有限モノドロミー群をもつ超幾何微分方程式の Schwarz map

Appell's Hypergeometric Systems F_2 with Finite Irreducible Monodromy Groups

Mitsuo KATO

1 Introduction.

H. A. Schwarz [Swz] determined Gauss' hypergeometric differential equation with finite irreducible monodromy group. The same problem for generalized hypergeometric differential equation of ${}_nF_{n-1}$ is solved by F. Beukers and G. Heckman [BH], for Appell's F_1 and Lauricella's F_D by T. Sasaki [Ssk], for Appell's F_4 by [Kt1], and for $E(3,6)$ by K. Matsumoto, T. Sasaki, N. Takayama and M. Yoshida [MSTY]. This paper solves the problem for Appell's F_2.

1.1 Notations.

Appell's hypergeometric function

$$F_2(a,b,b',c,c';x,y) = \sum_{m,n=0}^{\infty} \frac{(a,m+n)(b,m)(b',n)}{(c,m)(c',n)(1,m)(1,n)} x^m y^n,$$

where $(a,n) = \Gamma(a+n)/\Gamma(a)$, satisfies the following system of differential equations of rank four ([AK]):

$$\begin{cases} x(1-x)z_{xx} - xyz_{xy} + (c-(a+b+1)x)z_x - byz_y - abz = 0 \\ y(1-y)z_{yy} - xyz_{xy} + (c'-(a+b'+1)y)z_y - b'xz_x - ab'z = 0 \end{cases}$$

which we denote by $E_2(a,b,b',c,c')$. This is an extension of Gauss' hypergeometric differential equation

$$x(1-x)z'' + (c-(a+b+1)x)z' - abz = 0$$

which we denote by $E(a,b,c)$.

In this paper, we use the following notations:

$$\lambda = 1-c,\ \mu = c-a-b,\ \nu = b-a,$$
$$\lambda' = 1-c',\ \mu' = c'-a-b',\ \nu' = b'-a,$$
$$\mathcal{S} = (\lambda,\mu,\nu;\ \lambda',\mu',\nu'),\ \mathcal{S}' = (\lambda',\mu',\nu';\ \lambda,\mu,\nu),$$
$$e(x) = \exp(2\pi ix).$$

1.2 Main theorems.

The aim of this paper is to prove the following theorems.

Theorem 1.1. *The system $E_2(a,b,b',c,c')$ has finite irreducible monodromy group if and only if S or S' is one of the followings:*

$$\pm\left(\frac{2\epsilon}{3}+l,\frac{1}{3}+m,\frac{1}{3}+n;\ \frac{2}{3}+l',\frac{\epsilon}{3}+m',\frac{\epsilon}{3}+n'\right),\ \epsilon=\pm1, \tag{1.1}$$

$$\pm\left(\frac{2\epsilon}{3}+l,\frac{1}{4}+m,\frac{1}{4}+n;\ \frac{1}{2}+l',\frac{\epsilon}{3}+m',\frac{\epsilon}{3}+n'\right),\ \epsilon=\pm1, \tag{1.2}$$

$$\pm\left(\frac{2\epsilon}{5}+l,\frac{2}{5}+m,\frac{2}{5}+n;\ \frac{4}{5}+l',\frac{\epsilon}{5}+m',\frac{\epsilon}{5}+n'\right),\ \epsilon=\pm1, \tag{1.3}$$

$$\pm\left(\frac{2\epsilon}{5}+l,\frac{1}{3}+m,\frac{1}{3}+n;\ \frac{2}{3}+l',\frac{\epsilon}{5}+m',\frac{\epsilon}{5}+n'\right),\ 1\le\epsilon\le4, \tag{1.4}$$

$$\pm\left(\frac{2\epsilon}{3}+l,\frac{1}{4}+m,\frac{1}{4}+n;\ \frac{1}{2}+l',\frac{\epsilon}{2}+m',\frac{\epsilon}{6}+n'\right),\ \epsilon=\pm1, \tag{1.5}$$

$$\pm\left(\frac{2\epsilon}{3}+l,\frac{1}{4}+m,\frac{1}{4}+n;\ \frac{1}{2}+l',\frac{\epsilon}{6}+m',\frac{\epsilon}{2}+n'\right),\ \epsilon=\pm1, \tag{1.6}$$

where l,m,n,l',m',n' are arbitrary integers such that $l+m+n$ and $l'+m'+n'$ are equal to a common even number. The monodromy group does not depend on these integers l,m,n,l',m',n'.

Theorem 1.2. *Assume E_2 has a finite irreducible monodromy group M_2. Then M_2 is a semidirect product of a normal subgroup N_r (called the reflection subgroup of M_2) and an abelian subgroup A:*

$$M_2 = N_r \cdot A, \quad N_r \cap A = \{1\}.$$

If S or S' is (1.1), N_r is the group $G(2,2,4)$ in Shephard-Todd table in [ST], *S-T table, for short (a D_4-type Coxeter group) and $A \simeq \mathbf{Z}_3 \times \mathbf{Z}_3$ if $\epsilon = 1$, and $A \simeq \mathbf{Z}_3$ if $\epsilon = -1$.*

If S or S' is (1.2), N_r is the group of No. 28 in S-T table (a F_4-type Coxeter group) and $A \simeq \mathbf{Z}_3 \times \mathbf{Z}_2$.

If S or S' is (1.3), N_r is the group of No. 30 in S-T table (a H_4-type Coxeter group) and $A \simeq \mathbf{Z}_5$.

If S or S' is (1.4), N_r is the group of No. 30 in S-T table and $A \simeq \mathbf{Z}_5 \times \mathbf{Z}_3$.

If S or S' is (1.5) or (1.6), N_r is the symmetry group of the regular complex polytope 3(24)3(24)3(24)3, *the group of No. 32 in S-T table, and $A \simeq \mathbf{Z}_2$.*

Concerning to finite irreducible unitary reflection groups of degree 4, we give a sub-table of S-T table in Subsection 8.2.

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2 A monodromy representation of E_2.

We recall some results from [Kt2]. Put

$$X = \mathbf{C}^2 - \{(x,y) \mid xy(x-1)(y-1)(x+y-1) = 0\}, \ P_0 = (p_0, p_0) \qquad (2.1)$$

for sufficiently small positive number p_0. Then the fundamental group $\pi_1(X, P_0)$ with the base point P_0 is generated by the following five curves:

$$\begin{aligned}
\gamma_1 &= \{(x,y) \mid x = p_0 e^{it} \ 0 \le t \le 2\pi, \ y = p_0\}, \\
\gamma_2 &= \{(x,y) \mid x = p_0, \ y = p_0 e^{it} \ 0 \le t \le 2\pi\}, \\
\gamma_3 &= \{(x,y) \mid x = y = 1/2 - (1/2 - p_0)e^{it} \ 0 \le t \le 2\pi\}, \\
\gamma_4 &= C_{diag} C_{x=1} C_{diag}^{-1}, \quad \gamma_5 = C_{diag} C_{y=1} C_{diag}^{-1},
\end{aligned}$$

where

$$\begin{aligned}
C_{diag} &= \{x = y = 1/2 - (1/2 - p_0)e^{-it} \ 0 \le t \le \pi\}, \\
C_{x=1} &= \{x = 1 - p_0 e^{it} \ 0 \le t \le 2\pi, \ y = 1 - p_0\}, \\
C_{y=1} &= \{y = 1 - p_0 e^{it} \ 0 \le t \le 2\pi, \ x = 1 - p_0\}.
\end{aligned}$$

Let $V = V(P_0)$ be the set of germs of holomorphic solutions of E_2 at P_0. Then V is a four dimensional vector space. For $f \in V$ and $\gamma \in \pi_1(X, P_0)$, the analytic continuation $f\gamma_*$ of f along γ again belongs to $V(P_0)$. We write

$$f(\gamma\gamma')_* = (f\gamma_*)\gamma'_* = f\gamma_*\gamma'_*$$

if γ' is continued after γ. This defines a monodromy representation

$$\pi_1(X, P_0) \longrightarrow GL(V(P_0)). \qquad (2.2)$$

We denote its image by

$$M_2(a, b, b', c, c'; P_0) = M_2(a, b, b', c, c')$$

and call the monodromy group of $E_2(a, b, b', c, c')$. If S is obtained from parameters a, b, b', c, c', we denote

$$M_2(S) = M_2(a, b, b', c, c').$$

If φ_j; $1 \le j \le 4$ form a basis of $V(P_0)$, $GL(V(P_0))$ is identified with $GL(4, \mathbf{C})$ and we have a representation ρ_φ of $\pi_1(X, P_0)$:

$$\pi_1(X, P_0) \xrightarrow{\rho_\varphi} GL(4, \mathbf{C}).$$

We say that the monodromy group M_2 (or E_2) is irreducible if $V(P_0)$ does not have a non-trivial invariant subspace under the action of M_2. We know ([Kt2]) that $M_2(a, b, b', c, c')$ is irreducible if and only if

$$a, \ c-a, \ c'-a, \ c+c'-a, \ b, \ c-b, \ b', \ c'-b' \notin \mathbf{Z}. \qquad (2.3)$$

Assume that neither c nor c' is an integer. Then E_2 has the following linearly independent solutions ([AK], [Kmr]):

$$\begin{aligned} f_1 &= F_2(a,b,b',c,c';x,y), \\ f_2 &= x^{1-c}F_2(1+a-c,1+b-c,b',2-c,c';x,y), \\ f_3 &= y^{1-c'}F_2(1+a-c',b,1+b'-c',c,2-c';x,y), \\ f_4 &= x^{1-c}y^{1-c'}F_2(2+a-c-c',1+b-c,1+b'-c',2-c,2-c';x,y). \end{aligned} \tag{2.4}$$

Assume moreover the irreducibility condition (2.3) and we fix the basis φ_j; $1 \leq j \leq 4$ of $V(P_0)$ as follows:

$$\begin{aligned} \varphi_1 &= \frac{\Gamma(a)\Gamma(b)\Gamma(b')}{\Gamma(c)\Gamma(c')} f_1, \\ \varphi_2 &= \frac{\Gamma(1+a-c)\Gamma(1+b-c)\Gamma(b')}{\Gamma(2-c)\Gamma(c')} f_2, \\ \varphi_3 &= \frac{\Gamma(1+a-c')\Gamma(b)\Gamma(1+b'-c')}{\Gamma(c)\Gamma(2-c')} f_3, \\ \varphi_4 &= \frac{\Gamma(2+a-c-c')\Gamma(1+b-c)\Gamma(1+b'-c')}{\Gamma(2-c)\Gamma(2-c')} f_4. \end{aligned} \tag{2.5}$$

By use of this basis φ_j, we identify M_2 with $\rho_\varphi(\pi_1(X,P_0))$:

$$M_2 = \rho_\varphi(\pi_1(X,P_0)). \tag{2.6}$$

From [Kt2], we have

$$\rho_\varphi(\gamma_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e(1-c) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e(1-c) \end{pmatrix}, \tag{2.7}$$

$$\rho_\varphi(\gamma_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e(1-c') & 0 \\ 0 & 0 & 0 & e(1-c') \end{pmatrix}, \tag{2.8}$$

$$\rho_\varphi(\gamma_3) = I_4 + e_3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} (\gamma_{31}, \gamma_{32}, \gamma_{33}, \gamma_{34}), \tag{2.9}$$

$$\rho_\varphi(\gamma_4) = I_4 + e_4 \begin{pmatrix} \sin\pi(c'-b') \\ \sin\pi(c'-b') \\ e((1-c')/2)\sin\pi b' \\ e((1-c')/2)\sin\pi b' \end{pmatrix} (\gamma_{41}, \gamma_{42}, \gamma_{43}, \gamma_{44}), \tag{2.10}$$

$$\rho_\varphi(\gamma_5) = I_4 + e_5 \begin{pmatrix} \sin\pi(c-b) \\ e((1-c)/2)\sin\pi b \\ \sin\pi(c-b) \\ e((1-c)/2)\sin\pi b \end{pmatrix} (\gamma_{51}, \gamma_{52}, \gamma_{53}, \gamma_{54}) \tag{2.11}$$

where

$$\begin{aligned}
e_3 &= 2i\, e((c+c'-a-b-b')/2)/\sin\pi c \sin\pi c', \\
e_4 &= 2i\, e((c+b'-a-b)/2)/\sin\pi c \sin\pi c', \\
e_5 &= 2i\, e((c'+b-a-b')/2)/\sin\pi c \sin\pi c', \\
\gamma_{31} &= -\sin\pi a \sin\pi b \sin\pi b', \\
\gamma_{32} &= \sin\pi(c-a)\sin\pi(c-b)\sin\pi b', \\
\gamma_{33} &= \sin\pi(c'-a)\sin\pi b \sin\pi(c'-b'), \\
\gamma_{34} &= \sin\pi(c+c'-a)\sin\pi(c-b)\sin\pi(c'-b'), \\
\gamma_{41} &= -\sin\pi a \sin\pi b, \\
\gamma_{42} &= \sin\pi(c-a)\sin\pi(c-b), \\
\gamma_{43} &= e((c'-1)/2)\sin\pi(c'-a)\sin\pi b, \\
\gamma_{44} &= e((c'-1)/2)\sin\pi(c+c'-a)\sin\pi(c-b), \\
\gamma_{51} &= -\sin\pi a \sin\pi b', \\
\gamma_{52} &= e((c-1)/2)\sin\pi(c-a)\sin\pi b', \\
\gamma_{53} &= \sin\pi(c'-a)\sin\pi(c'-b'), \\
\gamma_{54} &= e((c-1)/2)\sin\pi(c+c'-a)\sin\pi(c'-b').
\end{aligned} \tag{2.12}$$

Put

$$\begin{aligned}
v_3 &= (\gamma_{31}, \gamma_{32}, \gamma_{33}, \gamma_{34}), \\
v_4 &= (\gamma_{41}, \gamma_{42}, \gamma_{43}, \gamma_{44}), \quad v_5 = (\gamma_{51}, \gamma_{52}, \gamma_{53}, \gamma_{54}).
\end{aligned} \tag{2.13}$$

Then, by direct calculations, we have

$$\begin{aligned}
v_3\, \rho_\varphi(\gamma_3) &= e(c+c'-a-b-b')v_3, \\
v_4\, \rho_\varphi(\gamma_4) &= e(c+b'-a-b)v_4, \\
v_5\, \rho_\varphi(\gamma_5) &= e(c'+b-a-b')v_5.
\end{aligned} \tag{2.14}$$

From the symmetry

$$(a, b, b', c, c'; x, y) \longleftrightarrow (a, b', b, c', c; y, x),$$

we have

$$M_2(S) \simeq M_2(S'). \tag{2.15}$$

Lemma 2.1.

$$M_2(-\mathcal{S}) \simeq M_2(\mathcal{S}). \tag{2.16}$$

Proof. Let parameters $a_+, b_+, b'_+, c_+, c'_+$ generate $\mathcal{S}$ and $a_-, b_-, b'_-, c_-, c'_-$ generate $-\mathcal{S}$. Then since

$$a_\pm = 1/2 \mp (\lambda + \mu + \nu)/2,$$
$$b_\pm = a_\pm \pm \nu,\ b'_\pm = a_\pm \pm \nu',\ c_\pm = 1 \mp \lambda,\ c'_\pm = 1 \mp \lambda',$$

we have

$$a_- = 1 - a_+,\ b_- = 1 - b_+,\ b'_- = 1 - b'_+,\ c_- = 2 - c_+,\ c'_- = 2 - c'_+. \tag{2.17}$$

Hence we have

$$M_2(a_-, b_-, b'_-, c_-, c'_-) \simeq M_2(-a_+, -b_+, -b'_+, -c_+, -c'_+).$$

From (2.7)–(2.12), we find that if the parameters (a, b, b', c, c') change their signs simultanuously then $\rho_\varphi(\gamma_j)$; $1 \le j \le 5$ change to their complex conjugate. Thus we have

$$M_2(-a_+, -b_+, -b'_+, -c_+, -c'_+) \simeq M_2(a_+, b_+, b'_+, c_+, c'_+).$$

This completes the proof. □

Theorem 2.2. *Assume that E_2 is irreducible and that $c, c' \notin \mathbf{Z}$. Assume moreover that a, b, b', c, c' are rational numbers and have a common denominator n. Let k be an odd integer satisfying $(k, n) = 1$. Then we have*

$$M_2(ka, kb, kb', kc, kc') \simeq M_2(a, b, b', c, c').$$

Proof. Put $\xi = \exp(\pi i/n)$. Then all the components of $\rho_\varphi(\gamma_j)$ belong to $\mathbf{Q}[\xi]$. Hence any relation among $\rho_\varphi(\gamma_1) \sim \rho_\varphi(\gamma_5)$ is expressed by $f_{ij}(\xi) = 0$ for some polynomials $f_{ij}(X) \in \mathbf{Q}[X]$; $1 \le i, j \le 4$. Since the minimal polynomials of ξ and ξ^k in $\mathbf{Q}[X]$ are the same, that is $X^n + 1$, $f(\xi) = 0$ if and only if $f(\xi^k) = 0$ for $f(X) \in \mathbf{Q}[X]$. This means that $\rho_\varphi(\gamma_1) \sim \rho_\varphi(\gamma_5)$ satisfy the same relation if the parameters a, b, b', c, c' change to ka, kb, kb', kc, kc'. This completes the proof. □

We denote by $N_r = N_r(a, b, b', c, c')$ the smallest normal subgroup of M_2 containing $\rho_\varphi(\gamma_3), \rho_\varphi(\gamma_4)$ and $\rho_\varphi(\gamma_5)$. That is,

$$N_r = \left\langle \rho_\varphi(\gamma_1^p \gamma_2^q \gamma_j^r \gamma_2^{-q} \gamma_1^{-p}) \,|\, p, q, r \in \mathbf{Z},\ j = 3, 4, 5 \right\rangle. \tag{2.18}$$

Then we have

$$M_2(a, b, b', c, c') = N_r(a, b, b', c, c') \cdot \langle \rho_\varphi(\gamma_1), \rho_\varphi(\gamma_2) \rangle. \tag{2.19}$$

From (2.9), (2.10), (2.11) and (2.14), we find that the eigenvalues of $\rho_\varphi(\gamma_3)$ are $1, 1, 1, e(c + c' - a - b - b')$, those of $\rho_\varphi(\gamma_4)$ are $1, 1, 1, e(c + b' - a - b')$, and those of $\rho_\varphi(\gamma_5)$ are $1, 1, 1, e(c' + b - a - b)$. Hence if $a, b, b', c, c' \in \mathbf{Q}$ and none of $c + c' - a - b - b'$, $c + b' - a - b'$, $c' + b - a - b$ is an integer then $\rho_\varphi(\gamma_3)$, $\rho_\varphi(\gamma_4)$ and $\rho_\varphi(\gamma_5)$ are reflections. So we call $N_r(a, b, b', c, c')$ the reflection subgroup of $M_2(a, b, b', c, c')$.

Theorem 2.3. *Assume that M_2 is irreducible and that $c, c' \notin \mathbf{Z}$. Then the reflection subgroup N_r of M_2 is also irreducible.*

Proof. In this proof we denote $\rho_\varphi(\gamma_j)$ by g_j and identify $u = (u_1, u_2, u_3, u_4)$ with $u\varphi$ in $V(P_0)$.

Assume that N_r acts on $V(P_0)$ reducibly, that is, there exists a non-trivial subspace W of $V(P_0)$ invariant under the action of N_r. We will derive a contradiction.

Recall that v_j is an eigenvector of g_j for $j = 3, 4, 5$ (see (2.13), (2.14)).

(Case 1) Assume $v_3 \in W$.
From (2.10) and (2.12), we have

$$v_3(g_4 - I_4) = -2i\, e^{-\pi i b'} \sin \pi b' \sin \pi(c' - b')\, v_4,$$

which is in W by the invariance of W. From the irreducibility condition (2.3), we have $v_4 \in W$. By the same way, the fact that $v_3(I_4 - g_5) \in W$ implies that $v_5 \in W$.

By the invariance of W, we have, for $j = 3, 4, 5$,

$$v_j \cdot ((g_1g_2)g_3(g_1g_2)^{-1} - I_4) = e_3\delta_j v_3(g_1g_2)^{-1} \in W,$$

where

$$\delta_j = \gamma_{j1} + e(-c)\gamma_{j2} + e(-c')\gamma_{j3} + e(-c - c')\gamma_{j4}.$$

(Case 1.1) Assume $\delta_j \neq 0$ for some j.
Then $v_3(g_1g_2)^{-1} \in W$. Since

$$\det \begin{pmatrix} v_3 \\ v_4 \\ v_5 \\ v_3(g_1g_2)^{-1} \end{pmatrix} = 4e^{\pi i(b+b'+c+c')}(\sin c \sin c')^2$$
$$\times \sin \pi a \sin \pi(a - c) \sin \pi(a - c') \sin \pi(a - c - c')$$
$$\times \sin \pi b \sin \pi(b - c) \sin \pi b' \sin \pi(b' - c'),$$

W must be the whole space. This is a contradiction.

(Case 1.2) Assume $\delta_j = 0$ for $j = 3, 4, 5$.
In this case, $(x_2, x_3, x_4) = (e(-c), e(-c'), e(-c - c'))$ is the solution of

$$\gamma_{j1} + \gamma_{j2}x_2 + \gamma_{j3}x_3 + \gamma_{j4}x_4 = 0 \quad j = 3, 4, 5.$$

Therefore, we have $x_4 - x_2x_3 = 0$. But, by direct computation, we have

$$x_4 - x_2x_3 = \frac{\sin \pi a \sin \pi b \sin \pi b' \sin \pi c \sin \pi c'}{\sin \pi(a - c - c') \sin \pi(a - c) \sin \pi(a - c') \sin \pi(b - c) \sin \pi(b' - c')}$$

which cannot be zero by the assumption of this theorem. This is a contradiction.

(Case 2) Assume $v_3 \notin W$.
First we know that v_4, $v_5 \notin W$. For, otherwise, if $v_4 \in W$, then

$$v_4(g_3 - I_4) = -e_3 e^{\pi i(c+c'-a-b)} \sin \pi c \sin \pi c'\, v_3 \in W$$

which implies $v_3 \in W$.

Let $u = (u_1, u_2, u_3, u_4)$ be an element of W. Then

$$u(g_3 - I_4) = (u_1 + u_2 + u_3 + u_4)e_3\, v_3$$

is in W. Hence we must have

$$u_1 + u_2 + u_3 + u_4 = 0. \tag{2.20}$$

Now

$$u(g_4 - I_4) = \delta e_4 v_4 \in W, \quad \text{where}$$
$$\delta = (u_1 + u_2)\sin\pi(c' - b') - (u_3 + u_4)e^{\pi i c'}\sin\pi b'.$$

From (2.20),

$$\delta = (u_1 + u_2)e^{\pi i b'}\sin\pi c'$$

which must be zero. Hence we have

$$u_1 + u_2 = 0. \tag{2.21}$$

By the same way, from $u(g_5 - I_4) \in W$, we have

$$u_1 + u_3 = 0. \tag{2.22}$$

Equalities (2.20), (2.21) and (2.22) imply that

$$u = (1, -1, -1, 1)$$

up to constant multiplication and that

$$W = \langle(1, -1, -1, 1)\rangle.$$

If $u = (1, -1, -1, 1)$, then

$$u(g_2 g_5 g_2^{-1} - I_4) = e_5(1 - e(-c'))e^{-\pi i b}\sin\pi c\, v_5 g_2^{-1} \in W.$$

Hence $v_5 g_2^{-1} \in W$. This implies, for example, that $\gamma_{51} : \gamma_{52} = 1 : -1$. This means

$$e^{\pi i c}\sin\pi(c - a) = -\sin\pi a,$$

which is equivalent to

$$e^{\pi i(c-a)}\sin\pi c = 0.$$

This is a contradiction.

In any case we have a contradiction. This completes the proof. □

We denote by $M(a, b, c)$ the monodromy group of Gauss' hypergeometric differential equation $E(a, b, c)$. It is well known that $M(a, b, c)$ is irreducible if and only if none of $a, b, c - a, c - b$ is an integer.

3 Restrictions of E_2 to $\{x = 0\}$ and $\{y = 0\}$.

Assume in this section that $M_2(a, b, b', c, c')$ is finite and irreducible.

It is known that $E_2(a, b, b', c, c')$ has characteristic exponents $0, 0, 1-c, 1-c$ along $L_x := \{(x, y) | x = 0\}$ and $0, 0, 1-c', 1-c'$ along $L_y := \{(x, y) | y = 0\}$. Concerning to these exponents we have the following lemma.

Lemma 3.1. $1-c, 1-c' \notin \mathbf{Z}$.

Proof. Assume $c \in \mathbf{Z}$. Then E_2 has a solution of the form $g_1(x, y) \log x + g_2(x, y)$ where g_j are holomorphic along L_x and $g_1 \neq 0$ ([Kt2, Section 7]). This contradicts the finiteness of M_2. Similarly, we have $c' \notin \mathbf{Z}$. □

Lemma 3.2. *Gauss' hypergeometric differential equations $E(a, b, c)$, $E(1 + a - c', b, c)$, $E(a, b', c')$ and $E(1 + a - c, b', c')$ all have finite irreducible monodromy groups.*

Proof. Since neither c nor c' is an integer by the previous lemma, $E_2(a, b, b', c, c')$ has solutions f_j; $1 \leq j \leq 4$ (2.4). The restrictions of f_1 and f_2 to L_y form a fundamental solutions of $E(a, b, c)$. Hence $M(a, b, c)$ must be finite. The restrictions of $f_3/y^{1-c'}$ and $f_2/y^{1-c'}$ to L_y form a fundamental solutions of $E(1 + a - c', b, c)$. Hence $M(1 + a - c', b, c)$ must be finite. By the same way, $M(a, b', c')$ and $M(1 + a - c, b', c')$ are also finite.

By the irreducibility condition (2.3), $M(a, b, c)$, $M(1+a-c', b, c)$, $M(a, b', c')$ and $M(1 + a - c, b', c')$ are all irreducible. □

4 Proof of the "only if" part of Theorem 1.1.

It is well known that Gauss' hypergeometric differential equation $E(a,b,c)$ has a finite irreducible monodromy group $M(a,b,c)$ if and only if the triple

$$(\lambda,\mu,\nu)=(1-c,c-a-b,b-a)$$

belongs to Schwarz' list (S-list) (after acting the following operations finite times:
permutations of λ,μ,ν,
individual change their signs,
replacing by $(\lambda+l,\mu+m,\nu+n)$ with $l,m,n\in\mathbf{Z}$ and $l+m+n$ even)
(see [Swz], [Iwn], [CW] and Section 8 of this paper).

By Lemma 3.2, we know that if $M_2(a,b,b',c,c')$ is finite irreducible then the following four conditions hold.

$$(\lambda,\mu,\nu) \text{ belongs to S-list,} \tag{4.1}$$
$$(\lambda',\mu',\nu') \text{ belongs to S-list,} \tag{4.2}$$
$$(\lambda,\mu-\lambda',\nu-\lambda') \text{ belongs to S-list,} \tag{4.3}$$
$$(\lambda',\mu'-\lambda,\nu'-\lambda) \text{ belongs to S-list.} \tag{4.4}$$

We always have

$$\lambda+\mu+\nu=\lambda'+\mu'+\nu'\ (=1-2a). \tag{4.5}$$

Lemma 4.1. *Assume that* $M_2(a,b,b',c,c')$ *is finite irreducible. Then* S *or* S' *is one of* (1.1) – (1.6) *of* Theorem 1.1.

Proof. Let

$$\lambda=p/s,\ \mu=q/t,\ \nu=r/u,\ \lambda'=p'/s',\ \mu'=q'/t',\ \nu'=r'/u'$$

be irreducible fractions.

(Case 1) We first deal with the case when

$$s,t,u,s',t',u'\in\{2,3,4,5\}.$$

(Case 1.1) We deal with the case when s or s' is 2.
We assume $s'=2$. Then (4.3) implies $t=u=4$. Then $s=3$ by (4.1). Then (4.4) implies that $t'=u'=3$ and that $q'\equiv r'$ mod 3. Now the denominator of the right hand side of (4.5) is 6 whence we have $q\equiv r$ mod 4 and moreover the equality (4.5) implies that $p\equiv 2q'$ mod 3 and $p'\equiv q$ mod 2. Thus S takes the form of (1.2).

If $s=2$ then S' takes the form of (1.2).

(Case 1.2) Assume that $s,s'\neq 2$ and s or s' is 4.
Assume, for example, that $s'=4$. Then (4.3) implies that t and u are even.

If $t=u=2$ then (4.3) implies $s=3$. Then (4.4) implies $t'=u'=3$ and (4.2) cannot happen.

If $t=u=4$ then (4.1) implies $s=3$. Then (4.2) cannot happen as above.

If $\{t,u\}=\{2,4\}$ then $s=2$ or 3 by (4.1), but s cannot be 3 as above. On the other hand, if $s=2$ then $t'=u'=4$ by (4.4) whence (4.2) cannot happen. This concludes that (Case 1.2) cannot happen.

(Case 1.3) Assume $s = s' = 3$.
Then (4.3) and (4.4) imply that $t = u = t' = u' = 3$ and that $q \equiv r, q' \equiv r', p \not\equiv q', p' \not\equiv q$ mod 3. Thus $\mathcal{S}$ takes the form of (1.1).

(Case 1.4) Assume $s = s' = 5$.
Then (4.3) and (4.4) imply that $t = u = t' = u' = 5$. Then (4.1),(4.2) and (4.3) imply that we have the following two possibilities. That is, $p, q, r \equiv \pm 1$, $p', q', r' \equiv \pm 2$ or $p, q, r \equiv \pm 2$, $p', q', r' \equiv \pm 1$ mod 5. Moreover (4.3) and (4.4) imply that $q \equiv r, q' \equiv r'$ mod 5. Finary (4.3) and (4.4) imply that $p \equiv 2\epsilon, q \equiv r \equiv 2\epsilon'$, $p' \equiv 4\epsilon', q' \equiv r' \equiv \epsilon$, or $p \equiv 4\epsilon, q \equiv r \equiv \epsilon'$, $p' \equiv 2\epsilon', q' \equiv r' \equiv 2\epsilon$ mod 5 and $\epsilon, \epsilon' = \pm 1$. Thus $\mathcal{S}$ takes the form of (1.3).

(Case 1.5) Assume $s = 5, s' = 3$.
Then (4.3) implies $t = u = 3$, $q \equiv r$ mod 3 and (4.4) implies $t' = u' = 5$. Then (4.5) implies $p \equiv q' + r'$ mod 5 and $p' \equiv q + r$ mod 3. Now as for the values of q' and r', there are three cases, that is,
(Case 1.5.1) $q', r' \equiv \pm 1$ mod 5,
(Case 1.5.2) $q', r' \equiv \pm 2$ mod 5 and
(Case 1.5.3) $q' \equiv \pm 1, r' \equiv \pm 2$ or $q' \equiv \pm 2, r' \equiv \pm 1$ mod 5.

As for (Case 1.5.1) and (Case 1.5.2), we have $q' \equiv r'$ mod 5 because $p \equiv q' + r'$ mod 5. Then $\mathcal{S}$ takes the form of (1.4).

We will next show that (Case 1.5.3) does not happen. If, for example, $q' \equiv \pm 1, r' \equiv \pm 2$ mod 5 then $p \equiv 3$ mod 5. Then $\mathcal{S}$ takes the form

$$\mathcal{S} = \left(\frac{3}{5} + l, \frac{\epsilon}{3} + m, \frac{\epsilon}{3} + n; \frac{2\epsilon}{3} + l', \frac{1}{5} + m', \frac{2}{5} + n' \right), \ \epsilon = \pm 1,$$

where $l, m, n, l', m', n' \in \mathbf{Z}$ with $l + m + n = l' + m' + n'$. The condition (4.1) implies that $l + m + n$ is odd and (4.2) implies that $l' + m' + n'$ is even. This is a contradiction. Hence (Case 1.5.3) does not happen.

(Case 2) We next deal with the case when some of s, t, u, s', t', u' is not in $\{2, 3, 4, 5\}$.

We note first that s, s' must be in $\{2, 3, 4, 5\}$. For, otherwise, if $s \notin \{2, 3, 4, 5\}$, then (4.1) and (4.3) imply that $\mu, \nu, \mu - \lambda', \nu - \lambda' \equiv 1/2$ mod $\mathbf{Z}$. This implies that λ' is an integer. This is a contradiction.

(Case 2.1) Assume $u \notin \{2, 3, 4, 5\}$.
The condition (4.1) implies that $\lambda, \mu \equiv 1/2$ mod $\mathbf{Z}$. Then (4.3) implies that $\lambda' \not\equiv 1/2$ mod $\mathbf{Z}$ and (4.4) implies that $t' = u' = 4$. Then (4.2) implies $s' = 3$. Since the denominator of $\mu - \lambda'$ is 6, (4.3) implies that $\nu - \lambda' \equiv 1/2$ mod $\mathbf{Z}$. This implies that $u = 6$.

(Case 2.1.1) Assume $\nu \equiv 1/6$ mod $\mathbf{Z}$.
Then $\lambda' \equiv 2/3$ mod $\mathbf{Z}$ because $\nu - \lambda' \equiv 1/2$ mod $\mathbf{Z}$. Then (4.5) implies that $\mu' + \nu' \equiv 1/2$ mod $\mathbf{Z}$. Hence there are two possibilities, that is, $\mu', \nu' \equiv 1/4$ mod $\mathbf{Z}$ and $\mu', \nu' \equiv -1/4$ mod $\mathbf{Z}$. In both cases, $\mathcal{S}'$ takes the form of (1.5) of Theorem 1.1.

(Case 2.1.2) Assume $\nu \equiv -1/6$ mod $\mathbf{Z}$.
Same as the above case (Case 2.1.1), $\mathcal{S}'$ takes the form of (1.5) of Theorem 1.1.

(Case 2.2) Assume $t \notin \{2, 3, 4, 5\}$.
By the same reasoning as in (Case 2.1), we know that $\mathcal{S}'$ takes the form of (1.6) of Theorem 1.1.

In case of t' or $u' \notin \{2,3,4,5\}$, $\mathcal{S}$ takes the form of (1.5) or (1.6) of Theorem 1.1.

This completes the proof. □

Lemma 4.2. *Let, for* $j = 1, 2$, $\mathcal{S}_j = (\lambda_j, \mu_j, \nu_j;\ \lambda'_j, \mu'_j, \nu'_j)$ *be obtained from the parameters* $(a_j, b_j, b'_j, c_j, c'_j)$ *satisfying the irreducibility condition* (2.3). *Assume*

$$\mathcal{S}_2 = \mathcal{S}_1 + (l, m, n;\ l', m', n'),$$

where l, m, n, l', m', n' *are integers such that* $l+m+n$ *and* $l'+m'+n'$ *are equall to a common even number. Then we have*

$$M_2(\mathcal{S}_1) \simeq M_2(\mathcal{S}_2).$$

Proof. Since

$$a_j = 1/2 - (\lambda_j + \mu_j + \nu_j)/2,$$
$$b_j = a_j + \nu_j,\ b'_j = a_j + \nu'_j,\ c_j = 1 - \lambda_j,\ c'_j = 1 - \lambda'_j,$$

$a_1 \equiv a_2, b_1 \equiv b_2, \dots c'_1 \equiv c'_2 \bmod \mathbf{Z}$. This proves the lemma. □

Proof of the "only if" part of Theorem 1.1.

Lemma 4.1 and 4.2 proves the "only if" part of Theorem 1.1.

5 The system E_2 with quadric property.

T. Sasaki and M. Yoshida ([SY]) proved that $E_2(a,b,b',c,c')$ has the quadric property (that is, four linearly independent solutions are quadratically related) if and only if

$$c = 2b, \;\; c' = 2b', \tag{5.1}$$

$$b + b' - a = 1/2. \tag{5.2}$$

The condition (5.1) is equivalent to

$$\mu = \nu, \;\; \mu' = \nu'. \tag{5.3}$$

Under the condition (5.1), the equality (5.2) is equivalent to one of the following four conditions:

$$\begin{gathered} c + c' - a - b - b' = 1/2, \; c + b' - a - b = 1/2, \; c' + b - a - b' = 1/2, \\ \lambda + \lambda' = \mu + \nu + \mu' + \nu'. \end{gathered} \tag{5.4}$$

Remark 5.1. *E_2 has the characteristic exponents $0,0,0,c+c'-a-b-b'$ along $\{x+y=1\}$, $0,0,0,c+b'-a-b$ along $\{x=1\}$, $0,0,0,c'+b-a-b'$ along $\{y=1\}$.*

We note that if S or S' is one of (1.1)–(1.4) of Theorem 1.1 then E_2 has the quadric property (5.1) and (5.2) if we put $l = m = n = l' = m' = n' = 0$.

Let

$$\psi : (x,y) \longmapsto (u,v), \; u = \left(\frac{x}{2-x-y}\right)^2, \; v = \left(\frac{y}{2-x-y}\right)^2 \tag{5.5}$$

be the $4:1$ mapping of $\mathbf{P}^2$ to $\mathbf{P}^2$ ramified along three lines $\{u=0\}$, $\{v=0\}$ and the line at infinity L_∞.

We denote by $E_4(\alpha,\beta,\gamma,\gamma')$ the system of differential equations of rank four satisfied by Appell's hypergeometric function $F_4(\alpha,\beta,\gamma,\gamma';u,v)$ and by $M_4(\alpha,\beta,\gamma,\gamma')$ the monodromy group of $E_4(\alpha,\beta,\gamma,\gamma')$. For these notations, see [Kt1].

Put as (2.1),

$$X = \mathbf{C}^2 - \{(x,y) \mid xy(x-1)(y-1)(x+y-1) = 0\}, \; P_0 = (p_0, p_0)$$

and

$$X' = X - \{(x,y) | x+y=2\}. \tag{5.6}$$

And put

$$Y = \mathbf{C}^2 - \{(u,v) \mid uv((u-v)^2 - 2(u+v)+1) = 0\}, \; Q_0 = \psi(P_0). \tag{5.7}$$

Then

$$\psi : X' \longrightarrow Y \tag{5.8}$$

is a 4-sheeted covering with $\psi(X') = Y$.

Since X' is Zariski open in X, the following lemma holds.

Lemma 5.1. *Let $\iota : X' \longrightarrow X$ be the inclusion. Then*

$$\iota_* : \pi_1(X', P_0) \longrightarrow \pi_1(X, P_0) \tag{5.9}$$

is onto.

We denote by $V(Q_0)$ the set of germs of solutions of $E_4(\alpha, \beta, \gamma, \gamma')$ at Q_0, where

$$\alpha = \frac{a}{2},\ \beta = \frac{a+1}{2},\ \gamma = b + \frac{1}{2},\ \gamma' = b' + \frac{1}{2}. \tag{5.10}$$

In [SY], the following lemma has been proved.

Lemma 5.2 (Sasaki-Yoshida). *Assume* (5.1) *and* (5.10) *then we have*

$$\psi^*(V(Q_0)) = (2 - x - y)^a V(P_0). \tag{5.11}$$

Put $Q_0 = (q_0, q_0)$. We define $\tilde{\gamma}_j \in \pi_1(Y, Q_0)$, $j = 1, 2, 3$ in the following way.

$$\begin{aligned}
\tilde{\gamma}_1 &= \{(u, v) \mid u = q_0 e^{it}\ 0 \le t \le 2\pi,\ v = q_0\}, \\
\tilde{\gamma}_2 &= \{(u, v) \mid u = q_0,\ v = q_0 e^{it}\ 0 \le t \le 2\pi\}, \\
\tilde{\gamma}_3 &= \{(u, v) \mid u = v = 1/4 - (1/4 - q_0)e^{it}\ 0 \le t \le 2\pi\}.
\end{aligned}$$

Since $\psi : X' \longrightarrow Y$ is a (4-sheeted) covering,

$$\psi_* : \pi_1(X', P_0) \longrightarrow \pi_1(Y, Q_0) \tag{5.12}$$

is injection. It is clear that

$$\tilde{\gamma}_j^2 \in \psi_*(\pi_1(X', P_0));\ j = 1, 2 \text{ and } \tilde{\gamma}_1\tilde{\gamma}_2 = \tilde{\gamma}_2\tilde{\gamma}_1. \tag{5.13}$$

Lemma 5.3. *$\psi_*(\pi_1(X', P_0))$ is a normal subgroup of $\pi_1(Y, Q_0)$ with index 4. And we have*

$$\pi_1(Y, Q_0) = \psi_*(\pi_1(X', P_0)) \cdot \langle \tilde{\gamma}_1, \tilde{\gamma}_2 \rangle. \tag{5.14}$$

Proof. Since $\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_1\tilde{\gamma}_2$ induce the covering transformations of (5.8) defined by

$$(x, y) \longmapsto (x/(x-1), -y/(x-1)), \tag{5.15}$$
$$(x, y) \longmapsto (-x/(y-1), y/(y-1)), \tag{5.16}$$
$$(x, y) \longmapsto (x/(x+y-1), y/(x+y-1)) \tag{5.17}$$

respectively, the covering transformation group acts transitively on any fiber of ψ and is isomorphic to $\langle \tilde{\gamma}_1, \tilde{\gamma}_2 \rangle$. This proves the lemma. □

We fix a basis $\tilde{\varphi}_j$; $1 \le j \le 4$ of $V(Q_0)$ such that

$$\varphi_j = (2 - x - y)^{-a} \psi^*(\tilde{\varphi}_j);\ 1 \le j \le 4 \tag{5.18}$$

hold (see (2.5) and (5.11)). We identify $M_4(\alpha, \beta, \beta', \gamma, \gamma')$ with $\rho_{\tilde{\varphi}}(\pi_1(Y, Q_0))$, where $\alpha, \beta, \beta', \gamma, \gamma'$ are as in (5.10). Thus we have

$$M_4(\alpha, \beta, \beta', \gamma, \gamma') = \rho_{\tilde{\varphi}}(\pi_1(Y, Q_0)). \tag{5.19}$$

We have the following commutative diagrams.

$$\begin{array}{ccc} \pi_1(X', P_0) & \xrightarrow[1:1]{\psi_*} & \pi_1(Y, Q_0) \\ {\scriptstyle \rho_{\psi^*(\tilde{\varphi})}} \downarrow & & {\scriptstyle \rho_{\tilde{\varphi}}} \downarrow \\ GL(4, \mathbf{C}) & \xrightarrow{id} & GL(4, \mathbf{C}) \end{array} \tag{5.20}$$

$$\begin{array}{ccc} \pi_1(X', P_0) & \xrightarrow[\text{onto}]{\iota_*} & \pi_1(X, P_0) \\ {\scriptstyle \rho_{\varphi}} \downarrow & & {\scriptstyle \rho_{\varphi}} \downarrow \\ GL(4, \mathbf{C}) & \xrightarrow{id} & GL(4, \mathbf{C}) \end{array} \tag{5.21}$$

Lemma 5.4. *As subsets of* $GL(4, \mathbf{C})$, *we have*

$$\rho_{\psi^*(\tilde{\varphi})}(\pi_1(X', P_0)) = \rho_{\varphi}(\pi_1(X, P_0)) \cdot \langle e(a) I_4 \rangle, \tag{5.22}$$

$$\rho_{\tilde{\varphi}}(\pi_1(Y, Q_0)) = \rho_{\psi^*(\tilde{\varphi})}(\pi_1(X', P_0)) \cdot \langle \rho_{\tilde{\varphi}}(\tilde{\gamma}_1), \rho_{\tilde{\varphi}}(\tilde{\gamma}_2) \rangle. \tag{5.23}$$

Proof. Since φ_j are holomorphic along $\{x + y = 2\}$, we have, from (5.18),

$$\rho_{\psi^*(\tilde{\varphi})}(\pi_1(X', P_0)) = \rho_{\varphi}(\pi_1(X', P_0)) \cdot \langle e(a) I_4 \rangle.$$

From (5.21), we have

$$\rho_{\varphi}(\pi_1(X', P_0)) = \rho_{\varphi}(\pi_1(X, P_0)).$$

Hence (5.22) holds. From (5.14), we have

$$\begin{aligned} \rho_{\tilde{\varphi}}(\pi_1(Y, Q_0)) &= \rho_{\tilde{\varphi}}(\psi_*(\pi_1(X', P_0)) \cdot \langle \tilde{\gamma}_1, \tilde{\gamma}_2 \rangle) \\ &= \rho_{\tilde{\varphi}}(\psi_*(\pi_1(X', P_0))) \cdot \rho_{\tilde{\varphi}}(\langle \tilde{\gamma}_1, \tilde{\gamma}_2 \rangle) \\ &= \rho_{\psi^*(\tilde{\varphi})}(\pi_1(X', P_0)) \cdot \langle \rho_{\tilde{\varphi}}(\tilde{\gamma}_1), \rho_{\tilde{\varphi}}(\tilde{\gamma}_2) \rangle. \end{aligned}$$

□

Lemma 5.5. *Assume* (5.1), (5.2) *and* (5.10). *Then there exists a subgroup* M *of* $GL(2, \mathbf{C})$ *such that*

$$M \simeq M(\alpha, \beta, \gamma) \simeq M(\alpha, \beta, \gamma'), \tag{5.24}$$

and that the subgroup $(M \otimes M) \cdot \langle \rho_{\tilde{\varphi}}(\tilde{\gamma}_3) \rangle$ *of* $GL(4, \mathbf{C})$ *is isomorphic to* M_4*:*

$$M_4(\alpha, \beta, \beta', \gamma, \gamma') \simeq (M \otimes M) \cdot \langle \rho_{\tilde{\varphi}}(\tilde{\gamma}_3) \rangle. \tag{5.25}$$

We have moreover

$$(M \otimes M) \cap \langle \rho_{\tilde{\varphi}}(\tilde{\gamma}_3) \rangle = \{I_4\}, \ \langle \rho_{\tilde{\varphi}}(\tilde{\gamma}_3) \rangle \simeq \mathbf{Z}_2. \tag{5.26}$$

Proof. From (5.1),(5.2) and (5.10), we know that $M_4(\alpha, \beta, \beta', \gamma, \gamma')$ is irreducible and that

$$\gamma + \gamma' - \alpha - \beta - 1 = 0.$$

Hence the lemma follows from Proposition 4.1 of [Kt1]. □

6 Proof of the "if" part of Theorem 1.1.

If $\mathcal{S}$ or $\mathcal{S}'$ is one of (1.1)–(1.6) then (2.3) holds whence E_2 is irreducible. We will prove the finiteness of M_2. From (2.15) and (2.16), we may assume that $\mathcal{S}$ is one of (1.1)–(1.6) with the sign " + ".

6.1 $\mathcal{S}$ is one of (1.1)–(1.4).

Assume that $\mathcal{S}$ is one of (1.1)–(1.4) of Theorem 1.1.

Since M_2 does not depend on the integers l, m, n, l', m', n' (Lemma 4.2), we put

$$l = m = n = l' = m' = n' = 0.$$

Then E_2 has the quadric property (5.1) and (5.2). Recall $\alpha, \beta, \beta', \gamma, \gamma'$ are defined by (5.10). Then we find that (α, β, γ) belongs to S-list. Hence $M(\alpha, \beta, \gamma)$ is finite. Then Lemma 5.4 and 5.5 imply that M_2 is finite.

6.2 $\mathcal{S}$ is (1.5) or (1.6).

Assume that $\mathcal{S}$ is (1.5) or (1.6) of Theorem 1.1.

We denote the generators $\rho_\varphi(\gamma_j)$ of $M_2 = \rho_\varphi(\pi_1(X, P_0))$ by g_j:

$$g_j = \rho_\varphi(\gamma_j); \ 1 \le j \le 5.$$

By considering the eigenvalues of g_j, we have

$$g_1^3 = 1, \ g_2^2 = 1, \ g_3^3 = g_4^3 = g_5^3 = 1. \tag{6.1}$$

Put

$$r_1 = g_5, \ r_2 = g_3^{-1}, \ r_3 = g_4^{-1}, \ r_4 = g_4(g_1g_2)g_4(g_1g_2)^{-1}g_4^{-1}, \tag{6.2}$$

and

$$R = \langle r_1, r_2, r_3, r_4 \rangle \tag{6.3}$$

a subgroup of N_r. These r_j; $1 \le j \le 4$ are chosen so that the following equalities hold (cf. [ST, p.300]):

$$\begin{gathered} r_1^3 = r_2^3 = r_3^3 = r_4^3 = 1, \\ r_1r_3 = r_3r_1, \ r_1r_4 = r_4r_1, \ r_2r_4 = r_4r_2, \\ r_1z_1 = z_1r_1, \ r_2z_2 = z_2r_2, \ r_3z_3 = z_3r_3, \\ z_1^2 = z_2^2 = z_3^2 = 1, \end{gathered}$$

where

$$z_1 = (r_1r_2)^2, \ z_2 = (r_2r_3)^2, \ z_3 = (r_3r_4)^2.$$

From (6.2), we have

$$g_3, \ g_4, \ g_5 \in R. \tag{6.4}$$

and by direct computations, we have

$$g_1 = r_3(r_2r_1)^2r_3r_1^2r_2^2r_4^2r_3^2r_2^2r_1^2r_2^2r_3^2, \tag{6.5}$$

$$g_2g_3g_2^{-1} = r_1r_2^2r_3^2r_1^2(r_2r_3)^2(r_1r_2)^2r_1, \tag{6.6}$$

$$g_2g_5g_2^{-1} = (r_3r_4)^2r_1r_2^2r_1^2(r_3r_4)^2. \tag{6.7}$$

From (6.2), (6.4) and (6.5), we have

$$g_2g_4g_2^{-1} \in R.$$

From (2.18), we have

$$N_r = \langle g_1^p g_2^q g_j^r g_2^{-q} g_1^{-p} \,|\, p, r = 0, 1, 2,\ q = 0, 1;\ j = 3, 4, 5\rangle$$

in this case. Consequently we have

$$N_r = R. \tag{6.8}$$

There exists a matrix U so that the following equalities hold:

$$Ur_1U^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Ur_2U^{-1} = \frac{-i}{\sqrt{3}} \begin{pmatrix} \omega & \omega^2 & \omega^2 & 0 \\ \omega^2 & \omega & \omega^2 & 0 \\ \omega^2 & \omega^2 & \omega & 0 \\ 0 & 0 & 0 & i\sqrt{3} \end{pmatrix},$$

$$Ur_3U^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Ur_4U^{-1} = \frac{-i}{\sqrt{3}} \begin{pmatrix} \omega & -\omega^2 & 0 & -\omega^2 \\ -\omega^2 & \omega & 0 & \omega^2 \\ 0 & 0 & i\sqrt{3} & 0 \\ -\omega^2 & \omega^2 & 0 & \omega \end{pmatrix},$$

$$\omega = e(1/3). \tag{6.9}$$

The matrix U above is determined in the following way. Put

$$U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}.$$

Then (6.9) implies, for example,
u_2 is an eigenvector of r_3 corresponding to the eigenvalue ω^2,
u_3 is an eigenvector of r_1 corresponding to the eigenvalue ω^2,
u_4 is an eigenvector of r_1, r_2 and r_3 corresponding to the eigenvalue 1,
$u_1 - u_3$ is an eigenvector of r_2 corresponding to the eigenvalue 1,
$u_2 - u_3$ is an eigenvector of r_2 corresponding to the eigenvalue 1,
$u_2 - u_4$ is an eigenvector of r_4 corresponding to the eigenvalue 1,
$u_1 + u_4$ is an eigenvector of r_1, r_3 and r_4 corresponding to the eigenvalue 1.
These determine $u_j;\ 1 \leq j \leq 4$ and we have

$$U = \begin{pmatrix} \frac{1-i\sqrt{3}}{2} & 1 & \frac{(1+\sqrt{3})^2(\sqrt{3}+i)}{4} & \frac{i(1+\sqrt{3})^2}{2} \\ i & \frac{i(1+\sqrt{3})^2}{2} & \frac{(1+\sqrt{3})^2}{2} & 1 \\ \frac{2-\sqrt{3}+i}{2} & \frac{(1+\sqrt{3})(1-i)}{2} & \frac{2(1+\sqrt{3})^2+i(1+\sqrt{3})^4}{8} & \frac{(1+\sqrt{3})(1-i)}{2} \\ \frac{(1+\sqrt{3})(1+i)}{2} & -\frac{(1+\sqrt{3})(1+i)}{2} & -\frac{(1+\sqrt{3})(1+i)}{2} & \frac{(1+\sqrt{3})(1+i)}{2} \end{pmatrix} \tag{6.10}$$

The equalities (6.9) imply (see [ST, 10.5]) that R (= N_r) is the group of No. 32 in S-T table, that is, the symmetry group of order 155520 of the regular complex polytope 3(24)3(24)3(24)3 ([ST, p. 300]).

From (2.19) and (6.5), we have

$$M_2 = N_r \cdot \langle \rho_\varphi(\gamma_2) \rangle. \tag{6.11}$$

This proves the finiteness of M_2.

7 Proof of Theorem 1.2.

We will determine the reflection subgroup N_r and the abelian subgroup A in Theorem 1.2, when $\mathcal{S}$ or $\mathcal{S}'$ is one of (1.1)–(1.6) of Theorem 1.1. In any case, N_r is irreducible reflection group (Theorem 2.3), whence it is one of the groups in S-T table ([ST, Table VII]) (see also Subsection 8.2). From (2.15) and (2.16), we may assume that $\mathcal{S}$ is one of (1.1)–(1.6) with the sign " + ". Recall again the identifications (2.6) and (5.19).

7.1 $\mathcal{S}$ is one of (1.1)–(1.4).

In this subsection, we assume that $\mathcal{S}$ is one of (1.1)–(1.4) of Theorem 1.1 with

$$l = m = n = l' = m' = n' = 0. \tag{7.1}$$

Hence E_2 has the quadric property (5.1) and (5.2). M_4 denotes the monodromy group of $E_4(\alpha, \beta, \gamma, \gamma')$ with (5.10).

Lemma 7.1. *If $\lambda = 2m/n$ (resp. $\lambda' = 2m/n$) with odd n then $\rho_{\tilde{\varphi}}(\tilde{\gamma}_1)$ (resp. $\rho_{\tilde{\varphi}}(\tilde{\gamma}_2)$) $\in \rho_{\psi^*(\tilde{\varphi})}(\pi_1(X', P_0))$ (cf. (5.23)).*

Proof. Assume, for example, that $\lambda = 2m/n$ with odd n. From (5.1) and (5.10), we have $1 - \gamma = m/n$. E_4 has linearly independent solutions of the form

$$f_1(u, v),\ f_2(u, v),\ u^{1-\gamma} f_3(u, v),\ u^{1-\gamma} f_4(u, v),$$

where $f_j(u, v)$ are holomorphic along $\{u = 0\}$ ([AP], [Kt1]). Hence we have $\rho_{\tilde{\varphi}}(\tilde{\gamma}_1)^n = I_4$. Choose $p, q \in \mathbf{Z}$ such that $2p + nq = 1$. Then

$$\rho_{\tilde{\varphi}}(\tilde{\gamma}_1) = \rho_{\tilde{\varphi}}(\tilde{\gamma}_1)^{2p+nq} = \rho_{\tilde{\varphi}}(\tilde{\gamma}_1^2)^p$$

which is in $\rho_{\tilde{\varphi}}(\psi_*(\pi_1(X', P_0)))$ by (5.12) whence in $\rho_{\psi^*(\tilde{\varphi})}(\pi_1(X', P_0))$ by the commutative diagram (5.20). □

Lemma 7.2. *We have*

$$e(a) I_4 \in \rho_{\varphi}(\pi_1(X, P_0)) \tag{7.2}$$

(cf. (5.22)).

Proof. $E_2(a, b, b', c, c')$ has characteristic exponents $a, a, a, b + b'$ along L_∞ with $b + b' - a = 1/2$ (see (5.2)). Hence, by considering a loop surrounding L_∞, we know that $e(2a) I_4 \in M_2$. So if $e(ka) I_4 \in M_2$ for some odd integer k, we conclude $e(a) I_4 \in M_2$.

We note that $E_2(a, b, b', c, c')$ has characteristic exponents $b, b, a, 1 + a - c'$ along $\{x = \infty\}$, $b', b', a, 1 + a - c$ along $\{y = \infty\}$. By considering a loop surrounding $\{x = \infty\}$ or $\{y = \infty\}$, we get the following facts ($\epsilon = \pm 1$).

If $\mathcal{S}$ is (1.1), $b - a = \epsilon/3$ and $1 - c' = 2\epsilon/3$. Hence $e(3a) I_4 \in M_2$.
If $\mathcal{S}$ is (1.2), $b' - a = \epsilon/3$ and $1 - c = 2\epsilon/3$. Hence $e(3a) I_4 \in M_2$.
If $\mathcal{S}$ is (1.3), $b - a = 2\epsilon/5$ and $1 - c' = 4\epsilon/5$. Hence $e(5a) I_4 \in M_2$.
If $\mathcal{S}$ is (1.4), $b - a = \epsilon/3$ and $1 - c' = 2\epsilon/3$. Hence $e(3a) I_4 \in M_2$.
This completes the proof. □

Lemma 7.3. *Let M be the subgroup of $GL(2,\mathbf{C})$ in Lemma 5.5.*
If S is one of (1.1), (1.3) *and* (1.4) *then we have*

$$M_2 \simeq (M \otimes M) \cdot \langle \rho_{\tilde{\varphi}}(\tilde{\gamma}_3) \rangle. \tag{7.3}$$

If S is (1.2) *then we have*

$$M_2 \cdot \langle \rho_{\tilde{\varphi}}(\tilde{\gamma}_2) \rangle \simeq (M \otimes M) \cdot \langle \rho_{\tilde{\varphi}}(\tilde{\gamma}_3) \rangle \tag{7.4}$$

In the right hand side of (7.3) *and* (7.4), *we have*

$$(M \otimes M) \cap \langle \rho_{\tilde{\varphi}}(\tilde{\gamma}_3) \rangle = \{I_4\}, \quad \langle \rho_{\tilde{\varphi}}(\tilde{\gamma}_3) \rangle \simeq \mathbf{Z}_2. \tag{7.5}$$

In the left hand side of (7.4), *we have*

$$\rho_{\tilde{\varphi}}(\tilde{\gamma}_2)^2 \in M_2. \tag{7.6}$$

Proof. Lemma 5.4, 5.5, 7.1 and 7.2 imply (7.3) and (7.4). (7.5) is nothing but (5.26). (7.6) follows from (5.12), (5.20) and (5.22). □

Remark 7.1. *Later (see* (7.28)*), we will see that*

$$M_2 \cap \langle \rho_{\tilde{\varphi}}(\tilde{\gamma}_2) \rangle = \{I_4\}$$

in (7.4).

Lemma 7.4. *M_2 with S being* (1.2) *of Theorem 1.1 are all isomorphic.*

Proof. It suffices to prove that M_2 with S being (1.2) with $\epsilon = 1$ is isomorphic to that with $\epsilon = -1$. We have the following correspondence:

$$\begin{array}{ccc} S & \longleftrightarrow & (a, b, b', c, c') \\ (2/3, 1/4, 1/4;\ 1/2, 1/3, 1/3) & \longleftrightarrow & (-1/12, 1/6, 1/4, 1/3, 1/2) \\ (-2/3, 1/4, 1/4;\ 1/2, -1/3, -1/3) & \longleftrightarrow & (7/12, 5/6, 1/4, 5/3, 1/2) \end{array}$$

By Theorem 2.2 ($n = 12$, $k = 5$), the corresponding two monodromy groups are mutually isomorphic. □

Lemma 7.5. *M_2 with S being* (1.3) *of Theorem 1.1 are all isomorphic.*

Proof. It suffices to prove that M_2 with S being (1.3) with $\epsilon = 1$ is isomorphic to that with $\epsilon = -1$. We have the following correspondence:

$$\begin{array}{ccc} S & \longleftrightarrow & (a, b, b', c, c') \\ (2/5, 2/5, 2/5;\ 4/5, 1/5, 1/5) & \longleftrightarrow & (-1/10, 3/10, 1/10, 3/5, 1/5) \\ (-2/5, 2/5, 2/5;\ 4/5, -1/5, -1/5) & \longleftrightarrow & (3/10, 7/10, 1/10, 7/5, 1/5) \end{array}$$

By Theorem 2.2 ($n = 10$, $k = 7$), the corresponding two monodromy groups are mutually isomorphic. □

Lemma 7.6. *M_2 with S being* (1.4) *of Theorem 1.1 are all isomorphic.*

Proof. It suffices to prove that M_2 with $\mathcal{S}$ being (1.4) with ϵ is 1,2,3 and 4 are all isomorphic. We have the following correspondence:

$$\begin{array}{ccc} \mathcal{S} & \longleftrightarrow & (a,b,b',c,c') \\ (2/5,1/3,1/3;\ 2/3,1/5,1/5) & \longleftrightarrow & (-1/30,3/10,1/6,3/5,1/3) \\ (4/5,1/3,1/3;\ 2/3,2/5,2/5) & \longleftrightarrow & (-7/30,1/10,1/6,1/5,1/3) \\ (6/5,1/3,1/3;\ 2/3,3/5,3/5) & \longleftrightarrow & (-13/30,-1/10,1/6,-1/5,1/3) \\ (8/5,1/3,1/3;\ 2/3,4/5,4/5) & \longleftrightarrow & (-19/30,-3/10,1/6,-3/5,1/3) \end{array}$$

By Theorem 2.2 ($n = 30,\ k = 7,13,19$), the corresponding four monodromy groups are mutually isomorphic. □

Concerning several monodromy groups $M(\alpha,\beta,\gamma)$, we have the following lemma by direct computations.

Lemma 7.7. *Let $|G|$ denote the order of a group G. We have*

$$\begin{array}{lcr} |M(-1/12,5/12,2/3)| &=& 72, \\ |M(1/4,3/4,4/3)| &=& 24, \\ |M(-1/24,11/24,2/3)| &=& 288, \\ |M(-1/20,9/20,4/5)| &=& 600, \\ |M(-1/60,29/60,4/5)| &=& 1800. \end{array}$$

Proof of Theorem 1.2 for $\mathcal{S}$ being one of (1.1)–(1.4).
From (2.19), we have

$$M_2 = N_r \cdot \langle \rho_\varphi(\gamma_1), \rho_\varphi(\gamma_2)\rangle.$$

From Lemma 7.4–7.6, it suffices to verify for the following cases:

$$\mathcal{S} = (2/3,1/3,1/3;\ 2/3,1/3,1/3), \tag{7.7}$$
$$\mathcal{S} = (-2/3,1/3,1/3;\ 2/3,-1/3,-1/3), \tag{7.8}$$
$$\mathcal{S} = (2/3,1/4,1/4;\ 1/2,1/3,1/3), \tag{7.9}$$
$$\mathcal{S} = (2/5,2/5,2/5;\ 4/5,1/5,1/5), \tag{7.10}$$
$$\mathcal{S} = (2/5,1/3,1/3;\ 2/3,1/5,1/5). \tag{7.11}$$

Recall M denote a subgroup of $GL(2,\mathbf{C})$ in Lemma 7.3.
The case of (7.7).
In this case, $M \simeq M(-1/12,5/12,2/3)$. From Lemma 7.3 and 7.7, we have $|M_2| = 72\cdot 12\cdot 2$. Since $\langle \rho_\varphi(\gamma_1), \rho_\varphi(\gamma_2)\rangle \simeq \mathbf{Z}_3\times\mathbf{Z}_3$, $|N_r| = 2^6\cdot 3^k$, where $k=1$ or 2 or 3. Hence $N_r = G(2,2,4)$ in S-T table with $|N_r| = 2^6\cdot 3$. This again implies that

$$M_2 = N_r\cdot A,\ A = \langle \rho_\varphi(\gamma_1), \rho_\varphi(\gamma_2)\rangle,\ N_r\cap A = \{1\},\ A\simeq \mathbf{Z}_3\times\mathbf{Z}_3.$$

The case of (7.8).
In this case, $M \simeq M(1/4,3/4,4/3)$. From Lemma 7.3 and 7.7, we have $|M_2| = 24\cdot 12\cdot 2$. By the same reason as above, $N_r = G(2,2,4)$ with $|N_r| = 2^6\cdot 3$. This implies that

$$M_2 = N_r\cdot A,\ A = \langle \rho_\varphi(\gamma_1)\rangle \text{ or } \langle \rho_\varphi(\gamma_2)\rangle,\ N_r\cap A = \{1\},\ A\simeq \mathbf{Z}_3.$$

The case of (7.9).
In this case, $M \simeq M(-1/24, 11/24, 2/3)$. From Lemma 7.3 and 7.7, we have $|M_2| = 288 \cdot 24 \cdot 2^k$, where $k = 0$ or 1. Since $\langle \rho_\varphi(\gamma_1), \rho_\varphi(\gamma_2) \rangle \simeq \mathbf{Z}_3 \times \mathbf{Z}_2$, $|N_r| = 2^l \cdot 3^k$, where $l = 7$ or 8 or 9 and $k = 2$ or 3. Hence N_r is the group of No. 28 in S-T table with $|N_r| = 2^7 \cdot 3^2$. This again implies that

$$M_2 = N_r \cdot A,\ A = \langle \rho_\varphi(\gamma_1), \rho_\varphi(\gamma_2) \rangle,\ N_r \cap A = \{1\},\ A \simeq \mathbf{Z}_3 \times \mathbf{Z}_2$$

and

$$M_2 \cap \langle \rho_{\tilde{\varphi}}(\tilde{\gamma}_2) \rangle = \{1\} \tag{7.12}$$

at (7.4) in Lemma 7.3.

The case of (7.10).
In this case, $M \simeq M(-1/20, 9/20, 4/5)$. From Lemma 7.3 and 7.7, we have $|M_2| = 600 \cdot 60 \cdot 2$. Since $\langle \rho_\varphi(\gamma_1), \rho_\varphi(\gamma_2) \rangle \simeq \mathbf{Z}_5 \times \mathbf{Z}_5$, $|N_r| = 2^6 \cdot 3^2 \cdot 5^k$, where $k = 1$ or 2 or 3. Hence N_r is the group of No. 30 in S-T table with $|N_r| = 2^6 \cdot 3^2 \cdot 5^2$. This again implies that

$$M_2 = N_r \cdot A,\ A = \langle \rho_\varphi(\gamma_1) \rangle \text{ or } \langle \rho_\varphi(\gamma_2) \rangle,\ N_r \cap A = \{1\},\ A \simeq \mathbf{Z}_5.$$

The case of (7.11).
In this case, $M \simeq M(-1/60, 29/60, 4/5)$. From Lemma 7.3 and 7.7, we have $|M_2| = 1800 \cdot 60 \cdot 2$. Since $\langle \rho_\varphi(\gamma_1), \rho_\varphi(\gamma_2) \rangle \simeq \mathbf{Z}_5 \times \mathbf{Z}_3$, $|N_r| = 2^6 \cdot 3^k \cdot 5^l$, where $k, l = 2$ or 3. Hence N_r is the group of No. 30 in S-T table with $|N_r| = 2^6 \cdot 3^2 \cdot 5^2$. This again implies that

$$M_2 = N_r \cdot A,\ A = \langle \rho_\varphi(\gamma_1), \rho_\varphi(\gamma_2) \rangle,\ N_r \cap A = \{1\},\ A \simeq \mathbf{Z}_5 \times \mathbf{Z}_3.$$

This completes the proof of Theorem 1.2 for $\mathcal{S}$ being one of (1.1)–(1.4).

7.2 $\mathcal{S}$ is (1.5) or (1.6).

In Subsection 6.2, we have proved that

$$M_2 = N_r \cdot A, \quad A = \langle \rho_\varphi(\gamma_2) \rangle,$$

where N_r is the symmetry group of regular complex polytope 3(24)3(24)3(24)3 (of order 155520).

Now we will prove

$$N_r \cap A = \{1\}. \tag{7.13}$$

The 240 vertices of the polytope 3(24)3(24)3(24)3 are given by

$$\pm\omega(0,0,0,\sqrt{-3}),\ \pm\omega(0,0,\sqrt{-3},0),\ \pm\omega(0,\sqrt{-3},0,0),\ \pm\omega(\sqrt{-3},0,0,0),$$
$$\pm\omega(1,\omega_1,\omega_2,0),\ \pm\omega(1,-\omega_1,0,-\omega_2),\ \pm\omega(1,0,-\omega_1,\omega_2),\ \pm\omega(0,1,-\omega_1,-\omega_2),$$

where $\omega, \omega_1, \omega_2$ are roots of $x^3 = 1$ (see [Shp, p. 95]). The generators Ur_jU^{-1}; $1 \le j \le 4$ (see (6.9)) of UN_rU^{-1} induce pemutations of these points. But it can be verified, by direct computations, that $U\rho_\varphi(\gamma_2)U^{-1}$ does not induce a permutation of these points. This prove (7.13).

From (6.1), we have

$$A \simeq \mathbf{Z}_2.$$

This completes the proof of Theorem 1.2 for $\mathcal{S}$ being (1.5) or (1.6).

8 Appendix.

8.1 Schwarz' list.

Gauss' hypergeometric differential equation $E(a,b,c)$ has a finite irreducible monodromy group $M(a,b,c)$ if and only if the triple $(\lambda,\mu,\nu) = (1-c, c-a-b, b-a)$ is one in the Schwarz' list after acting the following operations finite times:

permutations of λ, μ, ν,

changing their signs individually,

replacing by $(\lambda+l, \mu+m, \nu+n)$ with $l,m,n \in \mathbf{Z}$ and $l+m+n$ even

(see [Swz], [Iwn], [CW]).

Schwarz' list

λ	μ	ν	
$\frac{1}{2}$	$\frac{1}{2}$	r	$r \in \mathbf{Q}-\mathbf{Z}$, dihedral case
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	tetrahedral case
$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	octahedral case
$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$	icosahedral case
$\frac{2}{5}$	$\frac{1}{3}$	$\frac{1}{3}$	
$\frac{2}{3}$	$\frac{1}{5}$	$\frac{1}{5}$	
$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{5}$	
$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	
$\frac{3}{5}$	$\frac{1}{3}$	$\frac{1}{5}$	
$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{5}$	
$\frac{4}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	
$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$	
$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{3}$	

8.2 Table of finite irreducible unitary reflection groups of degree 4.

We extract the following table of all of the finite irreducible unitary reflection groups in $U(4, \mathbf{C})$ from [ST, Table VII].

No.	Symbol	order	order of the center	
1		5!	1	
2	$G(pq, p, 4)$	$q(pq)^3 4!$	$q \cdot GCD(p, 4)$	$pq > 1$
28	$[3, 4, 3]$	1152	2	
29		7680	4	
30	$[3, 3, 5]$	14400	2	
31		$64 \cdot 6!$	4	
32		$216 \cdot 6!$	6	

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Mitsuo KATO
Department of Mathematics
College of Education
University of the Ryukyus
Nishihara-cho, Okinawa 903-0213
JAPAN
(e-mail: mkato@edu.u-ryukyu.ac.jp)