対合同変正則写像空間の位相幾何

メタデータ	言語:
	出版者: 神山靖彦
	公開日: 2009-03-17
	キーワード (Ja): 有理関数, 複素共役対合, 多項式, 重根, 配置空間, ループ空間, 安定分解, 生成多様体
	キーワード (En): rational function, conjugation, polynomial, multiple root, configuration space, loop space, stable splitting, generating variety
	作成者: 神山, 靖彦, 志賀, 博雄, 手塚, 康誠, Kamiyama, Yasuhiko, Shiga, Hiroo, Tezuka, Michishige
	メールアドレス:
	所属:
URL	http://hdl.handle.net/20.500.12000/9229

Configuration spaces and rational functions

Yasuhiko Kamiyama
University of the Ryukyus

July 8, 2005

§1. Arnold's results

V.I. Arnold: On some topological invariants of algebraic functions.
Trudy Moscov. Mat. Obshch. 21, 27–46 (1970);
English transl. in Trans. Moscow Math. Soc. 21, 30–52 (1970).

Today I will consider a certain question which originates in this paper.

Roughly speaking, the question is:

To study the relationship between the space of polynomials with n-fold roots and the space of n-tuples of polynomials with common roots

Definition of the configuration space

We set

$$C_k(\mathbb{C}) = \{(\alpha_1, \ldots, \alpha_k) \in \mathbb{C}^k : \alpha_i \neq \alpha_j \quad \text{if} \quad i \neq j\}/\Sigma_k,$$

where Σ_k is the symmetric group on k letters.

Interpretation of $C_k(\mathbb{C})$

(1) $C_k(\mathbb{C}) = \{f(z) = z^k + a_1 z^{k-1} + \cdots + a_k :$ $a_i \in \mathbb{C}, f(z)$ does not have a multiple root$\}$.

(2) $C_k(\mathbb{C}) = K(\beta_k, 1)$,

where β_k is Artin's braid group on k-strings.

Today we use (1).

Example.

(1) $C_1(\mathbb{C}) = \mathbb{C}$.

(2) $C_2(\mathbb{C}) = \{(z+u)^2 + v : v \neq 0\}$
$\cong \mathbb{C} \times \mathbb{C}^* \simeq S^1$.

Arnold's results

Arnold first studied the homology of $C_k(\mathbb{C})$ systematically. The results are:

(1) Finiteness Theorem: For $q \geq 2$, $H_q(C_k(\mathbb{C});\mathbb{Z})$ is a finite group.

(2) Repetition Theorem: For $q \geq 0$, we have

$$H_q(C_{2d}(\mathbb{C});\mathbb{Z}) \cong H_q(C_{2d+1}(\mathbb{C});\mathbb{Z}).$$

(But $C_{2d}(\mathbb{C})$ and $C_{2d+1}(\mathbb{C})$ are not homotopy equivalent, because the fundamental groups are not isomorphic.)

(3) Stability Theorem: Fix q. Then for $k \geq 2q$, we have

$$H_q(C_k(\mathbb{C});\mathbb{Z}) \cong H_q(C_\infty(\mathbb{C});\mathbb{Z}).$$

Moreover, Arnold calculated $H_q(C_k(\mathbb{C});\mathbb{Z})$ for $1 \leq q \leq 5$. The result is given by the following table.

Table 1. The groups $H_q(C_k(\mathbb{C});\mathbb{Z})$ for $1 \leq q \leq 5$.

$k \setminus q$	1	2	3	4	5
$0,1$	0	0	0	0	0
$2,3$	$\mathbb{Z}$	0	0	0	0
$4,5$	$\mathbb{Z}$	$\mathbb{Z}/2$	0	0	0
$6,7$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	0
$8,9$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/6$	$\mathbb{Z}/3$
$10,11$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/6$	$\mathbb{Z}/6$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/6$	$\mathbb{Z}/6$

Note that the stability theorem indeed holds.

• Fred Cohen (1976), using another approach, determined both $H_*(C_k(\mathbb{C});\mathbb{Z}/p)$ (where p is a prime) as modules over the Steenrod algebra, and $H_*(C_k(\mathbb{C});\mathbb{Z})$.

• Using this, Brown and Peterson determined the stable homotopy type of $C_k(\mathbb{C})$. Let

$$\Omega^2 S^3 \underset{s}{\simeq} \bigvee_{1\leq q} D_q(S^1)$$

be Snaith's stable splitting. Then

<u>Theorem</u> (Brown-Peterson, 1978).

$$C_k(\mathbb{C}) \underset{s}{\simeq} \bigvee_{q=1}^{\left[\frac{k}{2}\right]} D_q(S^1),$$

where $\left[\frac{k}{2}\right]$ is the largest integer $\leq \frac{k}{2}$.

By the repetition theorem, the number $\left[\frac{k}{2}\right]$ is reasonable.

Consequently, the bottom row of Table 1 (i.e., when $k = \infty$) turns out to be $H_*(\Omega^2 S^3; \mathbb{Z})$.

$H_*(C_k(\mathbb{C}); \mathbb{Z})$ is known completely now. But we review Arnold's proof.

Arnold's proof

We set

$$P^l_{k,n} = \{f(z) = z^k + a_1 z^{k-1} + \cdots + a_k : \\ a_i \in \mathbb{C}, \text{the number of } n\text{-fold roots} \\ \text{of } f(z) \text{ is at most } l\}.$$

Here two n-fold roots may coincide. Hence,

$$f(z) \in P^l_{k,n} \Leftrightarrow (\alpha(z))^n \not| f(z) \text{ for any } \\ \alpha(z) \text{ of degree } l+1.$$

It is natural to assume $n \geq 2$. In particular,

$$P^0_{k,2} = C_k(\mathbb{C}).$$

Example 1.

(1) For $l \geq d$,

$$P^l_{nd,n} = \mathbb{C}^{nd} \simeq \{\text{a point}\}.$$

(2)

$$P^{d-1}_{nd,n} \cong \mathbb{C}^{nd} - \mathbb{C}^d \simeq S^{2(n-1)d-1}.$$

Proof. (1) is clear. For (2), we must exclude polynomials $f(z)$ of the form $f(z) = (\alpha(z))^n$ for some $\alpha(z)$. ■

Induction. Fix n. By induction with making k larger and l smaller, we obtain information on $P^l_{k,n}$ for all k, n and l. (For example, induction proceeds from Example 1 (1) to Example 1 (2).)

In particular, the case $n = 2$ and $l = 0$ is the above Arnold's results.

Remark. To be exact, Arnold considered the complement $S^{2k} - P^l_{k,n}$ instead of $P^l_{k,n}$.

A table in low dimensions

Arnold calculated

$$H_*(P^{k-1}_{2k+i,2};\mathbb{Z})\ (i \geq 0)$$

in low dimensions. The results are:

1. For $1 \leq q \leq 2k-2$,

$$H_q(P^{k-1}_{2k+i,2};\mathbb{Z}) = 0.$$

2. For $2k-1 \leq q \leq 2k+3$,

$$H_q(P^{k-1}_{2k+i,2};\mathbb{Z})$$

are cyclic and the orders are given by the following table.

Table 2. The orders of the groups

$$H_q(P^{k-1}_{2k+i,2};\mathbb{Z})$$

for $2k-1 \le q \le 2k+3$.

$i \setminus q$	$2k-1$	$2k$	$2k+1$
$0,1$	∞	0	0
$2,3$	∞	$k+1$	0
$4,5$	∞	$k+1$	$2/k$
$6,7$	∞	$k+1$	$2/k$
$8,9$	∞	$k+1$	$2/k$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	∞	$k+1$	$2/k$

$i \setminus q$	$2k+2$	$2k+3$
$0,1$	0	0
$2,3$	0	0
$4,5$	$(k+2)/2$	0
$6,7$	$((k+2)/2)(2/k)$	$3/k$
$8,9$	$((k+2)/2)(2/k)$	$6/kv$
$\vdots$	$\vdots$	$\vdots$
∞	$((k+2)/2)(2/k)$	$6/kv$

Here

1. We introduce the notation

$$a/b = \frac{a}{\text{gcd}(a,b)},$$

where $\text{gcd}(a,b)$ is the greatest common divisor of the integers a and b.

2. Stability Theorem: Fix k and q. In each column, we go downward. Then the homology stabilizes when

$$i \geq 2(q - 2k + 1).$$

3. We have

$$v = \begin{cases} 1 & \text{if} \quad k \not\equiv 1 (\text{mod}\, 4) \\ 1 \quad \text{or} \quad 2 & \text{if} \quad k \equiv 1 (\text{mod}\, 4). \end{cases}$$

But the exact value is left unknown.

Question. Is it possible to reconstruct Table 2 using standard techniques in algebraic topology?

Here "standard techniques in algebraic topology" means:

1. We allow to use the structure of $H_*(\Omega^2 S^3; \mathbb{Z}/p)$.

2. We allow to use spectral sequences for fibrations, e.g., Serre or Eilenberg-Moore.

3. We want to avoid inductive arguments.

§2. The space of rational functions

Definition of $\mathrm{Rat}_k(\mathbb{C}P^{n-1})$

We set

$\mathrm{Rat}_k(\mathbb{C}P^{n-1}) = \{(p_1(z), \ldots, p_n(z))$:
each $p_i(z)$ is a monic polynomial over $\mathbb{C}$
of degree k and such that there are
no roots common to all $p_i(z)\}$.

$\mathrm{Rat}_k(\mathbb{C}P^{n-1})$ is considered to be the space of holomorphic maps

$$S^2 \to \mathbb{C}P^{n-1}$$

of degree k with the basepoint condition

$$\infty \mapsto [1, \ldots, 1].$$

There is an inclusion

$$i_{k,n} : \mathrm{Rat}_k(\mathbb{C}P^{n-1}) \hookrightarrow \Omega^2_k \mathbb{C}P^{n-1} \simeq \Omega^2 S^{2n-1}.$$

Example.

$$\begin{aligned}\mathrm{Rat}_1(\mathbb{C}P^{n-1}) = & \quad \{(z+\alpha_1, \ldots, z+\alpha_n) : \\ & \quad \alpha_i \neq \alpha_j \quad \text{for some} \quad i, j\} \\ = & \quad \mathbb{C}^n - \text{diagonal set} \\ \simeq & \quad S^{2n-3}.\end{aligned}$$

Hence, the generator of $\pi_{2n-3}(\Omega^2 S^{2n-1})$ is constructed in $\mathrm{Rat}_1(\mathbb{C}P^{n-1})$. Moreover, the following theorem holds.

Theorem (Segal, 1979). $i_{k,n}$ is a homotopy equivalence up to dimension $k(2n-3)$.

That is, the homomorphism

$$i_{k,n,*} : \pi_q(\mathrm{Rat}_k(\mathbb{C}P^{n-1})) \to \pi_q(\Omega^2 S^{2n-1})$$

is

$$\begin{cases} \text{an isomorphism when } q < k(2n-3) \\ \text{an epimorphism when } q = k(2n-3). \end{cases}$$

This theorem implies that $\mathrm{Rat}_k(\mathbb{C}P^{n-1})$ is a good finite dimensional model which approximates an infinite dimensional manifold $\Omega^2 S^{2n-1}$.

Later, the stable homotopy type of $\text{Rat}_k(\mathbb{C}P^{n-1})$ was determined. Let

$$\Omega^2 S^{2n-1} \underset{s}{\simeq} \bigvee_{1 \leq q} D_q(S^{2n-3})$$

be Snaith's stable splitting. Then

Theorem (F.Cohen-R.Cohen- Mann-Milgram, 1991).

$$\text{Rat}_k(\mathbb{C}P^{n-1}) \underset{s}{\simeq} \bigvee_{q=1}^{k} D_q(S^{2n-3}).$$

In particular, the homomorphism

$$i_{k,n,*}: \quad H_*(\text{Rat}_k(\mathbb{C}P^{n-1}); \mathbb{Z}) \to H_*(\Omega^2 S^{2n-1}; \mathbb{Z})$$

is injective.

Relationship between P and Rat

We have the following 2 spaces:

$P^l_{k,n} = \{f(z) = z^k + a_1 z^{k-1} + \cdots + a_k :$
$a_i \in \mathbb{C}$, the number of n-fold roots
of $f(z)$ is at most $l\}$

and

$\mathrm{Rat}_k(\mathbb{C}P^{n-1}) = \{(p_1(z), \ldots, p_n(z)) :$
each $p_i(z)$ is a monic polynomial over $\mathbb{C}$
of degree k and such that there are
no roots common to all $p_i(z)\}$.

Concerning them, we have the following 2 theorems:

- Brown-Peterson: $P^0_{k,2} \underset{s}{\simeq} \bigvee_{q=1}^{\left[\frac{k}{2}\right]} D_q(S^1)$.

- Cohen et al. for $n = 2$:

$$\mathrm{Rat}_k(\mathbb{C}P^1) \underset{s}{\simeq} \bigvee_{q=1}^{k} D_q(S^1).$$

Combining these theorems, we obtain:

(1) $$P^0_{k,2} \underset{s}{\simeq} \mathrm{Rat}_{\left[\frac{k}{2}\right]}(\mathbb{C}P^1).$$

Remark: We <u>cannot</u> improve (1) to an unstable homotopy equivalence, since π_1 of the both sides are not isomorphic.

Later, (1) was generalized to the following:

Theorem (Vassiliev, 1992).

$$P^0_{k,n} \underset{s}{\simeq} \mathrm{Rat}_{\left[\frac{k}{n}\right]}(\mathbb{C}P^{n-1}). \tag{2}$$

Remark. For $n \geq 3$, we can improve Vassiliev's theorem to an unstable homotopy equivalence if we combine the theorems of R. Cohen-Shimamoto, and Guest-Kozlowski-Yamaguchi.

§3. Main results

Purpose

We have the following 3 theorems:

- Segal: The inclusion

$i_{k,n}$: $\mathrm{Rat}_k(\mathbb{C}P^{n-1}) \hookrightarrow \Omega^2_k \mathbb{C}P^{n-1} \simeq \Omega^2 S^{2n-1}$

is a homotopy equivalence up to dimension $k(2n-3)$.

- Cohen et al.:

$$\mathrm{Rat}_k(\mathbb{C}P^{n-1}) \underset{s}{\simeq} \bigvee_{q=1}^{k} D_q(S^{2n-3}).$$

- Vassiliev:

$$P^0_{k,n} \underset{s}{\simeq} \mathrm{Rat}_{\left[\frac{k}{n}\right]}(\mathbb{C}P^{n-1}).$$

We want to generalize these theorems.

About Vassiliev's theorem, we generalize as follows:

<u>The left-hand side</u>: Generalize to $P^l_{k,n}$.

$P^l_{k,n} = \{f(z) = z^k + a_1 z^{k-1} + \cdots + a_k :$
$a_i \in \mathbb{C}$, the number of n-fold roots of $f(z)$ is at most $l\}$.

The right-hand side: What is the space which generalizes $\mathrm{Rat}_k(\mathbb{C}P^{n-1})$?

Definition of $X_{k,n}^l$

We set

$X_{k,n}^l = \{(p_1(z), \ldots, p_n(z))$:
each $p_i(z)$ is a monic polynomial over $\mathbb{C}$ of degree k and such that there are at most l roots common to all $p_i(z)\}$.

Here two common roots may coincide. Clearly

$$X_{k,n}^0 = \mathrm{Rat}_k(\mathbb{C}P^{n-1}).$$

Remark. The space $X_{k,n}^l$ was suggested by Fred Cohen.

Example 2.

(1) For $l \geq d$,

$$X^l_{d,n} = (\mathbb{C}^d)^n \simeq \{\text{a point}\}.$$

(2) $X^{d-1}_{d,n} \cong (\mathbb{C}^d)^n$ – diagonal set
$\simeq S^{2(n-1)d-1}$.

Proof. (1) is clear. For (2), we must exclude n-tuples

$$(p_1(z), \ldots, p_n(z))$$

of polynomials which satisfy

$$p_1(z) = \cdots = p_n(z).$$

■

We want to give Segal type, Cohen et al. type and Vassiliev type theorems for $X^l_{k,n}$. For that purpose, we need some notations.

Homotopy fibre

We set

(1) $J^l(2n-2)$: the l-th stage of the James construction which builds ΩS^{2n-1}. That is,

$$J^l(2n-2) \simeq S^{2n-2} \cup e^{2(2n-2)} \cup \cdots \cup e^{l(2n-2)}.$$

(2) $W^l(n)$: the homotopy theoretic fibre of the inclusion

$$J^l(2n-2) \hookrightarrow \Omega S^{2n-1}.$$

In particular,

$$W^0(n) = \Omega^2 S^{2n-1}.$$

(3) Wong generalized Snaith's stable splitting as follows.

$$W^l(n) \underset{s}{\simeq} \bigvee_{1 \leq q} D_q \xi^l(n).$$

Theorem 1 [Segal type] (K, 2003).

There is an unstable map

$$\alpha^l_{k,n} : X^l_{k,n} \to W^l(n)$$

which is a homotopy equivalence up to dimension

$$\left[\frac{k}{l+1}\right](2(l+1)(n-1)-1).$$

We will not use this theorem later.

Theorem 2 [Cohen et al. type] (K, 2001).

$$X^l_{k,n} \underset{s}{\simeq} \bigvee_{q=1}^{k} D_q\xi^l(n).$$

From Theorem 2, we can calculate $H_*(X^l_{k,n}; \mathbb{Z}/p)$, where p is a prime. This is the subspace of $H_*(W^l(n); \mathbb{Z}/p)$ spanned by monomials of weight $\leq k$.

Theorem 3 [Vassiliev type] (K, 2003). Except when $(n, l) = (2, 0)$, there is a homotopy equivalence

$$P^l_{k,n} \simeq X^l_{\left[\frac{k}{n}\right],n}.$$

As mentioned above, this holds stably when $(n, l) = (2, 0)$.

Note that Theorem 3 indeed holds between Examples 1 and 2:

Example 1.

(1) For $l \geq d$,

$$P^{l}_{nd,n} = \mathbb{C}^{nd} \simeq \{\text{a point}\}.$$

(2)

$$P^{d-1}_{nd,n} \cong \mathbb{C}^{nd} - \mathbb{C}^{d} \simeq S^{2(n-1)d-1}.$$

Example 2.

(1) For $l \geq d$,

$$X^{l}_{d,n} = (\mathbb{C}^{d})^{n} \simeq \{\text{a point}\}.$$

(2) $X^{d-1}_{d,n} \cong (\mathbb{C}^{d})^{n} -$ diagonal set
$\simeq S^{2(n-1)d-1}$.

Table 2. The orders of the groups

$$H_q(P^{k-1}_{2k+i,2};\mathbb{Z})$$

for $2k-1 \leq q \leq 2k+3$.

$i \setminus q$	$2k-1$	$2k$	$2k+1$
$0,1$	∞	0	0
$2,3$	∞	$k+1$	0
$4,5$	∞	$k+1$	$2/k$
$6,7$	∞	$k+1$	$2/k$
$8,9$	∞	$k+1$	$2/k$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	∞	$k+1$	$2/k$

$i \setminus q$	$2k+2$	$2k+3$
$0,1$	0	0
$2,3$	0	0
$4,5$	$(k+2)/2$	0
$6,7$	$((k+2)/2)(2/k)$	$3/k$
$8,9$	$((k+2)/2)(2/k)$	$6/kv$
$\vdots$	$\vdots$	$\vdots$
∞	$((k+2)/2)(2/k)$	$6/kv$

Reconstruction of Table 2

(1) By Theorem 3,

$$P^{k-1}_{2k+i,2} \simeq X^{k-1}_{k+\left[\frac{i}{2}\right],2}.$$

Hence, we calculate the right-hand side.

(2) By Theorem 2, as a vector space,

$$H_*(X^{k-1}_{k+\left[\frac{i}{2}\right],2}; \mathbb{Z}/p)$$

is isomorphic to the subspace of

$$H_*(W^{k-1}(2); \mathbb{Z}/p)$$

spanned by monomials of
weight $\leq k + \left[\frac{i}{2}\right]$.

(3) We can determine

$$H_*(W^{k-1}(2); \mathbb{Z}/p)$$

from the mod p Serre spectral sequence for the fibration

$$\Omega^2 S^3 \to W^{k-1}(2) \to J^{k-1}(2).$$

(4) If we follow the steps (1)-(3), then we can prove that the value of the indeterminacy v in Table 2 is 1 when $k \equiv 1 (\mathrm{mod}\, 4)$.

Example

We calculate the case $i = \infty$ and $q = 2k$. By Theorems 2 and 3,

$$H_{2k}(P^{k-1}_{\infty,2}; \mathbb{Z}) \cong H_{2k}(W^{k-1}(2); \mathbb{Z}).$$

Hence, it suffices to prove

$$\boxed{H^{2k+1}(W^{k-1}(2); \mathbb{Z}) \cong \mathbb{Z}/(k+1)}$$

We can consider the Serre spectral sequence for the fibration

$$W^{k-1}(2) \to J^{k-1}(2) \to \Omega S^3.$$

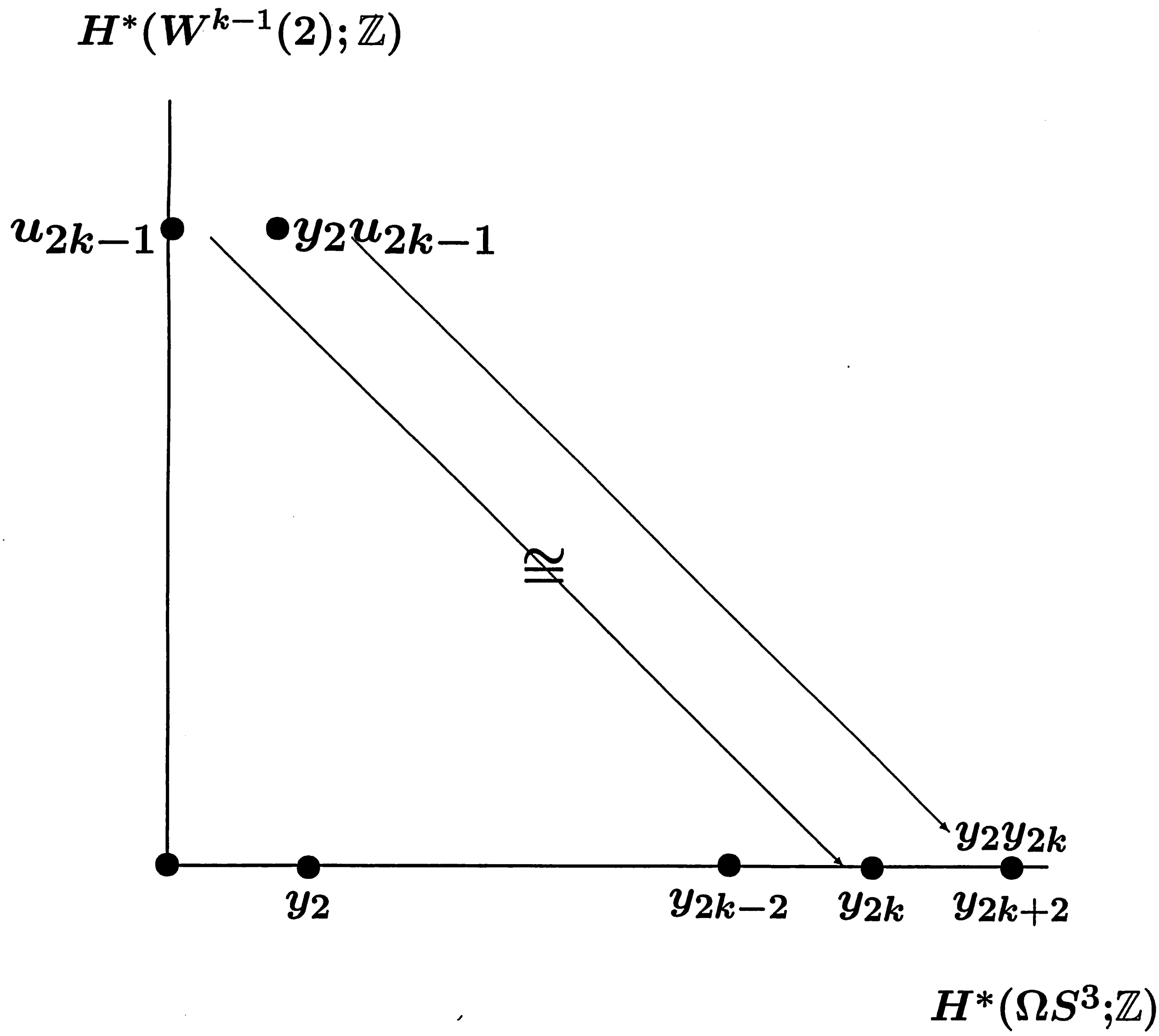
Recall that

$$H^*(\Omega S^3; \mathbb{Z}) \cong \Gamma(x_2),$$

the divided power algebra. That is,

$$y_{2j} := \frac{x_2^j}{j!}$$

is the generator of $H^{2j}(\Omega S^3; \mathbb{Z})$.



First, let

$$u_{2k-1} \in H^{2k-1}(W^{k-1}(2);\mathbb{Z}) \cong \mathbb{Z}$$

be the generator which kills y_{2k}.

Next, since

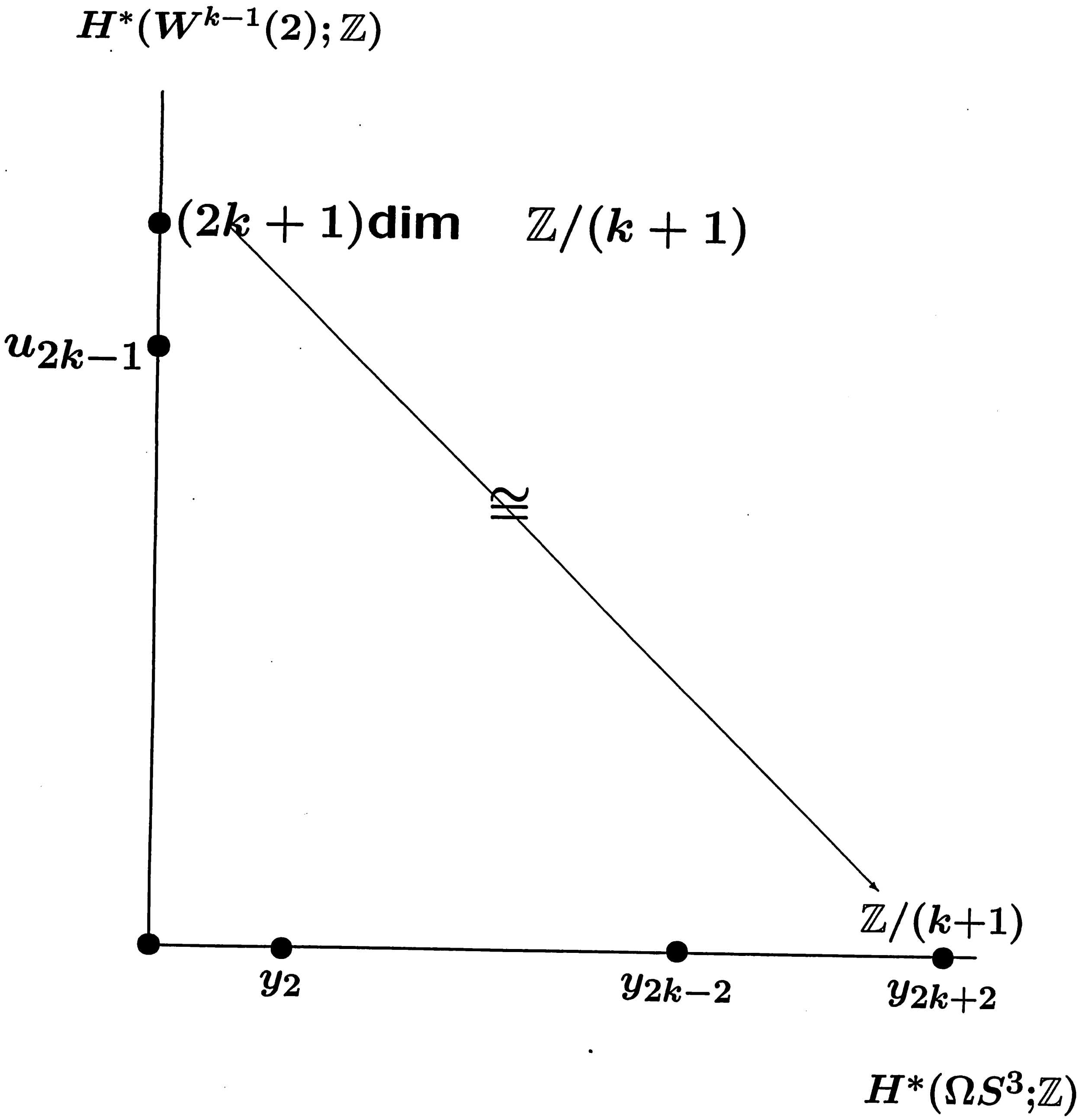
$$y_2 y_{2k} = x_2 \frac{x_2^k}{k!} = (k+1) y_{2k+2},$$

the spectral sequence becomes as follows.

Then we must have

$$H^{2k+1}(W^{k-1}(2); \mathbb{Z}) \cong \mathbb{Z}/(k+1).$$

This is what we wanted to prove. ■


$H^*(W^{k-1}(2);\mathbb{Z})$
$(2k+1)$dim
$\mathbb{Z}/(k+1)$
u_{2k-1}
$\cong$
$\mathbb{Z}/(k+1)$
y_2
y_{2k-2}
y_{2k+2}
$H^*(\Omega S^3;\mathbb{Z})$

A concluding remark

Today we considered one of 4 cases. That is, for $(p_1(z), \dots, p_n(z))$, there are cases

(1) $p_i(z)$ is a polynomial over $\mathbb{R}$ or $\mathbb{C}$.

(2) whether a point $\in \mathbb{C}$ off the real axis can be a common root.

For example, when $p_i(z)$ is a polynomial over $\mathbb{R}$ and $p_i(z)$ $(1 \le i \le n)$ may have common roots, but none of the common roots lie on the real axis. Then $(p_1(z), \dots, p_n(z))$ is considered to be an element of

$$\Omega_{k \bmod 2} \mathbb{R}P^{n-1} \simeq \Omega S^{n-1},$$

where $S^1 = \mathbb{R} \cup \{\infty\}$. Note that is a single loop space.

Today's theorems hold for these 4 cases under suitable modifications.