有限モノドロミー群をもつ超幾何微分方程式の Schwarz map

A Simple Pfaffian Form Representing the Hypergeometric Differential Equation of Type (3,6)

Mitsuo KATO

1 Introduction.

The system $E(k,n) = E(k,n;a_1,a_2,\cdots,a_m)$ of differential equations of type (k,n) is defined on the Grassmanian $G_{k,n}$ and has many symmetries. Thanks to these symmetries, we can reduce the system $E(k,n)$ to the system $E(k,n)' = E(k,n;a_1,a_2,\cdots,a_m)'$ of differential equations on $\mathbf{C}^{(k-1)(n-k-1)}$ of rank $\binom{n-2}{k-1}$.

In [MSY], Matsumoto, Sasaki and Yoshida studied the system $E(3,6) = E(3,6;a_1,a_2,\cdots,a_6)$ and explicitly got the reduced differential equations $E(3,6)' = E(3,6;a_1,a_2,\cdots,a_6)'$ on $\mathbf{C}^4$ with rank 6. They represented $E(3,6)'$ in the form

$$\frac{\partial^2 u}{\partial x^i \partial x^j} = G_{ij}\frac{\partial^2 u}{\partial x^1 \partial x^4} + \sum_{k=1}^{4} A_{ij}^k \frac{\partial u}{\partial x^k} + A_{ij}^0 u \quad (1 \le i,j \le 4) \tag{1.1}$$

(Theorem 1.7.1 in [MSY]). They also obtained an equivalent Pfaffian form

$$d\vec{u} = \omega \vec{u}, \tag{1.2}$$

where

$$\vec{u} = {}^t(u, u_1, u_2, u_3, u_4, u_{14}), \quad u_j = \frac{\partial u}{\partial x^j} \quad u_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}$$

(see [MSY, pp.52–55]). Unfortunately, this Pfaffian form is somewhat complicated.

In this paper, we replace $\vec{u}$ with $\tilde{u} := \Lambda \vec{u}$, where Λ is the diagonal matrix with elements 1, x^1, x^2, x^3, x^4, $D_2 := x^1x^4 - x^2x^3$. Then the Pfaffian form (1.2) changes to the following simple form :

$$d\tilde{u} = \tilde{\omega}\tilde{u}, \quad \tilde{\omega} = \sum_j P_j d\log f_j, \tag{1.3}$$

where P_j are constant 6 by 6 matrices and f_j are defining functions of irreducible components of singular locus of $E(3,6)'$ in $\mathbf{C}^4$ (see Theorem below).

2 Main theorem.

The system $E(3,6)$ has six parameters a_j; $1 \leq j \leq 6$ satisfying the relation $\sum a_j = 3$. We denote

$$a_{ij} = a_i + a_j \quad \text{and} \quad a_{ijk} = a_i + a_j + a_k.$$

The reduced system $E(3,6)'$ on $\mathbf{C}^4$ is written in the form of (1.1), where x^j $(1 \leq j \leq 4)$ are the coordinates on $\mathbf{C}^4$ and $u = u(x)$ is the unkown function. Let

$$u_j = \partial u/\partial x^j,\ u_{ij} = \partial^2 u/\partial x^i \partial x^j,$$
$$D_1 = (x^1-1)(x^4-1) - (x^2-1)(x^3-1),\ D_2 = x^1x^4 - x^2x^3.$$

The singular locus of $E(3,6)'$ are defined by

$$\prod_{j=1}^{4} x^j(x^j-1)\cdot(x^1-x^2)(x^1-x^3)(x^2-x^4)(x^3-x^4)D_1D_2 = 0$$

(see [MSY, p.51]).

Theorem . *The system $E(3,6;a_1,a_2,\cdots,a_6)'$ is equivalent to the following Pfaffian form:*

$$d\tilde{u} = \tilde{\omega}\tilde{u},$$

where

$$\tilde{u} = {}^t(u, x^1u_1, x^2u_2, x^3u_3, x^4u_4, D_2u_{14})$$

and

$$\begin{aligned}\tilde{\omega} = &\sum_{j=1}^{4} P_j\, d\log x^j + \sum_{j=1}^{4} Q_j\, d\log(x^j-1)\\ &+ P_{12}\, d\log(x^1-x^2) + P_{13}\, d\log(x^1-x^3) + P_{24}\, d\log(x^2-x^4)\\ &+ P_{34}\, d\log(x^3-x^4) + R_1\, d\log D_1 + R_2\, d\log D_2,\end{aligned}$$

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 2-a_{246} & 0 & 0 & 0 & 0\\ 0 & a_6-1 & 0 & 0 & 0 & 0\\ 0 & -a_3 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & a_3(a_6-1) & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & a_5-1 & 0 & 0 & 0\\ 0 & 0 & 2-a_{245} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -a_3 & 0 & 0 & 0\\ 0 & 0 & a_3(1-a_5) & 0 & 0 & 0 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2-a_{346} & 0 & 0 \\ 0 & 0 & 0 & a_6-1 & 0 & 0 \\ 0 & 0 & 0 & a_2(1-a_6) & 0 & 0 \end{pmatrix},$$

$$P_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_2 & 0 \\ 0 & 0 & 0 & 0 & a_5-1 & 0 \\ 0 & 0 & 0 & 0 & 2-a_{345} & 0 \\ 0 & 0 & 0 & 0 & a_2(a_5-1) & 0 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ a_2(a_5-1) & -(2-a_{345}) & a_5-1 & -a_2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ a_2(a_6-1) & a_6-1 & -(2-a_{346}) & 0 & -a_2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ a_3(a_5-1) & -a_3 & 0 & -(2-a_{245}) & a_5-1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Q_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ a_3(a_6-1) & 0 & -a_3 & a_6-1 & -(2-a_{246}) & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_6 - 1 & 1 - a_5 & 0 & 0 & -1 \\ 0 & 1 - a_6 & a_5 - 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_3(1 - a_6) & a_3(a_5 - 1) & 0 & 0 & a_3 \end{pmatrix},$$

$$P_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a_3 & 0 & a_2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_3 & 0 & -a_2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_3(1 - a_6) & 0 & -a_2(1 - a_6) & 0 & 1 - a_6 \end{pmatrix},$$

$$P_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_3 & 0 & a_2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & -a_2 & -1 \\ 0 & 0 & -a_3(1 - a_5) & 0 & a_2(1 - a_5) & 1 - a_5 \end{pmatrix},$$

$$P_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_6 - 1 & 1 - a_5 & 1 \\ 0 & 0 & 0 & 1 - a_6 & a_5 - 1 & -1 \\ 0 & 0 & 0 & a_2(a_6 - 1) & a_2(1 - a_5) & a_2 \end{pmatrix},$$

$$R_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_3(a_6 - 1) & a_3(1 - a_5) & a_2(1 - a_6) & a_2(a_5 - 1) & -a_{123} \end{pmatrix},$$

$$R_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 - a_{234} \end{pmatrix}.$$

Proof. Since $\tilde{u} = \Lambda\vec{u}$ (see Section 1 for the terminologies), the Pfaffian form (1.2) given in [MSY, pp.52–55] changes to

$$\tilde{u} = (d\Lambda + \Lambda\omega)\Lambda^{-1}\tilde{u}.$$

A direct computation proves the theorem. □

Remark . *The (5,5)-element ω_5^5 of ω given at* [MSY, p.55] *is wrong; it shoud be corrected as*

$$\omega_5^5 = -\alpha_{123}d\log D_1 - \alpha_{234}d\log D_2 + \alpha_2 d\log(x^4 - x^3) + \alpha_3 d\log(x^1 - x^2) \\ + (1-\alpha_5)d\log(x^4 - x^2) + (1-\alpha_6)d\log(x^1 - x^3).$$

The characteristic exponents along the singular locus of $E(3,6)'$ in $\mathbf{C}^4$ are given in [MSY]. Those along the plane at infinity L_∞ can be easily obtained by use of the Pfaffian form in the above theorem.

Corollary . *Assume $a_{23} + a_{56}$ is not an integer. Then the characteristic exponents of $E(3,6)'$ along L_∞ are*

$$a_{23},\ a_{23},\ a_{23},\ 2-a_{56},\ 2-a_{56},\ 2-a_{56}.$$

Acknowledgements. The author would like to thank Professors K. Matsumoto, T. Sasaki and M. Yoshida for their valuable advices.

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Mitsuo KATO
Department of Mathematics
College of Education
University of the Ryukyus
Nishihara-cho, Okinawa 903-0213
JAPAN
(e-mail: mkato@edu.u-ryukyu.ac.jp)