有限モノドロミー群をもつ超幾何微分方程式の Schwarz map

Hypergeometric function ${}_nF_{n-1}$ with imprimitive finite irreducible monodromy group

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1 Introduction.

A generalized hypergeometric function

$$ {}_nF_{n-1}(a_0, a_1, a_2, \dots, a_{n-1}; b_1, b_2, \dots, b_{n-1}; z) = \sum_{n=0}^{\infty} \frac{\prod_{j=0}^{n-1}(a_j, n)}{\prod_{j=1}^{n-1}(b_j, n) n!} z^n, $$

where $(a, n) = \Gamma(a+n)/\Gamma(a)$, satisfies a Fuchsian differential equation

$$ {}_nE_{n-1}(a_0, a_1, a_2, \dots, a_{n-1}; b_1, b_2, \dots, b_{n-1}) $$

of rank n with singularities at $z = 0$, 1 and ∞. F. Beukers and G. Heckman [B-H] determined ${}_nE_{n-1}$ with finite irreducible monodromy groups. In [Kt], for ${}_3E_2$ with finite irreducible primitive monodromy groups, Schwarz maps of $\mathbf{P}^1$ to $\mathbf{P}^2$ defined by linearly independent three solutions are studied. The images of Schwarz maps and their single valued inverse maps are determined.

As stated in Theorem 5.8 in [B-H], under some condition, ${}_nE_{n-1}$ with irreducible imprimitive monodromy group is essentially given by

$$ {}_nE_{n-1}\Big(\frac{-\alpha}{p}, \frac{-\alpha+1}{p}, \dots, \frac{-\alpha+p-1}{p}, \frac{\alpha}{q}, \frac{\alpha+1}{q}, \dots, \frac{\alpha+q-1}{q}; \frac{1}{n}, \dots, \frac{n-1}{n}\Big), \tag{1.1} $$

where $(p, q) = 1$ and $n = p + q$.

In this paper, for (1.1) with $\alpha = -1/(mn)$, $m \geq 2$, we will determine its Schwarz map and obtain its single valued inverse map. (If $\alpha = -1/n$, then the monodromy group of (1.1) is not irreducible.)

For this purpose, we use the generalized binomial function (see Section 2)

$$ \psi(\alpha, -p/n, x), \tag{1.2} $$

because if we put $z = (-p)^p q^q n^{-n} x^n$, then (1.2) is (as a multi-valued function of z) a solution of (1.1).

If $\alpha = -1/n$, then (1.2) is also a solution of the algebraic equation

$$ y^n + xy^p - 1 = 0. \tag{1.3} $$

This fact was first discovered by Lambert (see [Brn, p.307]), and studied by many mathematicians (for example, [Blr]). We also remark that the generalized binomial function is a typical example of quasi-hypergeometric function studied in [A-I].

In Section 2, for the sake of self-containedness, we give elementary proofs for several known results concerning to (1.2) and (1.3).

In Section 3, we consider the case of $\alpha = -1/n$. In this case, (1.1) is reducible and if moreover $p < n-1$, then (1.2) satisfies a differential equation of rank $n-1$ with the projective monodromy group isomorphic to the symmetric group S_n (Corollary 3.6). In Section 4, we put $\alpha = -1/(nm)$ $m \geq 2$. Take n solutions of (1.3) and choose m-th roots $f_j^{(1/m)}(x)$, $0 \leq j \leq n-1$ of these solutions. Then these are (as functions of z) linearly independent solutions of (1.1). The monodromy group induces all permutations on these solutions and multiplications of m-th roots of 1 to each $f_j^{(1/m)}(x)$ (up to multiplications of common constant numbers to all $f_j^{(1/m)}$). Thus (1.1) has imprimitive finite irreducible projective monodromy group of order $m^{n-1}n!$ (Corollary 4.5). The Schwarz map of (1.1) is defined by

$$z \longmapsto [f_0^{(1/m)} : f_1^{(1/m)} : \cdots : f_{n-1}^{(1/m)}].$$

The defining functions of its image in $\mathbf{P}^{n-1}$ and its single valued inverse map are expressed, consulting (1.3), by use of elementary symmetric functions of n-variables (Theorem 4.4).

Finally, in Section 5, we state several topics for $n = 3$ case. We give a proof of Cardano's formula for a cubic equation, using properties of generalized binomial functions. We also give a uniformization of ${}_3E_2$ by theta functions, that is, if we put $z = J(\tau)$, the elliptic modular function, then the solutions of (1.1) with $\alpha = -1/12, p = 1, q = 2$ are single valued functions of τ and are expressed by theta functions.

2 Generalized binomial function.

The statements in this section are found in [Brn], [Blr], etc, but the proofs here are elementary.

For any complex numbers α and s, put

$$c_0(\alpha, s) = 1,$$
$$c_k(\alpha, s) = \alpha(\alpha + ks + 1, k - 1)/k! \quad (k \geq 1), \tag{2.1}$$

and put

$$\psi(\alpha, s, x) = \sum_{k=0}^{\infty} c_k(\alpha, s) x^k. \tag{2.2}$$

We call $\psi(\alpha, s, x)$ a generalized binomial function because $\psi(\alpha, 0, x) = (1-x)^{-\alpha}$.

We will prove some properties of $\psi(\alpha, s, x)$.

Lemma 2.1.

$$\psi(\alpha, s, x) = \psi(-\alpha, -s - 1, -x). \tag{2.3}$$

Proof.

$$\begin{aligned}
&(-1)^k c_k(-\alpha, -s - 1) \\
&= (-1)^k(-\alpha)(-\alpha - (s + 1)k + 1, k - 1)/k! \\
&= \alpha(\alpha + sk + k - 1)(\alpha + sk + k - 2) \cdots (\alpha + sk + 1) \\
&= c_k(\alpha, s).
\end{aligned}$$

□

We note that $\psi(\alpha, -1, x) = (1 + x)^\alpha$ and $\psi(0, s, x) = 1$.

Proposition 2.2. *If $\alpha, s, s+1 \neq 0$, then the radious of convergence of $\psi(\alpha, s, x)$ is $|s^s/(s + 1)^{s+1}|$. Where z^z denotes the principal value.*

Proof. Put

$$\tilde{c}_k(\alpha, s) = (\alpha + 1 + sk, k - 1)/k! = \frac{\Gamma(\alpha + (s + 1)k)}{\Gamma(1 + k)\Gamma(\alpha + 1 + sk)}.$$

Then the radious of convergence of $\psi(\alpha, s, x)$ is the reciprocal of the upper limit of $|\tilde{c}_k|^{1/k}$.

First assume that s is not a negative real number. Then, from the Stirling's formula:

$$\Gamma(z) \sim \sqrt{2\pi} z^{z-1/2} e^{-z} \quad \text{as } z \to \infty \text{ and } |\arg z| < \pi - \delta, \delta > 0,$$

we have

$$\begin{aligned}
|\tilde{c}_m(\alpha, s)|^{1/m} &\sim \frac{|(\alpha + (s + 1)m)^{s+1}|}{(1 + m)|(\alpha + 1 + sm)^s|} \sim \left| \frac{\alpha + (s + 1)m}{1 + m} \left(\frac{\alpha + (s + 1)m}{\alpha + 1 + sm} \right)^s \right| \\
&\sim |(s + 1)^{s+1}/s^s|.
\end{aligned}$$

This proves the proposition for s which is not a negative real number.

Assume $-1 < s < 0$. For large $m \in \mathbf{N}$, choose $n_m \in \mathbf{N}$ and δ_m with $0 \le \delta_m < 1$ such that

$$Re(\alpha) + sm = -n_m - \delta_m.$$

Then

$$\begin{aligned}
|\tilde{c}_m(\alpha, s)| &= |(\alpha + 1 + sm, m-1)|/m! \\
&= |(\alpha+1+sm)\cdots(\alpha+1+sm+n_m-1)| \\
&\quad \times |(\alpha+1+sm+n_m)\cdots(\alpha+(s+1)m-1)|/m! \\
&= |(-\alpha - sm - n_m, n_m)| \cdot |(\alpha + sm + n_m + 1, m - 1 - n_m)|/m! \\
&= \frac{|\Gamma(-\alpha - sm)| \cdot |\Gamma(\alpha + (s+1)m)|}{|\Gamma(1+m)\Gamma(-\alpha - sm - n_m)\Gamma(\alpha + sm + n_m + 1)|}
\end{aligned}$$

Hence we have

$$\begin{aligned}
\limsup_{m\to\infty} |\tilde{c}_m(\alpha, s)|^{1/m} &= \lim_{m\to\infty} \left| \frac{(-\alpha - sm)^{-s}(\alpha + (s+1)m)^{s+1}}{1+m} \right| \\
&= \lim_{m\to\infty} \left| \left(\frac{-\alpha - sm}{1+m}\right)^{-s} \left(\frac{\alpha + (s+1)m}{1+m}\right)^{s+1} \right| \\
&= |(-s)^{-s}(s+1)^{s+1}| = |(s+1)^{s+1}/s^s|.
\end{aligned}$$

This proves the proposition for s with $-1 < s < 0$. From Lemma 2.1, the proposition holds for any negative real number s which is not -1.

This completes the proof. □

Proposition 2.3. *We have the following two equalities.*

$$c_k(\alpha, s) - c_k(\alpha - 1, s) = c_{k-1}(\alpha + s, s), \tag{2.4}$$

$$\psi(\alpha + \beta, s, x) = \psi(\alpha, s, x)\psi(\beta, s, x). \tag{2.5}$$

Proof. Proof of (2.4).

$$\begin{aligned}
&c_k(\alpha, s) - c_k(\alpha - 1, s) \\
&= \frac{\alpha(\alpha + ks + 1, k-1) - (\alpha - 1)(\alpha + ks, k-1)}{k!} \\
&= \frac{(\alpha + s)(\alpha + s + (k-1)s + 1, k-2)}{(k-1)!} = c_{k-1}(\alpha + s, s).
\end{aligned}$$

Proof of (2.5). It is sufficient to prove

$$c_k(\alpha + \beta, s) = \sum_{i+j=k} c_i(\alpha, s)c_j(\beta, s), \tag{2.6}$$

which is proved by induction for k. Consider

$$d_k(\beta) = c_k(\alpha + \beta, s) - \sum_{i+j=k} c_i(\alpha, s)c_j(\beta, s)$$

as a polynomial of β (α being a parameter) of degree at most k. From (2.4), we have

$$d_k(\beta) - d_k(\beta - 1) = d_{k-1}(\beta + s),$$

which vanishes by induction. Hence $d_k(\beta)$ must be constant C. Since $c_j(0, s) = 0$ for $j > 0$, we hace $C = d_k(0) = 0$. This completes the proof of (2.6) whence of (2.5). □

Corollary 2.4. *For any rational number $\beta \in \mathbf{Q}$, we have*

$$\psi(\alpha\beta, s, x) = \psi(\alpha, s, x)^{\beta},$$

where the right hand side is the branch which takes the value 1 at $x = 0$.

Proposition 2.5. *Let $\epsilon_k = e^{2\pi i/k}$. For positive integers p, q with $(p, q) = 1$, $n = p + q$, the equation (1.3) : $y^n + xy^p - 1 = 0$ has solutions*

$$\epsilon_n^j \psi(-1/n, -p/n, \epsilon_n^{pj} x), \quad 0 \le j \le n - 1, \tag{2.7}$$

in a neighborhood of $x = 0$,

$$\epsilon_p^{-j} x^{-1/p} \psi\left(1/p, q/p, -(\epsilon_p^{-j} x^{-1/p})^n\right), \quad 0 \le j \le p - 1, \tag{2.8}$$

$$\epsilon_q^{j} (-x)^{1/q} \psi\left(-1/q, p/q, -(\epsilon_q^{j} (-x)^{1/q})^{-n}\right), \quad 0 \le j \le q - 1, \tag{2.9}$$

in a neighborhood of $x = \infty$.

Proof. From (2.4), we have

$$\psi(\alpha, s, x) - \psi(\alpha - 1, s, x) = x\psi(\alpha + s, s, x). \tag{2.10}$$

Put $s = -p/n$ and $\alpha = 0$ then we have

$$1 - \psi(-1, s, x) = x\psi(-p/n, s, x),$$

which is equivalent to

$$\psi(-1/n, s, x)^n + x\psi(-1/n, s, x)^p - 1 = 0. \tag{2.11}$$

If we replace x by $\epsilon_n^{pj} x$, we know that (2.7) are solutions of (1.3).

Put $s = q/p$ and $\alpha = 1$ in (2.10). Then we have

$$\psi(1/p, s, x)^p - 1 = x\psi(1/p, s, x)^n,$$

which is equivalent to

$$\left[(-x)^{1/n}\psi(1/p, s, x)\right]^n + (-x)^{-p/n}\left[(-x)^{1/n}\psi(1/p, s, x)\right]^p - 1 = 0.$$

Put $x_1 = (-x)^{-p/n}$, and wright x instead of x_1, then we know that functions in (2.8) are solutions of (1.3).

Now put $s = p/q$ and $\alpha = -s$ in (2.10), then we have

$$\psi(-1/q, s, x)^n - \psi(-1/q, s, x)^p + x = 0.$$

Then, by the same way as above, we know that functions in (2.9) are solutions of (1.3). This completes the proof. □

Corollary 2.6. *If σ_k denotes the elementary symmetric polynomial of degree k of $\{\epsilon_n^j \psi(-1/n, -p/n, \epsilon_n^{pj} x),\ 0 \le j \le n-1\ \}$, then we have*

$$\sigma_k = 0, \quad 1 \le k \le n-2,\ k \ne n-p, \tag{2.12}$$

$$\sigma_{n-p} = (-1)^{n-p} x, \tag{2.13}$$

$$\sigma_n = (-1)^{n-1}. \tag{2.14}$$

For any $s = m/n$ with positive integer n, put

$$\varphi_j(\alpha, s, x) = x^j \sum_{l=0}^{\infty} c_{j+ln}(\alpha, s) x^{ln}, \tag{2.15}$$

then we have

$$\psi(\alpha, s, x) = \sum_{j=0}^{n-1} \varphi_j(\alpha, s, x). \tag{2.16}$$

Proposition 2.7. *Let $s = -p/n$ and $n = p + q$, then we have*

$$\begin{aligned} \varphi_j(\alpha, s, x) &= c_j(\alpha, s) x^j \\ &\times {}_nF_{n-1}\Big(\frac{-\alpha+\mu}{p} + \frac{j}{n},\ 0 \le \mu \le p-1,\ \frac{\alpha+\nu}{q} + \frac{j}{n},\ 0 \le \nu \le q-1; \\ &\quad \frac{j+1}{n}, \dots, \frac{n-1}{n}, \frac{n+1}{n}, \dots, \frac{n+j}{n};\ \frac{(-1)^p p^p q^q}{n^n} x^n\Big). \end{aligned} \tag{2.17}$$

Proof. If $k = nl$ $(l \ge 1)$, then we have

$$\begin{aligned} c_k(\alpha, s) &= \frac{1}{k!}\alpha(\alpha - pl + 1, nl - 1) = \frac{1}{k!}\alpha(\alpha - pl + 1, pl - 1)(\alpha, ql) \\ &= (-1)^{pl}\frac{(-\alpha, pl)(\alpha, ql)}{(1, nl)} = (-1)^{pl}\frac{p^{pl} q^{ql} \prod_{\mu=0}^{p-1}(-\frac{\alpha}{p} + \frac{\mu}{p}, l) \prod_{\nu=0}^{q-1}(\frac{\alpha}{q} + \frac{\nu}{q}, l)}{n^{nl} \prod_{\lambda=0}^{n-1}(\frac{1}{n} + \frac{\lambda}{n}, l)}. \end{aligned}$$

If $k = nl + j$ $(1 \le j \le n-1)$, then we have

$$\begin{aligned} &c_k(\alpha, s) \\ &= \frac{1}{k!}\alpha(\alpha - \frac{p}{n}(nl + j) + 1, nl + j - 1) \\ &= \frac{1}{j!(j+1, nl)}\alpha(\alpha - \frac{p}{n}(nl + j) + 1, pl)(\alpha - \frac{pj}{n} + 1, j - 1)(\alpha + \frac{qj}{n}, ql) \\ &= \frac{\alpha(\alpha + \frac{qj}{n} - j + 1, j - 1)}{j!}(-1)^{pl}\frac{(-\alpha + \frac{pj}{n}, pl)(\alpha + \frac{qj}{n}, ql)}{(j+1, nl)} \\ &= c_j(\alpha, s)(-1)^{pl}\frac{p^{pl} q^{ql} \prod_{\mu=0}^{p-1}(-\frac{\alpha}{p} + \frac{j}{n} + \frac{\mu}{p}, l) \prod_{\nu=0}^{q-1}(\frac{\alpha}{q} + \frac{j}{n} + \frac{\nu}{q}, l)}{n^{nl} \prod_{\lambda=0}^{n-1}(\frac{j+1}{n} + \frac{\lambda}{n}, l)}. \end{aligned}$$

This implies (2.17). □

Corollary 2.8. *Let $s = -p/n, n = p + q$ and $\epsilon_n = e^{2\pi i/n}$. Then $\psi(\alpha, s, \epsilon_n^k x)$ is, as a multi-valued function of $z = (-p)^p q^q n^{-n} x^n$, a solution of the differential equation (1.1). If $c_j(\alpha, s) \ne 0$ for $0 \le j \le n-1$, then $\psi(\alpha, s, \epsilon_n^k x)$ $0 \le k \le n-1$ are linearly independent.*

Proof. From (2.17), we know that $\varphi_j(\alpha, s, x)$ is a solution of (1.1) (see the lemma below). From (2.15) and (2.16), we have

$$\psi(\alpha, s, \epsilon_n^k x) = \sum_{j=0}^{n-1} \epsilon_n^{jk} \varphi_j(\alpha, s, x), \tag{2.18}$$

which is thus a solution of (1.1). If $c_j(\alpha, s) \neq 0$ then $\varphi_j(\alpha, s, x) \neq 0$ and $\psi(\alpha, s, \epsilon_n^k x)$ $0 \leq k \leq n-1$ are linearly independent from (2.18). $\square$

The following lemma is well known.

Lemma 2.9. *Let $b_0 = 1$, then differential equation*

$${}_nE_{n-1}(a_0, a_1, a_2, ..., a_{n-1}; b_1, b_2, ..., b_{n-1})$$

has solutions

$$\begin{aligned} &z^{1-b_j}\,{}_nF_{n-1}(a_0 + 1 - b_j, ..., a_{n-1} + 1 - b_j;\\ &\quad b_0 + 1 - b_j, ..., \widehat{b_j + 1 - b_j}, ..., b_{n-1} + 1 - b_j; z); \;\; 0 \leq j \leq n-1 \end{aligned}$$

at $z = 0$ and

$$\begin{aligned} &z^{-a_j}\,{}_nF_{n-1}(a_j + 1 - b_0, ..., a_j + 1 - b_{n-1};\\ &\quad a_j + 1 - a_0, ..., \widehat{a_j + 1 - a_j}, ..., a_j + 1 - a_{n-1}; 1/z); \;\; 0 \leq j \leq n-1 \end{aligned}$$

at $z = \infty$.

Proof. ${}_nE_{n-1}$ is defined by

$$\begin{aligned} &[\vartheta(\vartheta + b_1 - 1)(\vartheta + b_2 - 1) \cdots (\vartheta + b_{n-1} - 1)\\ &\qquad - z(\vartheta + a_0)(\vartheta + a_1) \cdots (\vartheta + a_{n-1})]u = 0, \end{aligned} \tag{2.19}$$

where $\vartheta = z\partial/\partial z$ (see [Bly]). It is easily verified that functions in Lemma satisfy (2.19). $\square$

Remark 2.1. *If $s = p/q$ with $n = p + q$, then we can prove, for $0 \leq j \leq q-1$,*

$$\begin{aligned} \varphi_j(\alpha, s, x) &= x^j \sum_{l=0}^{\infty} c_{j+lq}(\alpha, s) x^{lq}\\ &= c_j(\alpha, s) x^j\,{}_nF_{n-1}\Big(\frac{\alpha}{n} + \frac{j}{q}, \frac{\alpha+1}{n} + \frac{j}{q}, \cdots, \frac{\alpha+n-1}{n} + \frac{j}{q};\\ &\quad \frac{\alpha+1}{p} + \frac{j}{q}, \cdots, \frac{\alpha+p}{p} + \frac{j}{q}, \frac{1+j}{q}, \cdots, \frac{q-1}{q}, \frac{q+1}{q}, \cdots, \frac{q+j}{q}; \frac{n^n}{p^p q^q} x^q\Big). \end{aligned}$$

3 Global properties of solutions of $y^n + xy^p - 1 = 0$.

Put $r = |s^s/(s+1)^{s+1}|$, then $\psi(\alpha, s, x)$ is holomorphic in $\Delta_r := \{x \mid |x| < r\}$ (Proposition 2.2).

Lemma 3.1. *Assume $s \in \mathbf{R}$.*

(1) *If $\alpha \in \mathbf{R}$, then $\psi(\alpha, s, x) > 0$ for real x in Δ_r.*

(2) *$\psi(-1, s, x)$ does not take negative value in Δ_r, that is $|arg\,\psi(-1, s, x)| < \pi$.*

Proof. If $\alpha, x \in \mathbf{R}$, then $\psi(\alpha, s, x) \in \mathbf{R}$. Since $\psi(\alpha, s, x) \neq 0$ and $\psi(\alpha, s, 0) = 1$, we have $\psi(\alpha, s, x) > 0$.

Assume $\psi(-1, s, x_0) < 0$ for some $x_0 \in \Delta_r$. Put $\theta = \arg x_0$. Then there exist $t_0 < r$ and positive number b_0 such that

$$|\arg \psi(-1, s, e^{i\theta}t)| < \pi, \text{ for } 0 < t < t_0 \text{ and } \psi(-1, s, e^{i\theta}t_0) = -b_0.$$

From (2.10), we know $y = \psi(-1, s, x)$ satisfies

$$y + xy^{-s} - 1 = 0.$$

Since $\psi(-1, s, e^{i\theta}t_0)^{-s} = e^{\pm\pi i s}b_0^{-s}$, we have (put $x = e^{i\theta}t_0$)

$$-b_0 + e^{i\theta}t_0\, e^{\pm\pi i s}b_0^{-s} - 1 = 0.$$

Thus we have $e^{i(\theta\pm\pi s)} = (b_0+1)/(b_0^{-s}t_0) > 0$, which implies $\theta = (\pm s + 2n)\pi$, $n \in \mathbf{Z}$. Since $y = \psi(-1, s, x)$ defines an open map, $\psi(-1, s, e^{i\theta}t)$ maps some open interval $(t_0-\delta, t_0+\delta)$ onto some open interval$(-b_0-\delta', -b_0+\delta')$. This contradicts to the choice of t_0. □

We assume $(p, q) = 1$ and $n = p + q$. From Proposition 2.5,

$$f_j(x) := \epsilon_n^j \psi(-1/n, -p/n, \epsilon_n^{pj}x), \quad 0 \le j \le n-1, \tag{3.1}$$

are solutions of the equation (1.3): $y^n + xy^p - 1 = 0$.

The equation (1.3) has multiple roots at x with

$$(-1)^p p^p q^q n^{-n} x^n = 1$$

and at $x = \infty$. Let

$$x_j = e\left(\frac{-p(1+2j)}{2n}\right)(p^{-p}q^{-q})^{1/n}n, \quad 0 \le j \le n-1, \tag{3.2}$$

where $e(x) = e^{2\pi i x}$.

Lemma 3.2. *At $x = x_j$, the equation* (1.3) *has double root*

$$e((1+2j)/2n)(p/q)^{1/n} \tag{3.3}$$

and n-2 simple roots.

Proof. The double root of the equation (1.3) is uniquely determined by (1.3) and $ny^{n-1} + pxy^{p-1} = 0$. □

We know that $f_j(x)$ are holomorphic in $\Delta := \{x \,|\, |x| < r_{n,p}\}$ and continuous in the closure $\bar{\Delta}$ of Δ, where $r_{n,p} = (p/n)^{-p/n}(q/n)^{-q/n}$. They have analytic continuations along any curve not through x_k, $0 \leq k \leq n-1$.

Put

$$D_j = f_j(\bar{\Delta}), \tag{3.4}$$

then we have $D_j = e(j/n)D_0$ and put $D_n = D_0$.

Lemma 3.3.

$$\Big(\frac{-1+2j}{n}\Big)\pi \leq arg\, y \leq \Big(\frac{1+2j}{n}\Big)\pi \quad for\ y \in D_j, \tag{3.5}$$

$$D_j \cap D_{j+1} = \{f_j(x_j) = f_{j+1}(x_j)\} = \{e((1+2j)/2n)(p/q)^{1/n}\} \tag{3.6}$$

and $D_j \cap D_k = \emptyset$ *if* $j - k \neq \pm 1$.

Proof. The inequalities (3.5) follow from Corollary 2.4 and (2) of Lemma 3.1. These inequalities imply that $D_j \cap D_k = \emptyset$ if $j - k \neq \pm 1$. Since any element of $D_j \cap D_{j+1}$ is one of (3.3), we have

$$D_j \cap D_{j+1} = \{e((1+2j)/2n)(p/q)^{1/n}\}$$

from (3.5). From Lemma 3.2, (3.6) follows. □

Corollary 3.4. *Let γ_0 be a loop starting and ending at the origin and once surrounding x_0. Let $\gamma_j = e(-pj/n)\gamma_0$. Then, by the analytic continuation along γ_j, $f_j(x)$ and $f_{j+1}(x)$ are interchanged and other $f_k(x)$ are unchanged.*

Proof. Assume γ_0 (hence any γ_j) acts trivially on $\{f_0, \dots, f_{n-1}\}$, then $f_j(x)$ are entire functions. This contradicts Proposition 2.2. □

Definition 3.1. *Let E be a Fuchsian linear differential equation of rank n on $\mathbf{P}^1$. Let $Z = \mathbf{P}^1 - \{$singular points of $E\}$. Fix a base point $z_b \in Z$, and let V be the set of germs of holomorphic solutions of E at z_b. For any $\gamma \in \pi_1(Z, z_b)$ and $f \in V$, the analytic continuation $\gamma_* f$ of f along γ is again in V. We consider γ_* an element of $GL(V)$ and call the set $M(E)$ of all γ_* the* ***monodromy group*** *of E and $M(E)/($its center$)$ the* ***projective monodromy group*** *of E.*

We say that $M(E)$ is (or E is) ***reducible*** *if there exists a non trivial subspace V_1 of V which is invariant under the action of $M(E)$ and say $M(E)$ is (or E is)* ***irreducible*** *if $M(E)$ is not reducible.*

We say that $M(E)$ is (or E is) ***imprimitive*** *if V has a direct sum decomposition $V = V_1 + V_2 + \cdots + V_k$ such that any element of $M(E)$ induces a permutation of $\{V_1, V_2, \dots, V_k\}$.*

Choose a fundamental system $f_j(z)$, $1 \leq j \leq n$ of solutions of E and fix initial values of them at z_b. Then, by taking analytic continuations of $f_j(z)$, we have a multi-valued map

$$z \in Z \longmapsto [f_1(z) : f_2(z) : \cdots : f_n(z)] \in \mathbf{P}^{n-1},$$

which we call a ***Schwarz map*** *of E.*

Remark 3.1. *In the above definition, we have two remarks.*

If the characteristic exponents of E are real and do not differ by integers at each singular point then the Schwarz map above can be extended to a map from $\mathbf{P}^1$ to $\mathbf{P}^{n-1}$.

If the Schwarz map has a single valued inverse map π_E, then the projective monodromy group of E is isomorphic to the covering transformation group of π_E.

The map of Δ_r to $\mathbf{P}^{n-1}$ defined by $[f_0(x) : f_1(x) : \cdots : f_{n-1}(x)]$ is extended to a multi-valued map of $\mathbf{C} - \{x_0, \cdots, x_{n-1}\}$ to $\mathbf{P}^{n-1}$ by the analytic continuation. Take the closure of its image in $\mathbf{P}^{n-1}$ which we denote by $X_{n,p}$.

Proposition 3.5. *$X_{n,p}$ is equal to the set of common zeros of σ_k, $1 \le k \le n-1$, $k \neq q$, where σ_k are the elementary symmetric function of degree k. Put*

$$\pi_{n,p}([y_0 : y_1 : \cdots : y_{n-1}]) = (-1)^n \frac{p^p q^q}{n^n} \frac{(\sigma_q(y_0, \cdots, y_{n-1}))^n}{(\sigma_n(y_0, \cdots, y_{n-1}))^q}, \tag{3.7}$$

then we have

$$\pi_{n,p}([f_0(x) : f_1(x) : \cdots : f_{n-1}(x)]) = z := (-1)^p p^p q^q n^{-n} x^n. \tag{3.8}$$

$\pi_{n,p}$ is an $n! : 1$ map of $X_{n,p}$ to $\mathbf{P}^1$ ramifying at $z = 0, 1, \infty$. The ramification indices at these points are $n, 2, pq$ respectively. The covering transformation group is isomorphic to symmetric group S_n of order $n!$.

Proof. Denote $\hat{X}_{n,p}$ be the set of common zeros of σ_k, $0 \le k \le n-2$, $k \neq q$. From Bezout's theorem, $\pi_{n,p}|_{\hat{X}_{n,p}}$ is an $n! : 1$ map of $\hat{X}_{n,p}$ to $\mathbf{P}^1$. From Corollary 2.6, we have $X_{n,p} \subset \hat{X}_{n,p}$, that is $X_{n,p}$ is an irreducible component of $\hat{X}_{n,p}$. From Corollary 2.6, (3.8) holds and from Corollary 3.4, we know that S_n acts on each fiber of $\pi_{n,p}|_{X_{n,p}}$. Consequently we must have $\hat{X}_{n,p} = X_{n,p}$.

By definition of z, the ramification index is n at $z = 0$. From Corollary 3.4, the index at $z = 1$ is 2. From Proposition 2.5, we know that the ramification index at $z = \infty$ is pq. This completes the proof. □

The statement (2) of the following corollary is proved in [B-H, Proposition 2.6].

Corollary 3.6. (1) *If $p < n-1$, then $\psi(-1/n, -p/n, \epsilon_n^k x)$, $0 \le k \le n-1$ are solutions of a differential equation ${}_{n-1}E_{n-2}$, the projective monodromy group of which is isomorphic to the symmetric group S_n of order $n!$. Any $n-1$ of the above solutions are linearly independent.*

(2) *The projective monodromy group of*

$${}_{n-1}E_{n-2}\left(\frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}; \frac{1}{p}, \cdots, \frac{p-1}{p}, \frac{1}{q}, \cdots, \frac{q-1}{q}\right) \tag{3.9}$$

is isomorphic to S_n.

Proof. Proof of (1). Assume $p < n-1$ or equivalently $q > 1$. Put $\alpha = -1/n$ and $s = -p/n$. Let q^* be the integer such that

$$1 \le q^* \le n-1 \quad \text{and} \quad qq^* \equiv 1 \bmod n.$$

Then $p^* := n - q^*$ also satisfies $pp^* \equiv 1 \bmod n$. For $k = p$ or q, put $d_k = (kk^*-1)/n$. Note $q^* > 1$ and $d_q > 0$ because $q > 1$. We easily have $c_{q^*}(\alpha, s) = 0$, consequently $\varphi_{q^*}(\alpha, s, x) = 0$ (see Proposition 2.7). Since

$$(-\alpha + d_p)/p = (\alpha + q - d_q)/q = p^*/n,$$

we have

$$\begin{aligned}\varphi_0(\alpha, s, x) \\ = {}_{n-1}F_{n-2}\Big(&\frac{-\alpha}{p}, \cdots, \frac{-\alpha+p-1}{p}, \frac{\alpha}{q}, \cdots, \widehat{\frac{\alpha+q-d_q}{q}}, \cdots, \frac{\alpha+q-1}{q}; \\ &\frac{n-1}{n}, \cdots, \widehat{\frac{p^*}{n}}, \cdots, \frac{1}{n}; z\Big),\end{aligned}$$

where $z = (-1)^p p^p q^q n^{-n} x^n$ as before. By the same way, we know that $\{\varphi_j \,|\, 0 \le j \le n-1, j \ne q^*\}$ form a system of fundamental solutions of

$$\begin{aligned}{}_{n-1}E_{n-2}\Big(&\frac{-\alpha}{p}, \cdots, \frac{-\alpha+p-1}{p}, \frac{\alpha}{q}, \cdots, \widehat{\frac{\alpha+q-d_q}{q}}, \cdots, \frac{\alpha+q-1}{q}; \\ &\frac{n-1}{n}, \cdots, \widehat{\frac{p^*}{n}}, \cdots, \frac{1}{n}\Big).\end{aligned} \tag{3.10}$$

The equalities (2.18) imply that $\psi(-1/n, -p/n, \epsilon_n^k x)$, $0 \le k \le n-1$ are solutions of (3.10) and moreover any $n-1$ of them are linearly independent. Since the projective monodromy group of (3.10) is isomorphic to the covering transformation group of $\pi_{n,p}$ which is isomorphic to S_n from Proposition 3.5. This completes the proof of (1).

Proof of (2). In (3.9), p and q are symmetric so that we can remain the assumption of $p < n-1$. Put $r = (-\alpha + d_p)/p = (\alpha + q - d_q)/q = p^*/n$ then, from Lemma 2.9, the equation (3.10) has the special solution

$$\begin{aligned}z^{-r}{}_{n-1}F_{n-2}\Big(&r, r+\frac{1}{n}, \cdots, \widehat{r+\frac{q^*}{n}}, \cdots, r+\frac{n-1}{n};\ 1+\frac{d_p}{p}, \cdots, 1+\frac{1}{p}, \\ &\frac{p-1}{p}, \cdots, \frac{1+d_p}{p}, 1+\frac{q-d_q}{q} \cdots, 1+\frac{1}{q}, \frac{q-1}{q} \cdots, \frac{q-d_q-1}{q}; 1/z\Big).\end{aligned}$$

Thus the projective monodromy groups of (3.9) and (3.10) are mutually isomorphic. This proves (2).

This completes the proof. □

4 Schwarz map of a family of imprimitive ${}_nE_{n-1}$.

Assume $(p, q) = 1$ and put

$$n = p + q, \quad s = -p/n, \quad z = (-p)^p q^q n^{-n} x^n, \quad \epsilon_k = e(1/k) = e^{2\pi i/k}.$$

For an integer $m \geq 2$, put $\alpha = -1/(mn)$ and put

$$f_j^{(1/m)}(x) = \epsilon_{mn}^j \psi(\alpha, s, \epsilon_n^{pj} x), \quad 0 \leq j \leq n-1, \tag{4.1}$$

which is a m-th root of $f_j(x)$. When we consider $f_j^{(1/m)}(x)$ as a multi-valued function of z, we denote it by $f_j^{(1/m)}(z)$.

Lemma 4.1. $f_j^{(1/m)}(z)$, $0 \leq j \leq n-1$ *are linearly independent solutions of differential equation* (1.1).

Proof. Since $c_j(\alpha, s) \neq 0$, for $0 \leq j \leq n-1$, Corollary 2.8 proves the lemma. □

Similar to (3.4) we put

$$D_j^{(1/m)} = f_j^{(1/m)}(\bar{\Delta}).$$

Then we have $D_j^{(1/m)} = e(j/(mn)) D_0^{(1/m)}$ and can prove the following lemma and its corollary from Lemma 3.3 and Corollary 3.4.

Lemma 4.2.

$$\begin{aligned} D_j^{(1/m)} \cap D_{j+1}^{(1/m)} &= \{f_j^{(1/m)}(x_j) = f_{j+1}^{(1/m)}(x_j)\} \\ &= \{e((1+2j)/(2mn))(p/q)^{1/n}\}, \quad 0 \leq j \leq n-2, \\ D_{n-1}^{(1/m)} \cap e(1/m) D_0^{(1/m)} &= \{f_{n-1}^{(1/m)}(x_{n-1}) = e(1/m) f_0^{(1/m)}(x_{n-1})\} \\ &= \{e((2n-1)/(2mn))(p/q)^{1/n}\}. \end{aligned}$$

Corollary 4.3. (1) *Let γ_j be the loop defined in Corollary 3.4. Then by the analytic continuations along γ_j, $0 \leq j \leq n-2$, $f_j^{(1/m)}(x)$ and $f_{j+1}^{(1/m)}(x)$ are interchanged and other $f_k^{(1/m)}(x)$ are unchanged, by that along γ_{n-1}, $f_{n-1}^{(1/m)}(x)$ and $e(1/m) f_0^{(1/m)}(x)$ are interchanged and other $f_k^{(1/m)}(x)$ are unchanged.*

(2) *We have*

$$\begin{aligned} f_j^{(1/m)}(e(p/n)x) &= e(-1/(mn)) f_{j+1}^{(1/m)}(x), \quad \textit{for } 0 \leq j \leq n-2, \\ f_{n-1}^{(1/m)}(e(p/n)x) &= e((n-1)/(mn)) f_0^{(1/m)}(x). \end{aligned}$$

From Lemma 4.1 (see also Remark 3.1), a Schwarz map of (1.1) is given by

$$z \in \mathbf{P}^1 \longmapsto [f_0^{(1/m)}(z) : f_1^{(1/m)}(z) : \cdots : f_{n-1}^{(1/m)}(z)]. \tag{4.2}$$

We denote its image by $X_{n,p}^{(1/m)}$ which is an irreducible curve in $\mathbf{P}^{n-1}$.

Theorem 4.4. *Let $\alpha = -1/(mn)$, $m \geq 2$, $s = -p/n$, then we have*

$$X_{n,p}^{(1/m)} = \{[y_0 : y_1 : \cdots : y_{n-1}] \in \mathbf{P}^{n-1} \mid \sigma_k(y_0^m, y_1^m, \cdots, y_{n-1}^m) = 0, \quad 1 \leq k \leq n-1,\ k \neq q\}, \tag{4.3}$$

where σ_k is the elementary symmetric function of degree k. Put

$$\pi_{n,p}^{(1/m)}([y_0 : y_1 : \cdots : y_{n-1}]) = (-1)^n \frac{p^p q^q}{n^n} \frac{\left(\sigma_q(y_0^m, y_1^m, \cdots, y_{n-1}^m)\right)^n}{\left(\sigma_n(y_0^m, y_1^m, \cdots, y_{n-1}^m)\right)^q}, \tag{4.4}$$

then $\pi_{n,p}^{(1/m)}$ is an $m^{n-1}n! : 1$ map of $X_{n,p}^{(1/m)}$ to $\mathbf{P}^1$ and satisfies

$$\pi_{n,p}^{(1/m)}([f_0^{(1/m)}(z) : f_1^{(1/m)}(z) : \cdots : f_{n-1}^{(1/m)}(z)]) = z. \tag{4.5}$$

The branch points of $\pi_{n,p}^{(1/m)}$ are $z = 0, 1, \infty$ with ramification indices $n, 2, mpq$ respectively.

Proof. We denote the right hand side of (4.3) by $\hat{X}_{n,p}^{(1/m)}$ for a moment. Since

$$\left(f_j^{(1/m)}(x)\right)^m = f_j(x),$$

we have, from Proposition 3.5, $X_{n,p}^{(1/m)} \subset \hat{X}_{n,p}^{(1/m)}$. By definition, $\pi_{n,p}^{(1/m)}$ is an $m^{n-1}n! : 1$ map of $\hat{X}_{n,p}^{(1/m)}$ to $\mathbf{P}^1$ and from (3.8) it satisfies (4.5). On the other hand, $\pi_{n,p}^{(1/m)}$ restricted to $X_{n,p}^{(1/m)}$ has $m^{n-1}n!$ points in general fiber because the covering transformation group of $X_{n,p}^{(1/m)}$ includes S_n from (1) of Corollary 4.3 and multiplication of $e(1/m)$ to coordinate y_{n-1} from (2) of the same corollary. Hence we have $X_{n,p}^{(1/m)} = \hat{X}_{n,p}^{(1/m)}$. The ramification index at $z = \infty$ is mpq from Proposition 2.5.

This completes the proof. □

Corollary 4.5. *Let $\alpha = -1/(mn)$, $m \geq 2$, then the differential equation* (1.1) *has imprimitive finite irreducible projective monodromy group of order $m^{n-1}n!$.*

Proof. The order of the projective monodromy group of (1.1) is equal to the degree of $\pi_{n,p}^{(1/m)}$ which is $m^{n-1}n!$ from the above theorem. Let Γ_0 and Γ_1 be loops once surrounding $z = 0$ and $z = 1$ respectively. From Corollary 4.3, both Γ_0 and Γ_1 induce permutations on the set $\{\langle f_j^{(1/m)} \rangle \mid 0 \leq j \leq n-1\}$ of one dimensional subspaces $\langle f_j^{(1/m)} \rangle$ of V. Hence the monodromy group of (1.1) is imprimitive.

Since none of $\frac{-\alpha+k}{p} - \frac{l}{n}$, $\frac{\alpha+k}{q} - \frac{l}{n}$, is an integer for any integers k and l, (1.1) is irreducible from (the proof of) Proposition 3.3 of [B-H]. □

Corollary 4.6. *For any positive integer m and integer q with $1 \leq q \leq n-1$, the algebraic set*

$$\{[y_0 : y_1 : \cdots : y_{n-1}] \in \mathbf{P}^{n-1} \mid \sigma_k(y_0^m, y_1^m, \cdots, y_{n-1}^m) = 0,\ 1 \leq k \leq n-1,\ k \neq q\}$$

is irreducible.

Proof. The statement is true for $m = 1$ from Proposition 3.5 and for $m \geq 2$ from Theorem 4.4. □

5 $\psi(\alpha, -1/3, x)$.

Lemma 5.1.

$$\psi(-1/2, -1/2, x) = \frac{-x + \sqrt{x^2+4}}{2}, \tag{5.1}$$

$$\psi(-1, 1, x) = \frac{1 + \sqrt{1-4x}}{2}. \tag{5.2}$$

Proof. From (2.16) and (2.17), we have

$$\psi(-1/2, -1/2, x) = {}_2F_1\Big(\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}; -\frac{1}{4}x^2\Big) - \frac{1}{2}x\, {}_2F_1\Big(1, 0; \frac{3}{2}; -\frac{1}{4}x^2\Big).$$

Since ${}_2F_1(a, b; b; x) = (1-x)^{-a}$, (5.1) is proved.

If $k \geq 1$, then we have

$$\begin{aligned} c_k(-1, 1) &= -(k, k-1)/k! \\ &= k(k+1)\cdots(2k-2)/k! = -(2k-2)!/(k!(k-1)!) \\ &= -1 \cdot 3 \cdots (2k-3)2^{k-1}/k! = -(1/2, k-1)2^{2k-2}/k! \\ &= (-1/2, k)4^k/(2k!) \end{aligned}$$

Hence we have (5.2). □

Lemma 5.2.

$$\begin{aligned} &\psi(-1/3, -1/3, x) \\ &= \left(\frac{1}{2}\left(1 + \frac{4}{27}x^3\right)^{1/2} + \frac{1}{2}\right)^{1/3} - \frac{1}{3}x\left(\frac{1}{2}\left(1 + \frac{4}{27}x^3\right)^{1/2} + \frac{1}{2}\right)^{-1/3} \\ &= \left(\frac{1}{2}\left(1 + \frac{4}{27}x^3\right)^{1/2} + \frac{1}{2}\right)^{1/3} - \left(\frac{1}{2}\left(1 + \frac{4}{27}x^3\right)^{1/2} - \frac{1}{2}\right)^{1/3}, \end{aligned} \tag{5.3}$$

where cube roots take positive values if x is a positive small number.

Proof. From (2.16) and (2.17), we have

$$\begin{aligned} &\psi(-1/3, -1/3, x) \\ &= {}_3F_2\Big(\frac{1}{3}, -\frac{1}{6}, \frac{1}{3}; \frac{2}{3}, \frac{1}{3}; -\frac{4}{27}x^3\Big) - \frac{1}{3}x\, {}_3F_2\Big(\frac{2}{3}, \frac{1}{6}, \frac{2}{3}; \frac{4}{3}, \frac{2}{3}; -\frac{4}{27}x^3\Big) \\ &= {}_2F_1\Big(-\frac{1}{6}, \frac{1}{3}; \frac{2}{3}; -\frac{4}{27}x^3\Big) - \frac{1}{3}x\, {}_2F_1\Big(\frac{1}{6}, \frac{2}{3}; \frac{4}{3}; -\frac{4}{27}x^3\Big), \end{aligned}$$

which is equal to, from Remark 2.1,

$$\begin{aligned} &\varphi_0(-1/3, 1/1; -x^3/27) - 1/3\, x\, \varphi_0(1/3, 1/1; -x^3/27) \\ &= \psi(-1/3, 1; -x^3/27) - 1/3\, x\, \psi(1/3, 1; -x^3/27) \\ &= [\psi(-1, 1; -x^3/27)]^{1/3} - 1/3\, x\, [\psi(-1, 1; -x^3/27)]^{-1/3} \\ &= \left[\frac{1 + \sqrt{1 + 4x^3/27}}{2}\right]^{1/3} - \frac{1}{3}x\left[\frac{1 + \sqrt{1 + 4x^3/27}}{2}\right]^{-1/3} \end{aligned}$$

from (5.2). This proves the lemma. □

Theorem 5.3 (Cardano). *The equation*

$$X^3 + 3pX - 2q = 0$$

has roots

$$\epsilon_3^m \left(q + \sqrt{p^3+q^2}\right)^{1/3} + \epsilon_3^{2m} \left(q - \sqrt{p^3+q^2}\right)^{1/3}, \quad 0 \le m \le 2, \tag{5.4}$$

where $\epsilon_3 = e^{2\pi i/3}$ *and cube roots must be chosen such that*

$$\left(q + \sqrt{p^3+q^2}\right)^{1/3} \left(q - \sqrt{p^3+q^2}\right)^{1/3} = -p. \tag{5.5}$$

Proof. Theorem follows from Lemma 5.2 and Proposition 2.5. □

Lemma 5.4. *Let* $s = -p/n$. *Then for any* α, *we have*

$$\prod_{j=0}^{n-1} \psi(\alpha, s, \epsilon_n^j x) = 1. \tag{5.6}$$

Proof. From (2.18), we have

$$\psi(\alpha, s, \epsilon_n^j x) = \sum_{k=0}^{n-1} \epsilon_n^{jk} \varphi_k(\alpha, s, x).$$

First we note

$$\varphi_0(0,s,x) = 1, \ \frac{\partial \varphi_0}{\partial \alpha}(0,s,x) = 0 \text{ and } \varphi_k(0,s,x) = 0 \text{ for } k \ge 1.$$

Put $f(\alpha) = \prod_{j=0}^{n-1} \psi(\alpha, s, \epsilon_n^j x)$. Then $f(0) = 1$ and

$$\begin{aligned} \frac{df}{d\alpha}\bigg|_{\alpha=0} &= \sum_{k=0}^{n-1} \frac{\partial \psi}{\partial \alpha}(\alpha, s, \epsilon_n^k x) \prod_{j \ne k} \psi(\alpha, s, \epsilon_n^j x)\bigg|_{\alpha=0} = \sum_{k=0}^{n-1} \frac{\partial \psi}{\partial \alpha}(\alpha, s, \epsilon_n^k x)\bigg|_{\alpha=0} \\ &= \sum_{k=0}^{n-1}\sum_{j=0}^{n-1} \epsilon_n^{jk} \frac{\partial \varphi_j}{\partial \alpha}\bigg|_{\alpha=0} = \Big(\sum_{j=1}^{n-1} \frac{\partial \varphi_j}{\partial \alpha}\bigg|_{\alpha=0}\Big)\Big(\sum_{k=0}^{n-1} \epsilon_n^{jk}\Big) \\ &= 0. \end{aligned}$$

Since $f(\alpha + \beta) = f(\alpha) f(\beta)$, we have $f(\alpha) = f(0) \exp(\alpha \, df(0)/d\alpha)$. This proves (5.6) □

Let $\alpha = -1/(3m)$ and put $y_j = f_j^{(1/m)}\left(\alpha, -\frac{1}{3}, z\right)$ for $j = 0, 1, 2$ (as for $f_j^{(1/m)}$, see (4.1)). Then, from (4.3), (4.4) and (4.5), we have

$$y_0^m + y_1^m + y_2^m = 0, \quad \pi_{3,1}^{(1/m)}([y_0 : y_1 : y_2]) = \frac{\left(y_0^{2m} + y_1^{2m} + y_2^{2m}\right)^3}{54(y_0 y_1 y_2)^{2m}} = z. \tag{5.7}$$

Let

$$J(\tau) = 12^{-3} q^{-2} \left(1 + 744q^2 + 196884q^4 + 21493760q^6 + \cdots\right), \ q = e^{\pi i \tau}$$

be the elliptic modular function defined on the upper half plane. We have the following theorem.

Theorem 5.5. *Let $\alpha = -1/12$, $s = -1/3$, $z = J(\tau)$. Recall $z = -4x^3/27$ and define x as a single valued function of τ so that $x > 0$ for $\tau = (-1+\sqrt{3}i)/2 + ti$ with $t > 0$. Then we have*

$$f_0^{(1/4)} = C\vartheta_2(0,\tau),\ f_1^{(1/4)} = C\vartheta_0(0,\tau),\ f_2^{(1/4)} = e(1/8)C\vartheta_3(0,\tau), \tag{5.8}$$

where $C = 2^{-1/3}e(1/24)q^{-1/12}H_0^{-1}$, $H_0 = \prod_{k=1}^{\infty}(1-q^{2k})$.

Proof. Let $C_4 = \{[y_0 : y_1 : y_2] \in \mathbf{P}^2 \,|y_0^4 + y_1^4 + y_2^4 = 0\}$, then

$$\pi_{3,1}^{(1/4)} : C_4 \longrightarrow \mathbf{P}^1$$

satisfy, from (5.7),

$$\pi_{3,1}^{(1/4)}([y_0 : y_1 : y_2]) = \frac{(y_0^8 + y_1^8 + y_2^8)^3}{54(y_0y_1y_2)^8}.$$

It is well known (see, for example [Akh]) that

$$\pi_{3,1}^{(1/4)}\left([\vartheta_2(0,\tau) : \vartheta_0(0,\tau) : e(1/8)\vartheta_3(0,\tau)]\right) = J(\tau). \tag{5.9}$$

This and the equality (5.6) imply that both

$$[f_0^{(1/4)} : f_1^{(1/4)} : f_2^{(1/4)}] \quad \text{and} \quad [\vartheta_2(0,\tau) : \vartheta_0(0,\tau) : e(1/8)\vartheta_3(0,\tau)]$$

belong to the same fiber $\left(\pi_{3,1}^{(1/4)}\right)^{-1}(J(\tau))$. Hence for some fourth roots ϵ, ϵ' of 1 and some function $C' = C'(\tau)$, we have

$$\{f_0^{(1/4)}, f_1^{(1/4)}, f_2^{(1/4)}\} = \{C'\vartheta_2(0,\tau), C'\epsilon\vartheta_0(0,\tau), C'\epsilon' e(1/8)\vartheta_3(0,\tau)\}.$$

If we put $\tau = (-1+\sqrt{3}i)/2 + ti$ and let t to $+\infty$, then $z = J(\tau) < 0$ goes to $-\infty$. Since, from (5.3),

$$f_j^{(1/4)} = \epsilon_{12}^j 2^{-1/12}\left(\left((\sqrt{1-z})+1\right)^{1/3} - \epsilon_3^j\left(\sqrt{1-z}-1\right)^{1/3}\right)^{1/4},$$

we have (5.8) for some $C = C(\tau)$. Since $\vartheta_2(0,\tau)\vartheta_0(0,\tau)\vartheta_3(0,\tau) = 2q^{1/4}H_0^3$ ([Akh]), C takes the value in the statement of the theorem. □

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