psi_1$$
$$+\frac{\Gamma(2+a+b'-c-c')\Gamma(c'-1)\Gamma(2-c)\Gamma(c+c'-a-b-1)}{\Gamma(1+a-c)\Gamma(b')\Gamma(c'-a)\Gamma(1-b)}\psi_2$$
$$+\frac{\Gamma(2+a+b'-c-c')\Gamma(2-c)\Gamma(a+b-c)\Gamma(1+a+b-c-c')}{\Gamma(2+a-c-c')\Gamma(1+a-c)\Gamma(1+b-c)\Gamma(1+a+b+b'-c-c')}\psi_3,$$

$$\psi_3' = \frac{\Gamma(c+c'-a-b')\Gamma(1-c')\Gamma(1+c'-a-b')\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b')\Gamma(c-a)\Gamma(c+c'-a-b-b')}\psi_1$$
$$+\frac{\Gamma(c+c'-a-b')\Gamma(c'-1)\Gamma(1+c'-a-b')\Gamma(c+c'-a-b-1)}{\Gamma(c'-a)\Gamma(c'-b')\Gamma(c+c'-a-1)\Gamma(c+c'-a-b-b')}\psi_2$$
$$+\frac{\Gamma(c+c'-a-b')\Gamma(a+b-c)\Gamma(1+c'-a-b')\Gamma(1+a+b-c-c')}{\Gamma(1-b')\Gamma(c'-b')\Gamma(b)\Gamma(1+b-c)}\psi_3.$$

Formula 4.9 (The connection formula among κ and κ').

$$\kappa_1 = \Gamma(1+a+b-c)\left\{\frac{\Gamma(1-c)\Gamma(c')\Gamma(c'-a-b')}{\Gamma(1+a-c)\Gamma(1+b-c)\Gamma(c'-a)\Gamma(c'-b')}\kappa_1'\right.$$
$$+\frac{\Gamma(c-1)\Gamma(c')\Gamma(c+c'-a-b'-1)}{\Gamma(a)\Gamma(b)\Gamma(c+c'-a-1)\Gamma(c'-b')}\kappa_2'$$
$$+\frac{\Gamma(c')\Gamma(b-a-b'+c')\Gamma(a+b'-c')\Gamma(1+a+b'-c-c')}{\Gamma(a)\Gamma(b)\Gamma(b')\Gamma(1+a-c)\Gamma(1+b-c)}$$
$$\times e^{i\pi(c'-a-b')}\kappa_3'$$
$$\left.+\frac{\Gamma(c')\Gamma(a+b'-b-c')}{\Gamma(1+a-c)\Gamma(a)\Gamma(b')}e^{-i\pi b}\kappa_4'\right\},$$

$$\kappa_2 = \Gamma(2+a+b-c-c')\left\{\frac{\Gamma(1-c)\Gamma(2-c')\Gamma(c'-a-b')}{\Gamma(2+a-c-c')\Gamma(1+b-c)\Gamma(1-a)\Gamma(1-b')}\kappa_1'\right.$$
$$+\frac{\Gamma(c-1)\Gamma(2-c')\Gamma(c+c'-a-b'-1)}{\Gamma(1+a-c')\Gamma(b)\Gamma(c-a)\Gamma(1-b')}\kappa_2'$$
$$+\frac{\Gamma(2-c')\Gamma(b-a-b'+c')\Gamma(a+b'-c')\Gamma(1+a+b'-c-c')}{\Gamma(1+a-c')\Gamma(b)\Gamma(1+b'-c')\Gamma(2+a-c-c')\Gamma(1+b-c)}$$
$$\times e^{i\pi(c'-a-b')}\kappa_3'$$
$$\left.+\frac{\Gamma(2-c')\Gamma(a+b'-b-c')}{\Gamma(2+a-c-c')\Gamma(1+a-c')\Gamma(1+b'-c')}e^{-i\pi b}\kappa_4'\right\},$$

$$\kappa_3 = \Gamma(1+a+b-b'-c)\left\{\right.$$
$$\frac{\Gamma(1-c)\Gamma(c'-a-b')\Gamma(1+c-a-b)\Gamma(c+c'-a-b)}{\Gamma(1-a)\Gamma(1-b)\Gamma(1-b')\Gamma(c'-a)\Gamma(c'-b')}e^{i\pi(a+b-c)}\kappa_1'$$
$$+\frac{\Gamma(c-1)\Gamma(c+c'-a-b'-1)\Gamma(1+c-a-b)\Gamma(c+c'-a-b)}{\Gamma(c-a)\Gamma(c-b)\Gamma(1-b')\Gamma(c+c'-a-1)\Gamma(c'-b')}$$
$$\times e^{i\pi(a+b-c)}\kappa_2'$$
$$+\frac{1}{\pi^3}(\sin\pi a\sin\pi b\sin\pi b'$$
$$-\sin\pi(c-a-b)\sin\pi(c'-a-b')\sin\pi(c+c'-a))$$
$$\times\Gamma(1+a+b'-c-c')\Gamma(b+c'-a-b')\Gamma(a+b'-c')\Gamma(1+c-a-b)$$
$$\left.\times\Gamma(c+c'-a-b)e^{i\pi(c'-c+b-b')}\kappa_3'\right\}$$
$$+\frac{\Gamma(a+b'-b-c')\Gamma(c+c'-a-b)\Gamma(1+c-a-b)}{\Gamma(1-b)\Gamma(c-b)\Gamma(c-a-b+b')}e^{i\pi(a-c)}\kappa_4',$$

$$\kappa_4 = \Gamma(1+b'+c-a-b)\left\{ \frac{\Gamma(1-c)\Gamma(c'-a-b')}{\Gamma(1-a)\Gamma(1-b)\Gamma(c'-a)}\, e^{i\pi b'}\, \kappa_1' \right.$$

$$+ \frac{\Gamma(c-1)\Gamma(c+c'-a-b'-1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(c+c'-a-1)}\, e^{i\pi b'}\, \kappa_2'$$

$$\left. + \frac{\Gamma(a+b'-c')\Gamma(1+a+b'-c-c')}{\Gamma(b')\Gamma(1+b'-c')\Gamma(1+a+b'-b-c')}\, e^{i\pi(c'-a)}\, \kappa_3' \right\},$$

$$\kappa_1' = \Gamma(1+a+b'-c')\left\{ \frac{\Gamma(1-c')\Gamma(c)\Gamma(c-a-b)}{\Gamma(1+a-c')\Gamma(1+b'-c')\Gamma(c-a)\Gamma(c-b)}\, \kappa_1 \right.$$

$$+ \frac{\Gamma(c'-1)\Gamma(c)\Gamma(c+c'-a-b-1)}{\Gamma(a)\Gamma(b')\Gamma(c+c'-a-1)\Gamma(c-b)}\, \kappa_2$$

$$+ \frac{\Gamma(c)\Gamma(b'-a-b+c)\Gamma(a+b-c)\Gamma(1+a+b-c-c')}{\Gamma(a)\Gamma(b)\Gamma(b')\Gamma(1+a-c')\Gamma(1+b'-c')}$$

$$\times\, e^{i\pi(c-a-b)}\, \kappa_3$$

$$\left. + \frac{\Gamma(c)\Gamma(a+b-b'-c)}{\Gamma(1+a-c')\Gamma(a)\Gamma(b)}\, e^{-i\pi b'}\, \kappa_4 \right\},$$

$$\kappa_2' = \Gamma(2+a+b'-c-c')\left\{ \frac{\Gamma(1-c')\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(2+a-c-c')\Gamma(1+b'-c')\Gamma(1-a)\Gamma(1-b)}\, \kappa_1 \right.$$

$$+ \frac{\Gamma(c'-1)\Gamma(2-c)\Gamma(c+c'-a-b-1)}{\Gamma(1+a-c)\Gamma(b')\Gamma(c'-a)\Gamma(1-b)}\, \kappa_2$$

$$+ \frac{\Gamma(2-c)\Gamma(b'-a-b+c)\Gamma(a+b-c)\Gamma(1+a+b-c-c')}{\Gamma(1+a-c)\Gamma(b')\Gamma(1+b-c)\Gamma(2+a-c-c')\Gamma(1+b'-c')}$$

$$\times\, e^{i\pi(c-a-b)}\, \kappa_3$$

$$\left. + \frac{\Gamma(2-c)\Gamma(a+b-b'-c)}{\Gamma(2+a-c-c')\Gamma(1+a-c)\Gamma(1+b-c)}\, e^{-i\pi b'}\, \kappa_4 \right\},$$

$$
\begin{aligned}
\kappa_3' = \Gamma(1+a+b'-b-c')\Bigg\{ \\
&\frac{\Gamma(1-c')\Gamma(c-a-b)\Gamma(1+c'-a-b')\Gamma(c+c'-a-b')}{\Gamma(1-a)\Gamma(1-b)\Gamma(1-b')\Gamma(c-a)\Gamma(c-b)}\, e^{i\pi(a+b'-c')}\,\kappa_1 \\
&+ \frac{\Gamma(c'-1)\Gamma(c+c'-a-b-1)\Gamma(1+c'-a-b')\Gamma(c+c'-a-b')}{\Gamma(c'-a)\Gamma(c'-b')\Gamma(1-b)\Gamma(c+c'-a-1)\Gamma(c-b)} \\
&\times e^{i\pi(a+b'-c')}\,\kappa_2 \\
&+ \frac{1}{\pi^3}\,(\sin\pi a \sin\pi b \sin\pi b' \\
&\qquad - \sin\pi(c-a-b)\sin\pi(c'-a-b')\sin\pi(c+c'-a)) \\
&\times \Gamma(1+a+b-c-c')\Gamma(b'+c-a-b)\Gamma(a+b-c)\Gamma(1+c'-a-b') \\
&\times \Gamma(c+c'-a-b')e^{i\pi(c-c'+b'-b)}\,\kappa_3\Bigg\} \\
&+ \frac{\Gamma(a+b-b'-c)\Gamma(c+c'-a-b')\Gamma(1+c'-a-b')}{\Gamma(1-b')\Gamma(c'-b')\Gamma(c'-a-b'+b)}\, e^{i\pi(a-c')}\,\kappa_4,
\end{aligned}
$$

$$
\begin{aligned}
\kappa_4' = \Gamma(1+b+c'-a-b')\Bigg\{ &\frac{\Gamma(1-c')\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b')\Gamma(c-a)}\, e^{i\pi b}\,\kappa_1 \\
&+ \frac{\Gamma(c'-1)\Gamma(c+c'-a-b-1)}{\Gamma(c'-a)\Gamma(c'-b')\Gamma(c+c'-a-1)}\, e^{i\pi b}\,\kappa_2 \\
&+ \frac{\Gamma(a+b-c)\Gamma(1+a+b-c-c')}{\Gamma(b)\Gamma(1+b-c)\Gamma(1+a+b-b'-c)}\, e^{i\pi(c-a)}\,\kappa_3\Bigg\}.
\end{aligned}
$$

5 A monodromy representation of E_2.

We put

$$e(x) = e^{2\pi i x}$$

and assume in this section that

$$c,\ c',\ \notin \mathbf{Z} \tag{5.1}$$

and

$$a,\ c-a,\ c'-a,\ c+c'-a,\ b,\ c-b,\ b',\ c'-b' \ \notin \mathbf{Z} \tag{5.2}$$

so that the following system ω of fundamental solutions of $E_2(a, b, b', c, c')$ can be defined:

$$\begin{aligned}
\omega_1 &= \frac{\Gamma(a)\Gamma(b)\Gamma(b')}{\Gamma(c)\Gamma(c')} f_1, \\
\omega_2 &= \frac{\Gamma(1+a-c)\Gamma(1+b-c)\Gamma(b')}{\Gamma(2-c)\Gamma(c')} f_2, \\
\omega_3 &= \frac{\Gamma(1+a-c')\Gamma(b)\Gamma(1+b'-c')}{\Gamma(c)\Gamma(2-c')} f_3, \\
\omega_4 &= \frac{\Gamma(2+a-c-c')\Gamma(1+b-c)\Gamma(1+b'-c')}{\Gamma(2-c)\Gamma(2-c')} f_4.
\end{aligned} \tag{5.3}$$

Put

$$X = \mathbf{C}^2 - \{(x,y)\,|\, xy(x-1)(y-1)(x+y-1) = 0\},\ P_0 = (r,r) \tag{5.4}$$

for sufficiently small positive number r. Then the fundamental group $\pi_1(X, P_0)$ with the base point P_0 is generated by the following five curves:

$$\begin{aligned}
\gamma_1 &= \{(x,y)\,|\, x = re^{it}\ 0 \le t \le 2\pi,\ y = r\}, \\
\gamma_2 &= \{(x,y)\,|\, x = r,\ y = re^{it}\ 0 \le t \le 2\pi\}, \\
\gamma_3 &= \{(x,y)\,|\, x = y = 1/2 - (1/2 - r)e^{it}\ 0 \le t \le 2\pi\}, \\
\gamma_4 &= C_{diag} C_{x=1} C_{diag}^{-1}, \\
\gamma_5 &= C_{diag} C_{y=1} C_{diag}^{-1},
\end{aligned}$$

where

$$\begin{aligned}
C_{diag} &= \{x = y = 1/2 - (1/2 - r)e^{-it}\ 0 \le t \le \pi\}, \\
C_{x=1} &= \{x = 1 - re^{it}\ 0 \le t \le 2\pi,\ y = 1 - r\}, \\
C_{y=1} &= \{y = 1 - re^{it}\ 0 \le t \le 2\pi,\ x = 1 - r\}.
\end{aligned}$$

Let $V = V(P_0)$ be the set of germs of holomorphic solutions of E_2 at P_0. Then V is a four dimensional vector space. For $f \in V$ and $\gamma \in \pi_1(X, P_0)$, the analytic continuation $f\gamma_*$ of f along γ again belongs to $V(P_0)$. We write

$$f(\gamma\gamma')_* = (f\gamma_*)\gamma'_* = f\gamma_*\gamma'_*$$

if γ' is continued after γ. This defines a monodromy representation

$$\pi_1(X, P_0) \longrightarrow GL(V(P_0)). \tag{5.5}$$

We denote its image by

$$M_2(a, b, b', c, c'; P_0) = M_2(a, b, b', c, c')$$

and call it the monodromy group of $E_2(a, b, b', c, c')$.

We will express the representation (5.5) by use of the basis ω_j $1 \le j \le 4$ of V of (5.3).

It is easy to find the analytic continuation of ω_j along γ_1 and γ_2, that is;

$$\begin{aligned} \omega_k \gamma_{1*} &= \omega_k \quad \text{for} \quad k = 1, 3, \\ \omega_k \gamma_{1*} &= e^{-2\pi i c} \omega_k \quad \text{for} \quad k = 2, 4, \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} \omega_k \gamma_{2*} &= \omega_k \quad \text{for} \quad k = 1, 2, \\ \omega_k \gamma_{2*} &= e^{-2\pi i c'} \omega_k \quad \text{for} \quad k = 3, 4. \end{aligned} \tag{5.7}$$

By Formula 4.4, we have

$$\omega_k \gamma_{3*} - \omega_k = (e(c + c' - a - b - b') - 1)\Gamma(a + b + b' - c - c')\psi_4 \quad 1 \le k \le 4. \tag{5.8}$$

By Formula 4.5, we have

$$\begin{aligned} \omega_k \gamma_{4*} - \omega_k &= (e(c + c' - a - b - b') - 1)\frac{\Gamma(a + b - b' - c)\Gamma(b')}{\Gamma(c' - b')}\kappa_4 \quad k = 1, 2, \\ \omega_k \gamma_{4*} - \omega_k &= (e(c + c' - a - b - b') - 1) \\ &\quad \frac{\Gamma(a + b - b' - c)\Gamma(1 + b' - c')}{\Gamma(1 - b')} e((1 - c')/2)\kappa_4 \qquad k = 3, 4. \end{aligned} \tag{5.9}$$

Using again Formula 4.4 and 4.5, we have the following representation of γ_j $1 \le j \le 4$. By the symmetry ($x \longleftrightarrow y$, $b \longleftrightarrow b'$, $c \longleftrightarrow c'$), we also have the representation of γ_5.

Theorem 5.1. *Assume the non-integral condition (5.1). Then the analytic continuations $\omega\gamma_{j*}$ of*

$$\omega = {}^t(\omega_1, \omega_2, \omega_3, \omega_4)$$

along γ_j are as follows:

$$\omega\gamma_{1*} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e(1-c) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e(1-c) \end{pmatrix} \omega,$$

$$\omega\gamma_{2*} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e(1-c') & 0 \\ 0 & 0 & 0 & e(1-c') \end{pmatrix} \omega,$$

$$\omega\gamma_{3*} = \left(I_4 + e_3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} (\gamma_{31}, \gamma_{32}, \gamma_{33}, \gamma_{34}) \right) \omega,$$

$$\begin{aligned} e_3 &= 2i\, e\left((c + c' - a - b - b')/2\right) / \sin \pi c \sin \pi c', \\ \gamma_{31} &= -\sin \pi a \sin \pi b \sin \pi b', \quad \gamma_{32} = \sin \pi(c-a) \sin \pi(c-b) \sin \pi b', \\ \gamma_{33} &= \sin \pi(c'-a) \sin \pi b \sin \pi(c'-b'), \\ \gamma_{34} &= \sin \pi(c+c'-a) \sin \pi(c-b) \sin \pi(c'-b'), \end{aligned}$$

$$\omega\gamma_{4*} = \left(I_4 + e_4 \begin{pmatrix} \sin \pi(c'-b') \\ \sin \pi(c'-b') \\ e((1-c')/2) \sin \pi b' \\ e((1-c')/2) \sin \pi b' \end{pmatrix} (\gamma_{41}, \gamma_{42}, \gamma_{43}, \gamma_{44}) \right) \omega,$$

$$\begin{aligned} e_4 &= 2i\, e\left((c + b' - a - b)/2\right) / \sin \pi c \sin \pi c', \\ \gamma_{41} &= -\sin \pi a \sin \pi b, \quad \gamma_{42} = \sin \pi(c-a) \sin \pi(c-b), \\ \gamma_{43} &= e((c'-1)/2) \sin \pi(c'-a) \sin \pi b, \\ \gamma_{44} &= e((c'-1)/2) \sin \pi(c+c'-a) \sin \pi(c-b), \end{aligned}$$

$$\omega\gamma_{5*} = \left(I_4 + e_5 \begin{pmatrix} \sin \pi(c-b) \\ e((1-c)/2) \sin \pi b \\ \sin \pi(c-b) \\ e((1-c)/2) \sin \pi b \end{pmatrix} (\gamma_{51}, \gamma_{52}, \gamma_{53}, \gamma_{54}) \right) \omega,$$

$$\begin{aligned} e_5 &= 2i\, e\left((c' + b - a - b')/2\right) / \sin \pi c \sin \pi c', \\ \gamma_{51} &= -\sin \pi a \sin \pi b', \quad \gamma_{52} = e((c-1)/2) \sin \pi(c-a) \sin \pi b', \\ \gamma_{53} &= \sin \pi(c'-a) \sin \pi(c'-b'), \\ \gamma_{54} &= e((c-1)/2) \sin \pi(c+c'-a) \sin \pi(c'-b'). \end{aligned}$$

Remark. Since γ_4 and γ_5 commute in $\pi_1(X, P_0)$, we have

$$\gamma_{4*}\gamma_{5*} = \gamma_{5*}\gamma_{4*}.$$

6 Irreducibility condition of E_2 for $c, c' \notin \mathbf{Z}$.

We assume in this section that c, c' satisfy the non-integral condition (5.1). Under this assumption, f_j $1 \le j \le 4$ of (1.1) form a fundamental solutions of E_2.

Definition. E_2 is said to be **reducible** if there exists a proper subspace W of $V(P_0)$ which is invariant under the action of the monodromy group M_2. E_2 is said to be **irreducible** if it is not reducible.

We first consider the case when the condition (5.2) does not hold. Then we find several reducible cases.

Assume that a is a non-positive integer. Then $\langle f_1 \rangle$ is an invariant subspace of V because f_1 is a polynomial.

Assume that a is a positive integer. If other conditions in (5.2) hold, then ω_j $1 \le j \le 4$ of (5.3) form a fundamental solutions of E_2 and Theorem 5.1 holds. We know, from this theorem, that $\langle f_2, f_3, f_4 \rangle$ is an invariant subspace of V. By continuity, $\langle f_2, f_3, f_4 \rangle$ is an invariant subspace of V in general.

Assume that $b' = -n$ is a non-positive integer. Then we have

$$f_1 = \sum_{j=0}^{n} \frac{(a,j)(b',j)}{(c',j)(1,j)} F(a+j, b, c; x)\, y^j,$$

$$f_2 = x^{1-c} \sum_{j=0}^{n} \frac{(1+a-c,j)(b',j)}{(c',j)(1,j)} F(1+a+j-c, 1+b-c, 2-c; x)\, y^j,$$

where $F(a,b,c;x)$ is the Gauss' hypergeometric function. Hence $\langle f_1, f_2 \rangle$ is an invariant subspace of V.

Assume that b' is a positive integer. If other conditions in (5.2) hold, then we know, from Theorem 5.1, that $\langle f_3, f_4 \rangle$ is an invariant subspace of V. By continuity, $\langle f_3, f_4 \rangle$ is an invariant subspace of V in general.

In the same way, we have the list of invariant subspaces of $V = V(P_0)$. Let $\mathbf{Z}_{\le 0}$ denote the set of non-positive integers.

Table 6.1.

	$\mathbf{Z}_{\le 0}$	$\mathbf{N}$
a	$\langle f_1 \rangle$	$\langle f_2, f_3, f_4 \rangle$
$1+a-c$	$\langle f_2 \rangle$	$\langle f_1, f_3, f_4 \rangle$
$1+a-c'$	$\langle f_3 \rangle$	$\langle f_1, f_2, f_4 \rangle$
$2+a-c-c'$	$\langle f_4 \rangle$	$\langle f_1, f_2, f_3 \rangle$
b	$\langle f_1, f_3 \rangle$	$\langle f_2, f_4 \rangle$
b'	$\langle f_1, f_2 \rangle$	$\langle f_3, f_4 \rangle$
$1+b-c$	$\langle f_2, f_4 \rangle$	$\langle f_1, f_3 \rangle$
$1+b'-c'$	$\langle f_3, f_4 \rangle$	$\langle f_1, f_2 \rangle$

Rmark. The 3-dimensional invariant subspaces above can be expressed by Appell's F_1. For example, if $c + c' - a = 1$ then $\langle f_1, f_2, f_3 \rangle$ is the space of the solutions of the differential equations satisfied by $F_1(a,b,b',c'+b;1-x,y)$. This fact is equivalent to the classical functional equation ([Bly])

$$F_2(a,b,b',c,a;x,y) = (1-y)^{-b'} F_1(b,a-b',b',c;x,x/(1-y)).$$

Now we give the irreducibility condition of $E_2(a,b,b',c,c')$.

Theorem 6.1. *Assume that c, c' satisfy condition (5.1). Then $E_2(a,b,b',c,c')$ is irreducible if and only if condition (5.2) holds.*

Proof. We have proved the "only if" part by Table 6.1. So we will prove the "if" part. By (5.2), ω_j $1 \le j \le 4$ of (5.3) is a fundamental solutions of E_2 and we can use Theorem 5.1. Suppose that there exists a proper invariant subspace W of V under the actions γ_{j*} $1 \le j \le 5$. By the invariance under γ_{1*} and γ_{2*}, we know that W is spanned by a proper subset of $\{\omega_1, \omega_2, \omega_3, \omega_4\}$. Then by the invariance under γ_{3*}, W must contain all ω_j. This proves the "if" part and completes the proof of the theorem. □

Rmark. Assumption (5.1) on c, c' in the theorem above will be removed in Theorem 7.1.

7 Irreducibility condition of E_2 for general cases.

Theorem 7.1. *$E_2(a,b,b',c,c')$ is irreducible if and only if condition (5.2) holds.*

Proof. The "only if" part follows from Theorem 6.1 by continuity. So we will prove the "if" part.

Assume (5.2) holds. We put

$$\begin{aligned} f_1' &= \Gamma(a)\Gamma(b)\Gamma(b')\, f_1, \\ f_2' &= \Gamma(1+a-c)\Gamma(1+b-c)\Gamma(b')\, f_2, \\ f_3' &= \Gamma(1+a-c')\Gamma(b)\Gamma(1+b'-c')\, f_3, \\ f_4' &= \Gamma(2+a-c-c')\Gamma(1+b-c)\Gamma(1+b'-c')\, f_4, \end{aligned} \tag{7.1}$$

for $c, c' \notin \mathbf{Z}$.

We must prove the irreducibility in the case when c or c' is an integer. Since E_2 has the following "symmetries":

$$\begin{aligned} E_2(a,b,b',c,c') &\simeq E_2(a,b',b,c',c), \\ E_2(a,b,b',c,c') &\simeq E_2(1+a-c,1+b-c,b',2-c,c'), \\ E_2(a,b,b',c,c') &\simeq E_2(1+a-c',b,1-b'-c',c,2-c'), \\ E_2(a,b,b',c,c') &\simeq E_2(2+a-c-c',1+b-c,1+b'-c',2-c,2-c'), \end{aligned}$$

it suffices to consider two cases, that is, the case when $c \in \mathbf{N}$, $c' \notin \mathbf{Z}$ and the case when $c, c' \in \mathbf{N}$.

The case when $c \in \mathbf{N}$, $c' \notin \mathbf{Z}$.

Put

$$c = 1 + m - \delta,$$

where m is a non negative integer and δ is in a small neiborhood of 0. If δ tends to 0 then

$$\begin{aligned} \omega_2 &= f_2'/\Gamma(2-c)\Gamma(c') \\ &= \sum_{k,k'=0}^{\infty} \frac{\Gamma(a-m+\delta+k+k')\Gamma(b-m+\delta+k)\Gamma(b'+k')}{\Gamma(1-m+\delta+k)\Gamma(c'+k')\Gamma(1+k)\Gamma(1+k')} x^{k-m+\delta} y^{k'} \end{aligned}$$

tends to

$$\sum_{k\geq m,\, k'\geq 0} \frac{\Gamma(a-m+k+k')\Gamma(b-m+k)\Gamma(b'+k')}{\Gamma(1-m+k)\Gamma(c'+k')\Gamma(1+k)\Gamma(1+k')} x^{k-m} y^{k'}$$

which is equall to

$$\omega_1|_{c=1+m} = (f_1'/\Gamma(c)\Gamma(c'))\,|_{c=1+m}.$$

Hence $\Gamma(1-c)(f_2'/\Gamma(2-c) - f_1'/\Gamma(c))$ is holomorphic in c at $c = 1+m$. equivalently

$$\Gamma(c-1)f_2' + \Gamma(1-c)f_1'$$

is holomorphic at $c = 1 + m$. By the same reason,

$$\Gamma(c-1)f_4' + \Gamma(1-c)f_3'$$

is holomorphic at $c = 1 + m$.

Now we put

$$\begin{aligned} g_1 &= f_1', \quad g_2 = \Gamma(c-1)f_2' + \Gamma(1-c)f_1', \\ g_3 &= f_3', \quad g_4 = \Gamma(c-1)f_4' + \Gamma(1-c)f_3'. \end{aligned} \tag{7.2}$$

Since

$$\begin{aligned} g_2 = &-\frac{\pi\Gamma(c')}{\sin\pi c} \\ &\times \left(\sum_{k,k'=0}^{\infty} \frac{\Gamma(a-m+\delta+k+k')\Gamma(b-m+\delta+k)\Gamma(b'+k')}{\Gamma(1-m+\delta+k)\Gamma(c'+k')\Gamma(1+k)\Gamma(1+k')} x^{k-m+\delta} y^{k'} \right. \\ &\left. - \sum_{k,k'=0}^{\infty} \frac{\Gamma(a+k+k')\Gamma(b+k)\Gamma(b'+k')}{\Gamma(1+m+k-\delta)\Gamma(c'+k')\Gamma(1+k)\Gamma(1+k')} x^k y^{k'} \right), \end{aligned}$$

we have

$$g_2|_{c=1+m} = (-1)^{1+m} \left(\log x\, g_1/\Gamma(1+m) + \Gamma(c')x^{-m} g_{21}(x,y)\right), \tag{7.3}$$

where

$$\begin{aligned} &g_{21} \\ &= \sum_{k=0}^{m-1} (-1)^{1-m+k} \frac{\Gamma(m-k)\Gamma(a-m+k+k')\Gamma(b-m+k)\Gamma(b'+k')}{\Gamma(c'+k')\Gamma(1+k)\Gamma(1+k')} x^k y^{k'} \\ &+ \left(\frac{\partial}{\partial\delta} \sum_{k,k'=0}^{\infty} \frac{\Gamma(a+k+k'+\delta)\Gamma(b+k+\delta)\Gamma(b'+k')}{\Gamma(1+k+\delta)\Gamma(1+m+k+\delta)\Gamma(c'+k')\Gamma(1+k')} x^{k+m} y^{k'} \right) \Bigg|_{\delta=0}. \end{aligned} \tag{7.4}$$

By the same reason, we have

$$g_4|_{c=1+m} = (-1)^{1+m} \left(\log x\, g_3/\Gamma(1+m) + \Gamma(2-c')x^{-m}y^{1-c'} g_{41}(x,y)\right), \tag{7.5}$$

where

$$\begin{aligned} g_{41} = &\sum_{k=0}^{m-1} (-1)^{1-m+k} \\ &\frac{\Gamma(m-k)\Gamma(1+a-c'-m+k+k')\Gamma(b-m+k)\Gamma(1+b'-c'+k')}{\Gamma(2-c'+k')\Gamma(1+k)\Gamma(1+k')} x^k y^{k'} \\ &+ \left(\frac{\partial}{\partial\delta} \sum_{k,k'=0}^{\infty} \frac{\Gamma(1+a-c'+k+k'+\delta)\Gamma(b+k+\delta)\Gamma(1+b'-c'+k')}{\Gamma(1+k+\delta)\Gamma(1+m+k+\delta)\Gamma(2-c'+k')\Gamma(1+k')} \right. \\ &\left. \times\, x^{k+m} y^{k'} \right) \Bigg|_{\delta=0}. \end{aligned} \tag{7.6}$$

Hence g_j $1 \le j \le 4$ form a system of fundamental solutions of E_2 for $c = 1 + m$ (and $c' \notin \mathbf{Z}$).

By the base change from ω to $g = {}^t(g_1, g_2, g_3, g_4)$, Theorem 5.1 implies

$$g\gamma_{1*} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ (1-e(-c))\Gamma(1-c) & e(-c) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & (1-e(-c))\Gamma(1-c) & e(-c) \end{pmatrix} g,$$

$$g\gamma_{2*} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e(-c') & 0 \\ 0 & 0 & 0 & e(-c') \end{pmatrix} g,$$

$$g\gamma_{3*} =$$

$$\left(I_4 + \frac{2ie\left((c+c'-a-b-b')/2\right)}{\sin \pi c'} \begin{pmatrix} \Gamma(c)\Gamma(c') \\ 0 \\ \Gamma(c)\Gamma(2-c') \\ 0 \end{pmatrix} (\gamma_{31}, \gamma_{32}, \gamma_{33}, \gamma_{34}) \right) g,$$

where

$$\begin{aligned}
\gamma_{31} &= \sin \pi(c-a-b) \sin \pi b' / \Gamma(c)\Gamma(c'), \\
\gamma_{32} &= -\sin \pi(c-a) \sin \pi(c-b) \sin \pi b' / \pi\Gamma(c'), \\
\gamma_{33} &= \sin \pi(c+c'-a-b) \sin \pi(c'-b') / \Gamma(c)\Gamma(2-c'), \\
\gamma_{34} &= -\sin \pi(c+c'-a) \sin \pi(c-b) \sin \pi(c'-b') / \pi\Gamma(2-c').
\end{aligned}$$

It is easy to see that there is no proper subspace of $V = V(P_0)$ which is invariant under the actions γ_{1*}, γ_{2*} and γ_{3*}.

This proves that E_2 is irreducible when $c \in \mathbf{N}$ and $c' \notin \mathbf{Z}$.

The case when $c, c' \in \mathbf{N}$.

We consider the irreducibility for $c = 1 + m$, $c' = 1 + m'$. So we put

$$c = 1 + m - \delta, \quad c' = 1 + m' - \delta'$$

and

$$\begin{aligned}
g_1 &= f_1', \\
g_2 &= \Gamma(c-1) f_2' + \Gamma(1-c) f_1', \\
g_3 &= \Gamma(c'-1) f_3' + \Gamma(1-c') f_1', \\
g_4 &= \Gamma(c-1)\Gamma(c'-1) f_4' + \Gamma(1-c)\Gamma(c'-1) f_3' \\
&\quad + \Gamma(c-1)\Gamma(1-c') f_2' + \Gamma(1-c)\Gamma(1-c') f_1'.
\end{aligned}$$

At $c = 1 + m$, $c' = 1 + m'$, that is $\delta = \delta' = 0$, g_j $1 \le j \le 4$ are of the following form.

$$\begin{aligned}
g_1 &= \Gamma(a)\Gamma(b)\Gamma(b')F_2(a,b,b',1+m,1+m';x,y),\\
g_2 &= (-1)^{1+m}\left(\log x\, g_1/\Gamma(c) + \Gamma(c')x^{-m}g_{21}(x,y)\right),\\
g_3 &= (-1)^{1+m'}\left(\log y\, g_1/\Gamma(c') + \Gamma(c)y^{-m'}g_{31}(x,y)\right),\\
g_4 &= (-1)^{m+m'}\Big(\log x \log y\, g_1/\Gamma(c)\Gamma(c')\\
&\quad + y^{-m'}\log x\, g_{31}(x,y) + x^{-m}\log y\, g_{21}(x,y) + x^{-m}y^{-m'}g_{41}(x,y)\Big),
\end{aligned}$$

where g_{21} is the same as (7.4) and g_{31} is of similar form as g_{21} Finally, g_{41} is as follows:

$$\begin{aligned}
g_{41} = &\sum_{0\le k<m,\, 0\le k'<m'} (-1)^{m+m'}\\
&\frac{\Gamma(m-k)\Gamma(m'-k')\Gamma(a-m-m'+k+k')\Gamma(b-m+k)\Gamma(b'-m'+k')}{\Gamma(1+k)\Gamma(1+k')}\\
&\times x^k y^{k'}\\
&+\left(\frac{\partial}{\partial\delta'}\sum_{k\ge 0,\, 0\le k'<m',}(-1)^{1+m'-k'}\right.\\
&\left.\frac{\Gamma(m'-k')\Gamma(a-m'+k+k'+\delta')\Gamma(b'-m'+k'+\delta')\Gamma(b+k)}{\Gamma(1+k'+\delta')\Gamma(1+m'+k'+\delta')\Gamma(1+k)}x^k y^{k'+m'}\right)\Bigg|_{\delta'=0}\\
&+\left(\frac{\partial}{\partial\delta}\sum_{0\le k<m,\, k'\ge 0}(-1)^{1+m-k}\right.\\
&\left.\frac{\Gamma(m-k)\Gamma(a-m+k+k'+\delta)\Gamma(b-m+k+\delta)\Gamma(b'+k')}{\Gamma(1+k+\delta)\Gamma(1+m+k+\delta)\Gamma(1+k')}x^{k+m}y^{k'}\right)\Bigg|_{\delta=0}\\
&+\left(\frac{\partial^2}{\partial\delta\partial\delta'}\sum_{k=k'=0}^{\infty}\right.\\
&\quad\frac{\Gamma(a+k+k'+\delta+\delta')\Gamma(b+k+\delta)\Gamma(b'+k'+\delta')}{\Gamma(1+k+\delta)\Gamma(1+m+k+\delta)\Gamma(1+m'+k'+\delta')\Gamma(1+k'+\delta')}\\
&\quad\left.\times x^{k+m}y^{k'+m'}\right)\Bigg|_{\delta=\delta'=0}
\end{aligned}$$

Theorem 5.1 implies

$$g\gamma_{1*} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ (1-e(-c))\Gamma(1-c) & e(-c) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & (1-e(-c))\Gamma(1-c) & e(-c) \end{pmatrix} g,$$

$$g\gamma_{2*} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ (1-e(-c'))\Gamma(1-c') & 0 & e(-c') & 0 \\ 0 & 0 & (1-e(-c'))\Gamma(1-c') & e(-c') \end{pmatrix} g,$$

$$g\gamma_{3*} =$$

$$\left(I_4 + 2ie\left((c+c'-a-b-b')/2\right) \begin{pmatrix} \Gamma(c)\Gamma(c') \\ 0 \\ 0 \\ 0 \end{pmatrix} (\gamma_{31}, \gamma_{32}, \gamma_{33}, \gamma_{34}) \right) g,$$

where

$$\begin{aligned} \gamma_{31} &= \sin\pi(c+c'-a-b-b')/\Gamma(c)\Gamma(c'), \\ \gamma_{32} &= -\sin\pi(c+c'-a-b')\sin\pi(c-b)/\pi\Gamma(c'), \\ \gamma_{33} &= -\sin\pi(c+c'-a-b)\sin\pi(c'-b')/\pi\Gamma(c), \\ \gamma_{34} &= \sin\pi(c+c'-a)\sin\pi(c-b)\sin\pi(c'-b')/\pi^2. \end{aligned}$$

Then it is easy to see that there is no proper subspace of V which is invariant under the actions of γ_{1*}, γ_{2*} and γ_{3*}. This proves the irreducibility of E_2 with $c, c' \in \mathbf{N}$.

This completes the proof of Theorem 7.1. □

8 A monodromy representation of generalized hypergeometric function ${}_3F_2$.

The generalized hypergeometric function ${}_3F_2$ is defined by the following power series:

$$ {}_3F_2(\alpha_0, \alpha_1, \alpha_2; \beta_1, \beta_2; z) = \sum_{n=0}^{\infty} \frac{\prod_{j=0}^{2}(\alpha_j, n)}{\prod_{j=1}^{2}(\beta_j, n)} \frac{z^n}{(1,n)}. \tag{8.1} $$

This is a solution of a Fuchsian differential equation

$$ {}_3E_2 \;=\; {}_3E_2(\alpha_0, \alpha_1, \alpha_2; \beta_1, \beta_2) $$

of rank three with singularities at $z = 0, 1, \infty$. The characteristic exponents of ${}_3E_2$ at these singular points are

$$ \begin{array}{ll} 0,\ 1-\beta_1,\ 1-\beta_2 & \text{at } z = 0, \\ 0,\ 1,\ \sum \beta_j - \sum \alpha_j & \text{at } z = 1, \\ \alpha_0,\ \alpha_1,\ \alpha_2 & \text{at } z = \infty. \end{array} $$

We put

$$ \begin{array}{lll} \alpha_j' = \alpha_j + 1 - \beta_1, & \beta_2' = \beta_2 + 1 - \beta_1, & \beta_1' = 2 - \beta_1, \\ \alpha_j'' = \alpha_j + 1 - \beta_2, & \beta_1'' = \beta_1 + 1 - \beta_2, & \beta_2'' = 2 - \beta_2, \end{array} \tag{8.2} $$

and we assume in this section

$$ \alpha_j,\ \alpha_j',\ \alpha_j'' \notin \mathbf{Z}\,(j = 0, 1, 2), \quad \beta_j,\ \beta_j',\ \beta_j'' \notin \mathbf{Z}\,(j = 1, 2). \tag{8.3} $$

At $z = 0$, ${}_3E_2$ has the following fundamental solutions

$$ \begin{aligned} f_0^{(0)}(z) &= {}_3F_2(\alpha_0, \alpha_1, \alpha_2; \beta_1, \beta_2; z), \\ f_1^{(0)}(z) &= z^{1-\beta_1}{}_3F_2(\alpha_0', \alpha_1', \alpha_2'; \beta_1', \beta_2'; z), \\ f_2^{(0)}(z) &= z^{1-\beta_2}{}_3F_2(\alpha_0'', \alpha_1'', \alpha_2''; \beta_1'', \beta_2''; z). \end{aligned} \tag{8.4} $$

Assume

$$ \beta_1 + \beta_2 - \alpha_0 - \alpha_1 - \alpha_2 \notin \mathbf{Z}, \tag{8.5} $$

then, at $z = 1$, ${}_3E_2$ has two linearly independent holomorphic solutions and a solution of the following form:

$$ f^{(1)}(z) = (1-z)^{\beta_1+\beta_2-\alpha_0-\alpha_1-\alpha_2}\, \tilde{f}^{(1)}(z), \tag{8.6} $$

where $\tilde{f}^{(1)}(z)$ is holomorphic at $z = 1$ and take the value 1 there. Assume

$$ \alpha_j - \alpha_k \notin \mathbf{Z}\ (0 \le j < k \le 2), \tag{8.7} $$

then, at $z = \infty$, ${}_3E_2$ has the following fundamental solutions

$$\begin{aligned}
f_0^{(\infty)}(z) &= (-z)^{-\alpha_0} \\
&\quad \times {}_3F_2(\alpha_0, 1+\alpha_0-\beta_1, 1+\alpha_0-\beta_2; 1+\alpha_0-\alpha_1, 1+\alpha_0-\alpha_2; 1/z), \\
f_1^{(\infty)}(z) &= (-z)^{-\alpha_1} \\
&\quad \times {}_3F_2(\alpha_1, 1+\alpha_1-\beta_1, 1+\alpha_1-\beta_2; 1+\alpha_1-\alpha_2, 1+\alpha_1-\alpha_0; 1/z), \\
f_2^{(\infty)}(z) &= (-z)^{-\alpha_2} \\
&\quad \times {}_3F_2(\alpha_2, 1+\alpha_2-\beta_1, 1+\alpha_2-\beta_2; 1+\alpha_2-\alpha_0, 1+\alpha_2-\alpha_1; 1/z).
\end{aligned} \tag{8.8}$$

Now we return to Appell's F_2.

Theorem 8.1. *$E_2(a, b, b', c, c')$ has the solutions of the following forms along the singular locus $X_1 = \{x = 1\}$.*

(1). $f(x, y)$, where $f(x, y)$ is holomorphic along X_1 and $f(1, y)$ is a solution of ${}_3E_2(a, b', 1+a-c; c', 1+a+b-c)$.

(2). $(1-x)^{c-a-b+b'} y^{-b'} f(x, y)$ where $f(x, y)$ is holomorphic along X_1 everwhere and $f(1, y) \equiv 1$.

Proof. We apply Theorem 2.1 to (2.5). Then matricies A and Q in (2.3) and (2.4) are

$$A = P_3, \quad Q = \frac{1}{y}(P_2 + P_5) + \frac{1}{y-1} P_4.$$

The eigenvalues of A are $0, 0, 0, c-a-b+b'-1$. We assume that

$$c - a - b + b' \notin \mathbf{Z}$$

in this section.

Let $\lambda = c-a-b+b'-1$. Then Theorem 2.1 implies that E_2 has a solution u of the form $u = (x-1)^{\lambda} v(x, y)$ with $v_0(y) = v(1, y)$ satisfying (2.3) and (2.4). This implies that v_0 is of the form

$$v_0 = {}^t(0, 1, 0, -b')\, y^{-b'}$$

up to a constant multiplication. This implies (2).

Let $\lambda = 0$. Then E_2 has a solution of the form

$$u = u_0(y) + (x-1)u_1(y) + (x-1)^2 u_2(y) + \cdots,$$

where u_0 satisfies (2.3) and (2.4). (2.3) implies that u_0 is of the form

$$u_0 = {}^t(f(y), g(y), h(y), -abf + (c-a-b-1)g - bh)$$

and (2.4) implies that $f(y)$ satisfies the differential equation ${}_3E_2(a, b', 1+a-c; c', 1+a+b-c)$. This proves (1).

This completes the proof. □

Put

$$\alpha_0 = a,\ \alpha_1 = b',\ \alpha_2 = 1+a-c,\ \beta_1 = c',\ \beta_2 = 1+a+b-c \tag{8.9}$$

then Theorem 8.1 implies that (see (3.15))

$$\begin{aligned} \kappa_1(0,y) &= f_0^{(0)}(y) \\ \kappa_2(0,y) &= f_1^{(0)}(y) \\ \kappa_3(0,y) &= e^{i\pi(\beta_2-1)} f_2^{(0)}(y) \end{aligned} \tag{8.10}$$

By the symmetry (3.17), we know that $\kappa_4'(x,(1-y)/x)$ is holomorphic at $(x,y) = (1,1)$ and $\kappa_4'(1,0) = e^{i\pi b}$, whence we have

$$\kappa_4'(1,1-y) = e^{i\pi(\beta_2-\alpha_2)} f^{(1)}(y). \tag{8.11}$$

In the upper half plane

$$\mathbf{H} = \{z \,|\, \mathrm{Im} z > 0\},$$

we choose the arguments

$$0 < \arg(z) < \pi, \quad -\pi < \arg(1-z) < 0$$

to obtain the single valued holomorphic solutions $f_j^{(0)}$ (see (8.4)) and $f^{(1)}$ (see (8.6)) of ${}_3E_2(\alpha_0,\alpha_1,\alpha_2;\beta_1,\beta_2)$. Then Formula 4.9 implies the following theorem.

Theorem 8.2. *Assume the non-integral conditions (8.3) and (8.5), then we have*

$$\begin{aligned} f_0^{(0)} &\equiv \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_0+\alpha_1+\alpha_2-\beta_1-\beta_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} f^{(1)} \\ f_1^{(0)} &\equiv \frac{\Gamma(\beta_1')\Gamma(\beta_2')\Gamma(\alpha_0+\alpha_1+\alpha_2-\beta_1-\beta_2)}{\Gamma(\alpha_0')\Gamma(\alpha_1')\Gamma(\alpha_2')} f^{(1)} \\ f_2^{(0)} &\equiv \frac{\Gamma(\beta_1'')\Gamma(\beta_2'')\Gamma(\alpha_0+\alpha_1+\alpha_2-\beta_1-\beta_2)}{\Gamma(\alpha_0'')\Gamma(\alpha_1'')\Gamma(\alpha_2'')} f^{(1)}, \end{aligned}$$

modulo holomorphic functions at $z = 1$. And we have

$$\begin{aligned} f^{(1)} = &\frac{\Gamma(1-\beta_1)\Gamma(1-\beta_2)\Gamma(1+\beta_1+\beta_2-\alpha_0-\alpha_1-\alpha_2)}{\Gamma(1-\alpha_0)\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)} f_0^{(0)} \\ &+ \frac{\Gamma(1-\beta_1')\Gamma(1-\beta_2')\Gamma(1+\beta_1+\beta_2-\alpha_0-\alpha_1-\alpha_2)}{\Gamma(1-\alpha_0')\Gamma(1-\alpha_1')\Gamma(1-\alpha_2')} f_1^{(0)} \\ &+ \frac{\Gamma(1-\beta_1'')\Gamma(1-\beta_2'')\Gamma(1+\beta_1+\beta_2-\alpha_0-\alpha_1-\alpha_2)}{\Gamma(1-\alpha_0'')\Gamma(1-\alpha_1'')\Gamma(1-\alpha_2'')} f_2^{(0)}. \end{aligned}$$

We put

$$
\begin{aligned}
v_0 &:= \frac{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\beta_1)\Gamma(\beta_2)}\,{}_3F_2(\alpha_0,\alpha_1,\alpha_2;\beta_1,\beta_2;z),\\
v_1 &:= \frac{\Gamma(\alpha_0')\Gamma(\alpha_1')\Gamma(\alpha_2')}{\Gamma(\beta_1')\Gamma(\beta_2')}z^{1-\beta_1}\,{}_3F_2(\alpha_0',\alpha_1',\alpha_2';\beta_1',\beta_2';z),\\
v_2 &:= \frac{\Gamma(\alpha_0'')\Gamma(\alpha_1'')\Gamma(\alpha_2'')}{\Gamma(\beta_1'')\Gamma(\beta_2'')}z^{1-\beta_2}\,{}_3F_2(\alpha_0'',\alpha_1'',\alpha_2'';\beta_1'',\beta_2'';z).
\end{aligned}
\tag{8.12}
$$

Put

$$Z = \mathbf{C} - \{0,1\}, \quad z_0 = 1/2$$

and

$$
\begin{aligned}
C_0 &= \{z = z_0 e^{it}\ 0 \le t \le 2\pi\},\\
C_1 &= \{z = 1 - z_0 e^{it}\ 0 \le t \le 2\pi\}.
\end{aligned}
$$

Let $V = V(z_0)$ denotes the set of germs of holomorphic solutions of ${}_3E_2$ at z_0. Then V is a three dimensional vector space. In the following theorem, we have a monodromy representation of ${}_3E_2$. Other representations are known (see [B-H] and [Ohr]).

Theorem 8.3. *Assume the non-integral condition (8.3). Then the analytic continuation vC_{j*} of*

$$v = {}^t(v_0, v_1, v_2)$$

along C_j are as follows:

$$
vC_{0*} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e(1-\beta_1) & 0 \\ 0 & 0 & e(1-\beta_2) \end{pmatrix} v, \tag{8.13}
$$

$$
vC_{1*} = \left(I_3 + e_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (c_{10}, c_{11}, c_{12}) \right) v, \tag{8.14}
$$

$$
\begin{aligned}
e_1 &= -2ie^{i\pi(\beta_1+\beta_2-\alpha_0-\alpha_1-\alpha_2)},\\
c_{10} &= \sin\pi\alpha_0 \sin\pi\alpha_1 \sin\pi\alpha_2 / \sin\pi\beta_1 \sin\pi\beta_2,\\
c_{11} &= \sin\pi\alpha_0' \sin\pi\alpha_1' \sin\pi\alpha_2' / \sin\pi\beta_1' \sin\pi\beta_2',\\
c_{12} &= \sin\pi\alpha_0'' \sin\pi\alpha_1'' \sin\pi\alpha_2'' / \sin\pi\beta_1'' \sin\pi\beta_2''.
\end{aligned}
$$

Proof. (8.13) is clear. (8.14) can be derived from Theorem 8.2. □

9 Connection formulas for ${}_3F_2$ along $\{z < 0\}$.

Put

$$X_2 = \{(x, y) \in \mathbf{C}^2 \,|\, x + y = 1\},$$

which is an irreducible component of (2.8).

Theorem 9.1. *Assume that $c + c' - a - b - b'$ is not an integer. Then $E_2(a, b, b', c, c')$ has the solutions of the following forms along X_2.*

(1). $(1 - y)^{-a} f(x, y)$, where $f(x, y)$ is holomorphic along X_2 and if we put

$$y = z/(z - 1),$$

then $f(1 - y, y)$ is a solution of ${}_3E_2(a, 1 + a - c, c' - b'; c', 1 + a + b - c)$ as a function of z.

(2). $(x + y - 1)^{c+c'-a-b-b'} f(x, y)$, where $f(x, y)$ is holomorphic along X_2 and $f(1 - y, y) = y^{b'-c'}(1 - y)^{b-c}$ up to a constant multiplication.

Proof. We apply Theorem 2.1 to (2.5). Then matricies A and Q in (2.3) and (2.4) are

$$A = P_5, \quad Q = \frac{1}{y}(P_2 + P_3) + \frac{1}{y - 1}(P_1 + P_4).$$

The eigenvalues of A are $0, 0, 0, \lambda_1 - 2$, where $\lambda_1 = c + c' - a - b - b'$. We have assumed that

$$\lambda_1 \notin \mathbf{Z}.$$

Let $\lambda = \lambda_1 - 2$. Then Theorem 2.1 implies that E_2 has a solution u of the form $u = (x + y - 1)^{\lambda_1 - 2} v(x, y)$ with $v_0(y) = v(1 - y, y)$ satisfying (2.3) and (2.4). This implies that v_0 is of the form

$$v_0 = {}^t(0, 0, 0, 1)\, y^{1+b'-c'}(1 - y)^{1+b-c}$$

up to a constant multiplication. Then (2.1) implies that the first element z of u (see (2.6)) is of the form

$$z = (x + y - 1)^{\lambda_1} f(x, y),$$

where $f(1 - y, y) = y^{b'-c'}(1 - y)^{b-c}$ up to a constant multiplication. This proves (2).

Let $\lambda = 0$ and

$$u = v_0(y) + (x + y - 1)v_1(y) + (x + y - 1)^2 v_2(y) + \cdots$$

be a solution of (2.5). Put

$$v_0 = {}^t(g_1, g_2, g_3, g_4),$$

then (2.3) implies that

$$g_4 = (abb' g_1 + b'(1 + a + b - c)g_2 + b(1 + a + b' - c')g_3)/(\lambda_1 - 2).$$

If we put

$$y = z/(z-1),$$

then (2.4) implies

$$g' = \left\{ \frac{1}{z}\begin{pmatrix} 0 & 0 & 1 \\ -ab & c-a-b-1 & -b \\ 0 & 0 & 1-c' \end{pmatrix} + \frac{1}{z-1}\begin{pmatrix} 0 & -1 & -1 \\ ab & a+b & b \\ ab' & b' & a+b' \end{pmatrix} \right\} g, \tag{9.1}$$

where $g = {}^t(g_1, g_2, g_3)$. The equation (9.1) above implies that $(z-1)^{-a}g_1$ is a solution of ${}_3E_2(a, 1+a-c, c'-b'; c', 1+a+b-c)$. This proves (1) and completes the proof. □

In this section we use the variable

$$z = y/(y-1)$$

for functions ${}_3F_2$ and induce connection formulas. The connection formulas for F_2 are derived under the condition that

$$\mathrm{Im}\, x,\ \mathrm{Im}\, y\ > 0.$$

Hence we must assume that

$$\mathrm{Im}\, z\ < 0 \tag{9.2}$$

in this section.

Put

$$\alpha_0 = a,\ \alpha_1 = 1+a-c,\ \alpha_2 = c'-b',\ \beta_1 = c',\ \beta_2 = 1+a+b-c, \tag{9.3}$$

then the restrictions of ψ_j and ψ'_j $(1 \le j \le 3)$ (see (3.14) and (3.19)) on X_2 are expressed by ${}_3F_2$ in the following way.

Lemma 9.2. *Put $y = z/(z-1)$. Let $f_j^{(0)}$ and $f_j^{(\infty)}$ be as in (8.4) and (8.8), where α_j and β_j are defined by (9.2). Then we have the following equalities.*

$$\begin{aligned}
\psi_1|_{X_2} &= (1-z)^{\alpha_0} f_0^{(0)}(z), \\
\psi_2|_{X_2} &= e^{i\pi(1-\beta_1)}(1-z)^{\alpha_0} f_1^{(0)}(z), \\
\psi_3|_{X_2} &= e^{i\pi(1-\beta_2)}(1-z)^{\alpha_0} f_1^{(0)}(z), \\
\psi'_1|_{X_2} &= (1-z)^{\alpha_0} f_0^{(\infty)}(z) \\
\psi'_2|_{X_2} &= (1-z)^{\alpha_0} f_1^{(\infty)}(z) \\
\psi'_3|_{X_2} &= (1-z)^{\alpha_0} f_2^{(\infty)}(z)
\end{aligned}$$

where for $z < 0$, we choose

$$arg(-z) = arg(1-z) = 0, \quad arg(z) = -\pi.$$

Proof. We will prove the equalities for $\psi_2|_{X_2}$ and $\psi_2'|_{X_2}$. From (3.14), we see that

$$\begin{aligned}\psi_2|_{X_2} &= y^{1-\beta_1}\tilde{\psi}_2|_{X_2}\\ &= (-z/(1-z))^{1-\beta_1}\tilde{\psi}_2|_{X_2}\\ &= (1-y)^{-a}(1-z)^{-\alpha_0}e^{i\pi(1-\beta_1)}z^{1-\beta_1}(1-z)^{\beta_1-1}\tilde{\psi}_2|_{X_2}\end{aligned}$$

Since $f(z) := (1-z)^{\beta_1-\alpha_0-1}z^{1-\beta_1}\tilde{\psi}_2|_{X_2}$ is a solution of ${}_3E_2(\alpha_0,\alpha_1,\alpha_2;\beta_1,\beta_2)$, we have

$$f(z) = z^{1-\beta_1}{}_3F_2(\alpha_0',\alpha_1',\alpha_2';\beta_1',\beta_2';z).$$

This proves the equality for $\psi_2|_{X_2}$.

Since $-z = y/(1-y)$, we have

$$\begin{aligned}\psi_2|_{X_2} &= (1-y)^{-a}\left(\frac{y}{1-y}\right)^{1-c'}\\ &\times {}_3F_2\left(1+a-c, 2+a-c-c', 1-b'; 2-c', 2+a+b-c-c'; \frac{y}{y-1}\right).\end{aligned}$$

By the symmetry (3.17), we have

$$\begin{aligned}\psi_2'|_{X_2} &= (1-x)^{-a}\left(\frac{x}{1-x}\right)^{1-c}\\ &\times {}_3F_2\left(1+a-c', 2+a-c-c', 1-b; 2-c, 2+a+b'-c-c'; \frac{x}{x-1}\right),\end{aligned}$$

which proves the equality for $\psi_2'|_{X_2}$. Other equalities can be proved similarly. □

In the lower half plane

$$-\mathbf{H} = \{z \mid \operatorname{Im} z < 0\},$$

we define the arguments of z and $-1/z$ by

$$-\pi < \arg(z) < 0, \quad \arg(-1/z) = -\arg(z) - \pi,$$

to make $f_j^{(0)}$ and $f_j^{(\infty)}$ single valued holomorphic functions on $-\mathbf{H}$. Then we have the following theorem.

Formula 9.1. *Between the fundamental systems (9.3) and (9.4), we have the* *following connection formulas.*

$$f_0^{(0)} = \frac{\Gamma(\alpha_1-\alpha_0)\Gamma(\alpha_2-\alpha_0)\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1-\alpha_0)\Gamma(\beta_2-\alpha_0)} f_0^{(\infty)}$$
$$+\frac{\Gamma(\alpha_2-\alpha_1)\Gamma(\alpha_0-\alpha_1)\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_2)\Gamma(\alpha_0)\Gamma(\beta_1-\alpha_1)\Gamma(\beta_2-\alpha_1)} f_1^{(\infty)}$$
$$+\frac{\Gamma(\alpha_0-\alpha_2)\Gamma(\alpha_1-\alpha_2)\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\beta_1-\alpha_2)\Gamma(\beta_2-\alpha_2)} f_2^{(\infty)},$$

$$f_1^{(0)} = e^{i\pi(\beta_1-1)}\left(\frac{\Gamma(\alpha_1-\alpha_0)\Gamma(\alpha_2-\alpha_0)\Gamma(2-\beta_1)\Gamma(1+\beta_2-\beta_1)}{\Gamma(1+\alpha_1-\beta_1)\Gamma(1+\alpha_2-\beta_1)\Gamma(1-\alpha_0)\Gamma(\beta_2-\alpha_0)} f_0^{(\infty)}\right.$$
$$+\frac{\Gamma(\alpha_2-\alpha_1)\Gamma(\alpha_0-\alpha_1)\Gamma(2-\beta_1)\Gamma(1+\beta_2-\beta_1)}{\Gamma(1+\alpha_2-\beta_1)\Gamma(1+\alpha_0-\beta_1)\Gamma(1-\alpha_1)\Gamma(\beta_2-\alpha_1)} f_1^{(\infty)}$$
$$\left.+\frac{\Gamma(\alpha_0-\alpha_2)\Gamma(\alpha_1-\alpha_2)\Gamma(2-\beta_1)\Gamma(1+\beta_2-\beta_1)}{\Gamma(1+\alpha_0-\beta_1)\Gamma(1+\alpha_1-\beta_1)\Gamma(1-\alpha_2)\Gamma(\beta_2-\alpha_2)} f_2^{(\infty)}\right),$$

$$f_2^{(0)} = e^{i\pi(\beta_2-1)}\left(\frac{\Gamma(\alpha_1-\alpha_0)\Gamma(\alpha_2-\alpha_0)\Gamma(1+\beta_1-\beta_2)\Gamma(2-\beta_2)}{\Gamma(1+\alpha_1-\beta_2)\Gamma(1+\alpha_2-\beta_2)\Gamma(\beta_1-\alpha_0)\Gamma(1-\alpha_0)} f_0^{(\infty)}\right.$$
$$+\frac{\Gamma(\alpha_2-\alpha_1)\Gamma(\alpha_0-\alpha_1)\Gamma(1+\beta_1-\beta_2)\Gamma(2-\beta_2)}{\Gamma(1+\alpha_2-\beta_2)\Gamma(1+\alpha_0-\beta_2)\Gamma(\beta_1-\alpha_1)\Gamma(1-\alpha_1)} f_1^{(\infty)}$$
$$\left.+\frac{\Gamma(\alpha_0-\alpha_2)\Gamma(\alpha_1-\alpha_2)\Gamma(1+\beta_1-\beta_2)\Gamma(2-\beta_2)}{\Gamma(1+\alpha_0-\beta_2)\Gamma(1+\alpha_1-\beta_2)\Gamma(\beta_1-\alpha_2)\Gamma(1-\alpha_2)} f_2^{(\infty)}\right),$$

$$f_0^{(\infty)} = \frac{\Gamma(1+\alpha_0-\alpha_1)\Gamma(1+\alpha_0-\alpha_2)\Gamma(1-\beta_1)\Gamma(1-\beta_2)}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)\Gamma(1+\alpha_0-\beta_1)\Gamma(1+\alpha_0-\beta_2)} f_0^{(0)}$$
$$+\frac{\Gamma(1+\alpha_0-\alpha_1)\Gamma(1+\alpha_0-\alpha_2)\Gamma(\beta_1-1)\Gamma(\beta_1-\beta_2)}{\Gamma(\beta_1-\alpha_1)\Gamma(\beta_1-\alpha_2)\Gamma(\alpha_0)\Gamma(1+\alpha_0-\beta_2)} e^{i\pi(1-\beta_1)} f_1^{(0)}$$
$$+\frac{\Gamma(1+\alpha_0-\alpha_1)\Gamma(1+\alpha_0-\alpha_2)\Gamma(\beta_2-\beta_1)\Gamma(\beta_2-1)}{\Gamma(\beta_2-\alpha_1)\Gamma(\beta_2-\alpha_2)\Gamma(1+\alpha_0-\beta_1)\Gamma(\alpha_0)} e^{i\pi(1-\beta_2)} f_2^{(0)},$$

$$f_1^{(\infty)} = \frac{\Gamma(1+\alpha_1-\alpha_2)\Gamma(1+\alpha_1-\alpha_0)\Gamma(1-\beta_1)\Gamma(1-\beta_2)}{\Gamma(1-\alpha_2)\Gamma(1-\alpha_0)\Gamma(1+\alpha_1-\beta_1)\Gamma(1+\alpha_1-\beta_2)} f_0^{(0)}$$
$$+\frac{\Gamma(1+\alpha_1-\alpha_2)\Gamma(1+\alpha_1-\alpha_0)\Gamma(\beta_1-1)\Gamma(\beta_1-\beta_2)}{\Gamma(\beta_1-\alpha_2)\Gamma(\beta_1-\alpha_0)\Gamma(\alpha_1)\Gamma(1+\alpha_1-\beta_2)} e^{i\pi(1-\beta_1)} f_1^{(0)}$$
$$+\frac{\Gamma(1+\alpha_1-\alpha_2)\Gamma(1+\alpha_1-\alpha_0)\Gamma(\beta_2-\beta_1)\Gamma(\beta_2-1)}{\Gamma(\beta_2-\alpha_2)\Gamma(\beta_2-\alpha_0)\Gamma(1+\alpha_1-\beta_1)\Gamma(\alpha_1)} e^{i\pi(1-\beta_2)} f_2^{(0)},$$

$$f_2^{(\infty)} = \frac{\Gamma(1+\alpha_2-\alpha_0)\Gamma(1+\alpha_2-\alpha_1)\Gamma(1-\beta_1)\Gamma(1-\beta_2)}{\Gamma(1-\alpha_0)\Gamma(1-\alpha_1)\Gamma(1+\alpha_2-\beta_1)\Gamma(1+\alpha_2-\beta_2)} f_0^{(0)}$$
$$+\frac{\Gamma(1+\alpha_2-\alpha_0)\Gamma(1+\alpha_2-\alpha_1)\Gamma(\beta_1-1)\Gamma(\beta_1-\beta_2)}{\Gamma(\beta_1-\alpha_0)\Gamma(\beta_1-\alpha_1)\Gamma(\alpha_2)\Gamma(1+\alpha_2-\beta_2)} e^{i\pi(1-\beta_1)} f_1^{(0)}$$
$$+\frac{\Gamma(1+\alpha_2-\alpha_0)\Gamma(1+\alpha_2-\alpha_1)\Gamma(\beta_2-\beta_1)\Gamma(\beta_2-1)}{\Gamma(\beta_2-\alpha_0)\Gamma(\beta_2-\alpha_1)\Gamma(1+\alpha_2-\beta_1)\Gamma(\alpha_2)} e^{i\pi(1-\beta_2)} f_2^{(0)}.$$

Theorem 9.3. *Assume* $Im\, z > 0$ *and choose* $arg(z) = 0$ *for* $0 < z < 1$,

$arg(-1/z)=0$ *for* $z<0$. *Then we have*

$$
\begin{aligned}
f_0^{(0)} &= \frac{\Gamma(\alpha_1-\alpha_0)\Gamma(\alpha_2-\alpha_0)\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1-\alpha_0)\Gamma(\beta_2-\alpha_0)} f_0^{(\infty)} \\
&+ \frac{\Gamma(\alpha_2-\alpha_1)\Gamma(\alpha_0-\alpha_1)\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_2)\Gamma(\alpha_0)\Gamma(\beta_1-\alpha_1)\Gamma(\beta_2-\alpha_1)} f_1^{(\infty)} \\
&+ \frac{\Gamma(\alpha_0-\alpha_2)\Gamma(\alpha_1-\alpha_2)\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\beta_1-\alpha_2)\Gamma(\beta_2-\alpha_2)} f_2^{(\infty)},
\end{aligned}
$$

$$
\begin{aligned}
f_1^{(0)} &= e^{i\pi(1-\beta_1)} \left(\frac{\Gamma(\alpha_1-\alpha_0)\Gamma(\alpha_2-\alpha_0)\Gamma(2-\beta_1)\Gamma(1+\beta_2-\beta_1)}{\Gamma(1+\alpha_1-\beta_1)\Gamma(1+\alpha_2-\beta_1)\Gamma(1-\alpha_0)\Gamma(\beta_2-\alpha_0)} f_0^{(\infty)} \right. \\
&+ \frac{\Gamma(\alpha_2-\alpha_1)\Gamma(\alpha_0-\alpha_1)\Gamma(2-\beta_1)\Gamma(1+\beta_2-\beta_1)}{\Gamma(1+\alpha_2-\beta_1)\Gamma(1+\alpha_0-\beta_1)\Gamma(1-\alpha_1)\Gamma(\beta_2-\alpha_1)} f_1^{(\infty)} \\
&\left. + \frac{\Gamma(\alpha_0-\alpha_2)\Gamma(\alpha_1-\alpha_2)\Gamma(2-\beta_1)\Gamma(1+\beta_2-\beta_1)}{\Gamma(1+\alpha_0-\beta_1)\Gamma(1+\alpha_1-\beta_1)\Gamma(1-\alpha_2)\Gamma(\beta_2-\alpha_2)} f_2^{(\infty)} \right),
\end{aligned}
$$

$$
\begin{aligned}
f_2^{(0)} &= e^{i\pi(1-\beta_2)} \left(\frac{\Gamma(\alpha_1-\alpha_0)\Gamma(\alpha_2-\alpha_0)\Gamma(1+\beta_1-\beta_2)\Gamma(2-\beta_2)}{\Gamma(1+\alpha_1-\beta_2)\Gamma(1+\alpha_2-\beta_2)\Gamma(\beta_1-\alpha_0)\Gamma(1-\alpha_0)} f_0^{(\infty)} \right. \\
&+ \frac{\Gamma(\alpha_2-\alpha_1)\Gamma(\alpha_0-\alpha_1)\Gamma(1+\beta_1-\beta_2)\Gamma(2-\beta_2)}{\Gamma(1+\alpha_2-\beta_2)\Gamma(1+\alpha_0-\beta_2)\Gamma(\beta_1-\alpha_1)\Gamma(1-\alpha_1)} f_1^{(\infty)} \\
&\left. + \frac{\Gamma(\alpha_0-\alpha_2)\Gamma(\alpha_1-\alpha_2)\Gamma(1+\beta_1-\beta_2)\Gamma(2-\beta_2)}{\Gamma(1+\alpha_0-\beta_2)\Gamma(1+\alpha_1-\beta_2)\Gamma(\beta_1-\alpha_2)\Gamma(1-\alpha_2)} f_2^{(\infty)} \right),
\end{aligned}
$$

$$
\begin{aligned}
f_0^{(\infty)} &= \frac{\Gamma(1+\alpha_0-\alpha_1)\Gamma(1+\alpha_0-\alpha_2)\Gamma(1-\beta_1)\Gamma(1-\beta_2)}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)\Gamma(1+\alpha_0-\beta_1)\Gamma(1+\alpha_0-\beta_2)} f_0^{(0)} \\
&+ \frac{\Gamma(1+\alpha_0-\alpha_1)\Gamma(1+\alpha_0-\alpha_2)\Gamma(\beta_1-1)\Gamma(\beta_1-\beta_2)}{\Gamma(\beta_1-\alpha_1)\Gamma(\beta_1-\alpha_2)\Gamma(\alpha_0)\Gamma(1+\alpha_0-\beta_2)} e^{i\pi(\beta_1-1)} f_1^{(0)} \\
&+ \frac{\Gamma(1+\alpha_0-\alpha_1)\Gamma(1+\alpha_0-\alpha_2)\Gamma(\beta_2-\beta_1)\Gamma(\beta_2-1)}{\Gamma(\beta_2-\alpha_1)\Gamma(\beta_2-\alpha_2)\Gamma(1+\alpha_0-\beta_1)\Gamma(\alpha_0)} e^{i\pi(\beta_2-1)} f_2^{(0)},
\end{aligned}
$$

$$
\begin{aligned}
f_1^{(\infty)} &= \frac{\Gamma(1+\alpha_1-\alpha_2)\Gamma(1+\alpha_1-\alpha_0)\Gamma(1-\beta_1)\Gamma(1-\beta_2)}{\Gamma(1-\alpha_2)\Gamma(1-\alpha_0)\Gamma(1+\alpha_1-\beta_1)\Gamma(1+\alpha_1-\beta_2)} f_0^{(0)} \\
&+ \frac{\Gamma(1+\alpha_1-\alpha_2)\Gamma(1+\alpha_1-\alpha_0)\Gamma(\beta_1-1)\Gamma(\beta_1-\beta_2)}{\Gamma(\beta_1-\alpha_2)\Gamma(\beta_1-\alpha_0)\Gamma(\alpha_1)\Gamma(1+\alpha_1-\beta_2)} e^{i\pi(\beta_1-1)} f_1^{(0)} \\
&+ \frac{\Gamma(1+\alpha_1-\alpha_2)\Gamma(1+\alpha_1-\alpha_0)\Gamma(\beta_2-\beta_1)\Gamma(\beta_2-1)}{\Gamma(\beta_2-\alpha_2)\Gamma(\beta_2-\alpha_0)\Gamma(1+\alpha_1-\beta_1)\Gamma(\alpha_1)} e^{i\pi(\beta_2-1)} f_2^{(0)},
\end{aligned}
$$

$$
\begin{aligned}
f_2^{(\infty)} &= \frac{\Gamma(1+\alpha_2-\alpha_0)\Gamma(1+\alpha_2-\alpha_1)\Gamma(1-\beta_1)\Gamma(1-\beta_2)}{\Gamma(1-\alpha_0)\Gamma(1-\alpha_1)\Gamma(1+\alpha_2-\beta_1)\Gamma(1+\alpha_2-\beta_2)} f_0^{(0)} \\
&+ \frac{\Gamma(1+\alpha_2-\alpha_0)\Gamma(1+\alpha_2-\alpha_1)\Gamma(\beta_1-1)\Gamma(\beta_1-\beta_2)}{\Gamma(\beta_1-\alpha_0)\Gamma(\beta_1-\alpha_1)\Gamma(\alpha_2)\Gamma(1+\alpha_2-\beta_2)} e^{i\pi(\beta_1-1)} f_1^{(0)} \\
&+ \frac{\Gamma(1+\alpha_2-\alpha_0)\Gamma(1+\alpha_2-\alpha_1)\Gamma(\beta_2-\beta_1)\Gamma(\beta_2-1)}{\Gamma(\beta_2-\alpha_0)\Gamma(\beta_2-\alpha_1)\Gamma(1+\alpha_2-\beta_1)\Gamma(\alpha_2)} e^{i\pi(\beta_2-1)} f_2^{(0)}.
\end{aligned}
$$

Remark. If we use the notations α_j', α_j'' and β_j', β_j'' in (8.2), the connection formulas in the theorem above have the following expressions.

$$\begin{aligned}
f_0^{(0)} &= \frac{\Gamma(\alpha_1-\alpha_0)\Gamma(\alpha_2-\alpha_0)\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1-\alpha_0)\Gamma(\beta_2-\alpha_0)} f_0^{(\infty)} \\
&+ \frac{\Gamma(\alpha_2-\alpha_1)\Gamma(\alpha_0-\alpha_1)\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_2)\Gamma(\alpha_0)\Gamma(\beta_1-\alpha_1)\Gamma(\beta_2-\alpha_1)} f_1^{(\infty)} \\
&+ \frac{\Gamma(\alpha_0-\alpha_2)\Gamma(\alpha_1-\alpha_2)\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\beta_1-\alpha_2)\Gamma(\beta_2-\alpha_2)} f_2^{(\infty)},
\end{aligned}$$

$$\begin{aligned}
f_1^{(0)} &= e^{i\pi(1-\beta_1)} \left(\frac{\Gamma(\alpha_1'-\alpha_0')\Gamma(\alpha_2'-\alpha_0')\Gamma(\beta_1')\Gamma(\beta_2')}{\Gamma(\alpha_1')\Gamma(\alpha_2')\Gamma(\beta_1'-\alpha_0')\Gamma(\beta_2'-\alpha_0')} f_0^{(\infty)} \right. \\
&+ \frac{\Gamma(\alpha_2'-\alpha_1')\Gamma(\alpha_0'-\alpha_1')\Gamma(\beta_1')\Gamma(\beta_2')}{\Gamma(\alpha_2')\Gamma(\alpha_0')\Gamma(\beta_1'-\alpha_1')\Gamma(\beta_2'-\alpha_1')} f_1^{(\infty)} \\
&+ \left. \frac{\Gamma(\alpha_0'-\alpha_2')\Gamma(\alpha_1'-\alpha_2')\Gamma(\beta_1')\Gamma(\beta_2')}{\Gamma(\alpha_0')\Gamma(\alpha_1')\Gamma(\beta_1'-\alpha_2')\Gamma(\beta_2'-\alpha_2')} f_2^{(\infty)} \right),
\end{aligned}$$

$$\begin{aligned}
f_2^{(0)} &= e^{i\pi(1-\beta_2)} \left(\frac{\Gamma(\alpha_1''-\alpha_0'')\Gamma(\alpha_2''-\alpha_0'')\Gamma(\beta_1'')\Gamma(\beta_2'')}{\Gamma(\alpha_1'')\Gamma(\alpha_2'')\Gamma(\beta_1''-\alpha_0'')\Gamma(\beta_2''-\alpha_0'')} f_0^{(\infty)} \right. \\
&+ \frac{\Gamma(\alpha_2''-\alpha_1'')\Gamma(\alpha_0''-\alpha_1'')\Gamma(\beta_1'')\Gamma(\beta_2'')}{\Gamma(\alpha_2'')\Gamma(\alpha_0'')\Gamma(\beta_1''-\alpha_1'')\Gamma(\beta_2''-\alpha_1'')} f_1^{(\infty)} \\
&+ \left. \frac{\Gamma(\alpha_0''-\alpha_2'')\Gamma(\alpha_1''-\alpha_2'')\Gamma(\beta_1'')\Gamma(\beta_2'')}{\Gamma(\alpha_0'')\Gamma(\alpha_1'')\Gamma(\beta_1''-\alpha_2'')\Gamma(\beta_2''-\alpha_2'')} f_2^{(\infty)} \right),
\end{aligned}$$

$$\begin{aligned}
f_0^{(\infty)} &= \frac{\Gamma(1+\alpha_0-\alpha_1)\Gamma(1+\alpha_0-\alpha_2)\Gamma(1-\beta_1)\Gamma(1-\beta_2)}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)\Gamma(1+\alpha_0-\beta_1)\Gamma(1+\alpha_0-\beta_2)} f_0^{(0)} \\
&+ \frac{\Gamma(1+\alpha_0'-\alpha_1')\Gamma(1+\alpha_0'-\alpha_2')\Gamma(1-\beta_1')\Gamma(1-\beta_2')}{\Gamma(1-\alpha_1')\Gamma(1-\alpha_2')\Gamma(1+\alpha_0'-\beta_1')\Gamma(1+\alpha_0'-\beta_2')} e^{i\pi(\beta_1-1)} f_1^{(0)} \\
&+ \frac{\Gamma(1+\alpha_0''-\alpha_1'')\Gamma(1+\alpha_0''-\alpha_2'')\Gamma(1-\beta_1'')\Gamma(1-\beta_2'')}{\Gamma(1-\alpha_1'')\Gamma(1-\alpha_2'')\Gamma(1+\alpha_0''-\beta_1'')\Gamma(1+\alpha_0''-\beta_2'')} e^{i\pi(\beta_2-1)} f_2^{(0)},
\end{aligned}$$

$$\begin{aligned}
f_1^{(\infty)} &= \frac{\Gamma(1+\alpha_1-\alpha_2)\Gamma(1+\alpha_1-\alpha_0)\Gamma(1-\beta_2)\Gamma(1-\beta_0)}{\Gamma(1-\alpha_2)\Gamma(1-\alpha_0)\Gamma(1+\alpha_1-\beta_2)\Gamma(1+\alpha_1-\beta_0)} f_0^{(0)} \\
&+ \frac{\Gamma(1+\alpha_1'-\alpha_2')\Gamma(1+\alpha_1'-\alpha_0')\Gamma(1-\beta_2')\Gamma(1-\beta_0')}{\Gamma(1-\alpha_2')\Gamma(1-\alpha_0')\Gamma(1+\alpha_1'-\beta_2')\Gamma(1+\alpha_1'-\beta_0')} e^{i\pi(\beta_1-1)} f_1^{(0)} \\
&+ \frac{\Gamma(1+\alpha_1''-\alpha_2'')\Gamma(1+\alpha_1''-\alpha_0'')\Gamma(1-\beta_2'')\Gamma(1-\beta_0'')}{\Gamma(1-\alpha_2'')\Gamma(1-\alpha_0'')\Gamma(1+\alpha_1''-\beta_2'')\Gamma(1+\alpha_1''-\beta_0'')} e^{i\pi(\beta_2-1)} f_2^{(0)},
\end{aligned}$$

$$\begin{aligned}
f_2^{(\infty)} &= \frac{\Gamma(1+\alpha_2-\alpha_0)\Gamma(1+\alpha_2-\alpha_1)\Gamma(1-\beta_0)\Gamma(1-\beta_1)}{\Gamma(1-\alpha_0)\Gamma(1-\alpha_1)\Gamma(1+\alpha_2-\beta_0)\Gamma(1+\alpha_2-\beta_1)} f_0^{(0)} \\
&+ \frac{\Gamma(1+\alpha_2'-\alpha_0')\Gamma(1+\alpha_2'-\alpha_1')\Gamma(1-\beta_0')\Gamma(1-\beta_1')}{\Gamma(1-\alpha_0')\Gamma(1-\alpha_1')\Gamma(1+\alpha_2'-\beta_0')\Gamma(1+\alpha_2'-\beta_1')} e^{i\pi(\beta_1-1)} f_1^{(0)} \\
&+ \frac{\Gamma(1+\alpha_2''-\alpha_0'')\Gamma(1+\alpha_2''-\alpha_1'')\Gamma(1-\beta_0'')\Gamma(1-\beta_1'')}{\Gamma(1-\alpha_0'')\Gamma(1-\alpha_1'')\Gamma(1+\alpha_2''-\beta_0'')\Gamma(1+\alpha_2''-\beta_1'')} e^{i\pi(\beta_2-1)} f_2^{(0)}.
\end{aligned}$$

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Mitsuo KATO
Department of Mathematics
College of Education
University of the Ryukyus
Nishihara-cho, Okinawa 903-0213
JAPAN
(e-mail: mkato@edu.u-ryukyu.ac.jp)