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CONVERGENCE RATES OF EQUI-UNIFORM APPROXIMATION PROCESSES OF INTEGRAL OPERATORS IN BANACH SPACES

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ABSTRACT. We establish quantitative pointwise estimates of the rate of convergence of equi-uniform approximation processes of integral operators in Banach spaces in terms of the modulus of continuity of functions to be approximated and higher order absolute moments of approximate kernels with respect to certain test systems of approximating functions. Furthermore, applications are presented for various equi-uniform summation processes, interpolation type operators, convolution type operators, and several concrete examples of approximating operators are also provided.

1. Introduction

Let $\mathbb{N}$ be the set of all natural numbers, and put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A bounded sequence $\{a_n\}_{n \in \mathbb{N}_0}$ of real numbers is said to be almost convergent to a if

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=m}^{n+m} a_k = a \qquad \text{uniformly in } m \in \mathbb{N}_0$$

(cf. [10]). If $\{a_n\}$ converges to a, then it is almost convergent to a, but not conversely.

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Let $\{B_n\}$ be the sequence of Bernstein operators defined by

$$B_n(f)(x) = \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big) \binom{n}{k} x^k (1-x)^{n-k}$$

$$(f \in C[0,1],\ x \in [0,1]).$$

Then for all $f \in C[0,1]$, $\{B_n(f)(x)\}$ is almost convergent to $f(x)$ uniformly on $[0,1]$ (cf. [8]).

Let $\{\sigma_n\}$ be the sequence of Fejér operators defined by

$$\sigma_n(f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(x-t) f(t)\, dt$$

$$(f \in C_{2\pi},\ x \in \mathbb{R}),$$

where

$$F_n(u) = \sum_{j=-n}^{n} \Big(1 - \frac{|j|}{n+1}\Big) e^{iju} \qquad (u \in \mathbb{R})$$

is the nth Fejér kernel. Then for every $f \in C_{2\pi}$, $\{\sigma_n(f)(x)\}$ is almost convergent to $f(x)$ uniformly on the real line $\mathbb{R}$ (cf. [8]).

In view of these results, we generally make the following situation:

Let $(E, \|\cdot\|)$ be a Banach space and let (X, d) be a metric space. Let $B(X,E)$ denote the Banach space of all E-valued bounded functions on X with the supremum norm. $BC(X,E)$ stands for the closed linear subspace of $B(X,E)$ consisting of all E-valued bounded continuous functions on X. Also, we denote by $C(X,E)$ the linear space consisting of all E-valued continuous functions on X. Let X_0 be a subset of X. Let $\mathfrak{K} = \{K_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of operators of $BC(X,E)$ into $B(X_0,E)$, where D is a directed set and Λ is an index set. Then $\mathfrak{K}$ is called an equi-uniform approximation process on $BC(X,E)$ if for all $F \in BC(X,E)$,

$$\lim_{\alpha} \|K_{\alpha,\lambda}(F)(x) - F(x)\| = 0 \qquad \text{uniformly in } \ \lambda \in \Lambda,\ x \in X_0.$$

We here consider a family $\mathfrak{K}$ of integral operators on $BC(X,E)$ defined as follows:

Let $\{Y_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of separable topological spaces with a Borel measure $\mu_{\alpha,\lambda}$ on $Y_{\alpha,\lambda}$. For each $\alpha \in D, \lambda \in \Lambda$ and each $x \in X$, let $\xi_{\alpha,\lambda}$ be a continuous mapping of $Y_{\alpha,\lambda}$ into X and let $\chi_{\alpha,\lambda}(x;\cdot)$ be a function in $L^1(Y_{\alpha,\lambda}, \mu_{\alpha,\lambda})$, which denotes the Banach space of all $\mu_{\alpha,\lambda}$-integrable functions χ on $Y_{\alpha,\lambda}$ with the norm

$$\|\chi\|_1 = \int_{Y_{\alpha,\lambda}} |\chi(y)| \, d\mu_{\alpha,\lambda}(y).$$

Then we define an integral operator by the form

$$(1) \qquad K_{\alpha,\lambda}(F)(x) = \int_{Y_{\alpha,\lambda}} \chi_{\alpha,\lambda}(x;y) F(\xi_{\alpha,\lambda}(y)) \, d\mu_{\alpha,\lambda}(y)$$

$$(F \in BC(X,E)),$$

which exists as a Bochner integral.

In case $Y_{\alpha,\lambda} = Y$ for all $\alpha \in D$ and all $\lambda \in \Lambda$, where Y is a separable topological space, in [17] we studied the convergence of equi-uniform approximation processes of integral operators defined by (1) and in [21] (cf. [18], [19], [20]) we established quantitative estimates of the rate of their convergence in terms of the modulus of continuity of function F.

The purpose of this paper is to refine the rate of convergence of equi-uniform approximation processes given in [21] by means of pointwise estimates. Furthermore, applications are presented for various summation processes, interpolation type operators, convolution type operators, and several concrete examples of approximating operators are also provided.

2. Preliminary results

(X,d) is said to be quasi-convex if $x, y \in X, d(x,y) \le a+b, a, b \ge 0, (a,b) \ne (0,0)$, then there exists a point $z \in X$ such that $d(x,z) \le a$ and $d(z,y) \le b$. Let (T,τ) be a metric linear space. If $\tau(x,y) = \tau(x+z, y+z)$ for all $x, y, z \in T$, then τ is called a translation invariant metric function. A real-valued function φ on a linear space V is said to be starshaped if $\varphi(\beta x) \le \beta\varphi(x)$ for all $x \in V$ and all $\beta \in [0,1]$.

Let $F \in B(X,E)$ and let $\delta \ge 0$. Then we define

$$\omega(F,\delta) = \sup\{\|F(x) - F(t)\| : x, t \in X, d(x,t) \le \delta\},$$

which is called the modulus of continuity of F. Obviously, $\omega(F,\cdot)$ is a monotone increasing function on $[0,\infty)$ and

$$\omega(F,0) = 0, \quad \omega(F,\delta) \le 2\sup\{\|F(x)\| : x \in X\} \qquad (\delta \ge 0).$$

Note that if X is bounded, then

$$\omega(F,\delta) = \omega(F,\delta(X)) \qquad (\delta \geq \delta(X)),$$

where $\delta(X)$ denotes the diameter of X, and F is uniformly continuous on X if and only if

$$\lim_{\delta \to +0} \omega(F,\delta) = 0.$$

Now in order to achieve our purpose, we always suppose that there exist constants $C \geq 1$ and $K > 0$ such that

$$\omega(F,\xi\delta) \leq (C + K\xi)\omega(F,\delta) \tag{2}$$

for every $\xi, \delta \geq 0$ and for every $F \in B(X,E)$.

The following lemma gives sufficient conditions such that (2) holds with $C = K = 1$, which can be more convenient for later applications and generalizes [15, Lemma 3]:

Lemma 1. *([21, Lemma 2.4]) (a) If (X,d) is quasi-convex, then (2) holds with $C = K = 1$.*

(b) If X is a convex subset of a metric linear space with the translation invariant metric function d and if $d(\cdot,0)$ is starshaped, then (2) holds with $C = K = 1$. In particular, if X is a convex subset of a normed linear space, then (2) holds with $C = K = 1$.

Let $V(X,E)$ denote the linear space of all E-valued functions on X. For any scalar-valued function v on X and $a \in E$, we define $(v \otimes a)(x) = v(x)a$ for all $x \in X$. 1_X stands for the unit function defined by $1_X(x) = 1$ for all $x \in X$. Let $A(X,E)$ be a linear subspace of $V(X,E)$ and let φ be a mapping of $A(X,E)$ into E. A positive linear functional ν on $A(X,\mathbb{R})$ is called a majorant (or dominant) functional of φ if $F \in A(X,E), v \in A(X,\mathbb{R})$ and

$$||F(t)|| \leq v(t) \qquad \text{for all } t \in X, \tag{3}$$

then $||\varphi(F)|| \leq \nu(v)$. Let L be a mapping of $A(X,E)$ into $V(X_0,E)$. A positive linear operator S of $A(X,\mathbb{R})$ into $V(X_0,\mathbb{R})$ is called a majorant (or dominant) operator of L if (3) implies that

$$||L(F)(x)|| \leq S(v)(x) \qquad \text{for all } x \in X_0.$$

Lemma 2. *([21, Lemma 2.5]) Let φ be a mapping of $A(X,E)$ into E having a majorant functional ν. Let $p \geq 1$ and $x \in X$. Suppose that*

$$\{1_X \otimes a : a \in E\} \subseteq A(X,E), \quad \{1_X, d(x,\cdot), d^p(x,\cdot)\} \subseteq A(X,\mathbb{R}).$$

Then for all $F \in A(X,E) \cap B(X,E)$ *and all* $\delta > 0$,

$$\|\varphi(F - 1_X \otimes F(x))\| \leq (C\nu(1_X) + Km_\nu(x;p,\delta))\omega(F,\delta),$$

where

$$m_\nu(x;p,\delta) = \min\{\delta^{-p}\nu(d^p(x,\cdot)),\ \delta^{-1}\nu(1_X)^{1-1/p}\nu(d^p(x,\cdot))^{1/p}\}.$$

Let $x \in X_0$ be fixed. Then applying Lemma 2 to $\varphi(\cdot) = L(\cdot)(x)$ and $\nu(\cdot) = S(\cdot)(x)$, we have the following result, which generalizes [5, Theorem 2.1] and improves the estimate by means of higher order absolute moments.

Lemma 3. *([21, Lemma 2.6]) Let* L *be a mapping of* $A(X,E)$ *into* $V(X_0,E)$ *having a majorant operator* S *and let* $p \geq 1$. *Suppose that*

$$\{1_X \otimes a : a \in E\} \subseteq A(X,E)$$

and

$$\{1_X\} \cup \{d(x,\cdot), d^p(x,\cdot) : x \in X_0\} \subseteq A(X,\mathbb{R}).$$

Then for all $F \in A(X,E) \cap B(X,E), x \in X_0$ *and all* $\delta > 0$,

$$\|L(F - 1_X \otimes F(x))(x)\| \leq (CS(1_X)(x) + Km_S(x;p,\delta))\omega(F,\delta),$$

where

$$m_S(x;p,\delta)$$

$$= \min\{\delta^{-p}S(d^p(x,\cdot))(x), \delta^{-1}(S(1_X)(x))^{1-1/p}(S(d^p(x,\cdot))(x))^{1/p}\}.$$

Now we make use of the following key estimate for integral operators on $BC(X,E)$.

Lemma 4. *([21, Lemma 2.7]) Let* $\{\chi(x;\cdot) : x \in X\}$ *be a family of functions in* $L^1(Y,\mu)$, τ *a continuous mapping of* Y *into* X *and* $p \geq 1$. *Assume that* $\chi(x;\cdot)d^p(x,\tau(\cdot)) \in L^1(Y,\mu)$ *for each* $x \in X_0$. *Then for all* $F \in BC(X,E), x \in X_0$ *and all* $\delta > 0$,

$$\left\|\int_Y \chi(x;y)(F(\tau(y)) - F(x))\, d\mu(y)\right\| \leq (C\|\chi(x;\cdot)\|_1 + Kc(x;p,\delta))\omega(F,\delta),$$

where

$$c(x;p,\delta)$$

$$= \min\{\delta^{-p}\|\chi(x;\cdot)d^p(x,\tau(\cdot))\|_1,\ \delta^{-1}\|\chi(x;\cdot)\|_1^{1-1/p}\|\chi(x;\cdot)d^p(x,\tau(\cdot))\|_1^{1/p}\}.$$

3. Pointwise estimates of equi-uniform convergence

Let $p \geq 1$. Let $\{s_1, s_2, \dots, s_r\}$ be a finite set of positive real numbers and let $\{\Phi_1, \Phi_2, \dots, \Phi_r\}$ be a finite set of nonnegative real-valued functions on $X_0 \times X$. Assume that

$$\chi_{\alpha,\lambda}(x;\cdot)\Phi_i^{s_i}(x, \xi_{\alpha,\lambda}(\cdot)) \in L^1(Y_{\alpha,\lambda}, \mu_{\alpha,\lambda})$$
$$(\alpha \in D, \ \lambda \in \Lambda, \ x \in X_0, \ i = 1, 2, \dots, r)$$

and we define

$$\mu_{\alpha,\lambda,i}(x; s_i) = \|\chi_{\alpha,\lambda}(x;\cdot)\Phi_i^{s_i}(x, \xi_{\alpha,\lambda}(\cdot))\|_1,$$

which is called the s_ith absolute moment of $\chi_{\alpha,\lambda}(x;\cdot)$ at x with respect to Φ_i.

Suppose that there exists a constant $L > 0$ such that

$$d^p(x,t) \leq L \sum_{i=1}^{r} \Phi_i^{s_i}(x,t) \tag{4}$$

for all $(x,t) \in X_0 \times X$. Let $\mathfrak{A} := \{\chi_{\alpha,\lambda}(x;\cdot) : \alpha \in D, \lambda \in \Lambda, x \in X\}$. For any $\alpha \in D, \lambda \in \Lambda$ and $x \in X_0$ we define

$$\beta_{\alpha,\lambda}(x) = \left| \int_{Y_{\alpha,\lambda}} \chi_{\alpha,\lambda}(x;y)\, d\mu_{\alpha,\lambda}(y) - 1 \right|.$$

If for all $\alpha \in D, \lambda \in \Lambda$ and all $x \in X_0$,

$$\chi_{\alpha,\lambda}(x;y) \geq 0 \quad (\mu\text{-a.e. } y \in Y_{\alpha,\lambda}),$$

then $\mathfrak{A}$ is said to be positive. Also, if for all $\alpha \in D, \lambda \in \Lambda$ and all $x \in X_0$,

$$\int_{Y_{\alpha,\lambda}} \chi_{\alpha,\lambda}(x;y)\, d\mu_{\alpha,\lambda}(y) = 1,$$

then $\mathfrak{A}$ is said to be normal.

For any $\alpha \in D, F \in BC(X,E)$ we define

$$E_\alpha(F) = \sup\{\|K_{\alpha,\lambda}(F)(x) - F(x)\| : \lambda \in \Lambda, x \in X_0\}.$$

Note that $\mathfrak{K}$ is an equi-uniform approximation process on $BC(X,E)$ if and only if

$$\lim_\alpha E_\alpha(F) = 0$$

for every $F \in BC(X,E)$.

From now on, let $\{\epsilon_\alpha\}_{\alpha \in D}$ be a net of positive real numbers.

Theorem 1. *For all $\alpha \in D, \lambda \in \Lambda, F \in BC(X,E)$ and all $x \in X_0$,*

(5) $\quad \|K_{\alpha,\lambda}(F)(x) - F(x)\| \le \|F(x)\|\beta_{\alpha,\lambda}(x) + \gamma_{\alpha,\lambda}(x)\omega(F,\epsilon_\alpha),$

where

$$\gamma_{\alpha,\lambda}(x) = C\|\chi_{\alpha,\lambda}(x;\cdot)\|_1 + Kc_{\alpha,\lambda}(x)$$

and

$$c_{\alpha,\lambda}(x) = \min\Big\{L\epsilon_\alpha^{-p}\sum_{i=1}^{r}\mu_{\alpha,\lambda,i}(x;s_i),$$
$$L^{1/p}\epsilon_\alpha^{-1}\Big(\sum_{i=1}^{r}\mu_{\alpha,\lambda,i}(x;s_i)\Big)^{1/p}\|\chi_{\alpha,\lambda}(x;\cdot)\|_1^{1-1/p}\Big\}.$$

In particular, if $\mathfrak{A}$ is positive and normal, then (5) reduces to

$$\|K_{\alpha,\lambda}(F)(x) - F(x)\| \le (C + Kc_{\alpha,\lambda}(x))\omega(F,\epsilon_\alpha),$$

and

$$c_{\alpha,\lambda}(x) = \min\Big\{L\epsilon_\alpha^{-p}\sum_{i=1}^{r}\mu_{\alpha,\lambda,i}(x;s_i),\ L^{1/p}\epsilon_\alpha^{-1}\Big(\sum_{i=1}^{r}\mu_{\alpha,\lambda,i}(x;s_i)\Big)^{1/p}\Big\}.$$

Proof. We have

(6) $\quad \|K_{\alpha,\lambda}(F)(x) - F(x)\| \le \Big|\int_{Y_{\alpha,\lambda}}\chi_{\alpha,\lambda}(x;y)\,d\mu_{\alpha,\lambda}(y) - 1\Big|\|F(x)\|$

$$+ \Big\|\int_{Y_{\alpha,\lambda}}\chi_{\alpha,\lambda}(x;y)(F(\xi_{\alpha,\lambda}(y)) - F(x))\,d\mu_{\alpha,\lambda}(y)\Big\| = J_{\alpha,\lambda}^{(1)}(x) + J_{\alpha,\lambda}^{(2)}(x),$$

say. We have $J_{\alpha,\lambda}^{(1)}(x) = \|F(x)\|\beta_{\alpha,\lambda}(x)$. Now, put

$$\mu_{\alpha,\lambda}(x) := \|\chi_{\alpha,\lambda}(x;\cdot)d^p(x,\xi_{\alpha,\lambda}(\cdot))\|_1 \le L\sum_{i=1}^{r}\mu_{\alpha,\lambda,i}(x;s_i)$$

on account of (4). Applying Lemma 4 to $\chi(x;\cdot) = \chi_{\alpha,\lambda}(x;\cdot)$ and $\tau = \xi_{\alpha,\lambda}$, we obtain

$$J_{\alpha,\lambda}^{(2)}(x) \le (C\|\chi_{\alpha,\lambda}(x;\cdot)\|_1 + Kc_{\alpha,\lambda}(x;p,\delta))\omega(F,\delta) \qquad (\delta > 0),$$

where

$$c_{\alpha,\lambda}(x;p,\delta) = \min\{\delta^{-p}\mu_{\alpha,\lambda}(x),\ \delta^{-1}\|\chi_{\alpha,\lambda}(x;\cdot)\|_1^{1-1/p}\mu_{\alpha,\lambda}(x)^{1/p}\}$$
$$\le \min\Big\{\delta^{-p}L\sum_{i=1}^{r}\mu_{\alpha,\lambda,i}(x;s_i),\ \delta^{-1}\|\chi_{\alpha,\lambda}(x;\cdot)\|_1^{1-1/p}\Big(L\sum_{i=1}^{r}\mu_{\alpha,\lambda,i}(x;s_i)\Big)^{1/p}\Big\}.$$

Putting $\delta = \epsilon_\alpha$ in the above inequality, (6) establishes the desired estimate (5).

In the rest of this section, we restrict the integral operators $K_{\alpha,\lambda}$ defined by (1) to the subclass of $BC(X,E)$ defined as follows:

Let E_0 be a subset of E and $\mathfrak{T} = \{T(x) : x \in X\}$ a family of mappings of E_0 into E such that for each $f \in E_0$, the orbit mapping $x \mapsto T(x)(f)$ is strongly continuous and bounded on X and let $L_{\alpha,\lambda}$ denote the restriction of the set $K_{\alpha,\lambda}$ to $\{T(\cdot)(f) : f \in E_0\}$, i.e.,

$$(7)\quad L_{\alpha,\lambda}(x)(f) = \int_{Y_{\alpha,\lambda}} \chi_{\alpha,\lambda}(x;y)T(\xi_{\alpha,\lambda}(y))(f)\, d\mu_{\alpha,\lambda}(y) \qquad (f \in E_0).$$

Shaw [25] considered the special case of (7) in the setting of certain spaces of operator-valued functions and obtained several representation formulas for strongly continuous semigroups of bounded linear operators on Banach spaces.

The family $\mathfrak{L} = \{L_{\alpha,\lambda}(x) : \alpha \in D, \lambda \in \Lambda, x \in X\}$ is called an equi-uniform $\mathfrak{T}$-approximation process on E_0 if for every $f \in E_0$,

$$(8)\quad \lim_{\alpha} \|L_{\alpha,\lambda}(x)(f) - T(x)(f)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda, x \in X_0.$$

Concerning the rate of convergence behavior of (8), we define

$$\omega_{\mathfrak{T}}(f,\delta) = \sup\{\|T(x)(f) - T(t)(f)\| : x,t \in X, d(x,t) \le \delta\}$$
$$(f \in E_0,\ \delta \ge 0),$$

which is called the modulus of continuity of f associated with $\mathfrak{T}$, and

$$e_\alpha(f) := \sup\{\|L_{\alpha,\lambda}(x)(f) - T(x)(f)\| : \lambda \in \Lambda, x \in X_0\}.$$

Note that $\mathfrak{L}$ is an equi-uniform $\mathfrak{T}$-approximation process on E_0 if and only if

$$\lim_{\alpha} e_\alpha(f) = 0$$

for every $f \in E_0$.

Now, since for all $f \in E_0, \delta \ge 0$ and all $\alpha \in D$

$$\omega_{\mathfrak{T}}(f,\delta) = \omega(T(\cdot)(f),\delta), \quad e_\alpha(f) = E_\alpha(T(\cdot)(f)),$$

Theorem 1 yields the following result which establishes the estimate for the rate of convergence of the equi-uniform $\mathfrak{T}$-approximation process $\mathfrak{L}$ on E_0:

Corollary 1. *For all $\alpha \in D, \lambda \in \Lambda, f \in E_0$ and all $x \in X_0$,*

$$\|L_{\alpha,\lambda}(x)(f) - T(x)(f)\| \le \|T(x)(f)\|\beta_{\alpha,\lambda}(x) + \gamma_{\alpha,\lambda}(x)\omega_{\mathfrak{T}}(f,\epsilon_\alpha).$$

In particular, if $\mathfrak{A}$ is positive and normal, then

$$\|L_{\alpha,\lambda}(x)(f) - T(x)(f)\| \le (C + Kc_{\alpha,\lambda}(x))\omega_{\mathfrak{T}}(f,\epsilon_\alpha),$$

and

$$c_{\alpha,\lambda}(x) = \min\Big\{L\epsilon_\alpha^{-p}\sum_{i=1}^{r}\mu_{\alpha,\lambda,i}(x;s_i),\ L^{1/p}\epsilon_\alpha^{-1}\Big(\sum_{i=1}^{r}\mu_{\alpha,\lambda,i}(x;s_i)\Big)^{1/p}\Big\}.$$

4. Equi-uniform summation processes

Let $\mathcal{A} = \{a_{\alpha,m}^{(\lambda)} : \alpha \in D, m \in \mathbb{N}_0, \lambda \in \Lambda\}$ be a family of scalars. $\mathcal{A}$ is said to be regular if it satisfies the following conditions:

(A-1) For each $m \in \mathbb{N}_0$, $\lim_\alpha a_{\alpha,m}^{(\lambda)} = 0$ uniformly in $\lambda \in \Lambda$.

(A-2) $\lim_\alpha \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} = 1$ uniformly in $\lambda \in \Lambda$.

(A-3) For each $\alpha \in D, \lambda \in \Lambda$,

$$a_\alpha^{(\lambda)} := \sum_{m=0}^{\infty} |a_{\alpha,m}^{(\lambda)}| < \infty,$$

and there exists $\alpha_0 \in D$ such that

$$\sup\{a_\alpha^{(\lambda)} : \alpha \geq \alpha_0, \alpha \in D, \lambda \in \Lambda\} < \infty.$$

$\mathcal{A}$ is said to be stochastic if

$$a_{\alpha,m}^{(\lambda)} \geq 0 \qquad (\alpha \in D,\ m \in \mathbb{N}_0,\ \lambda \in \Lambda)$$

and

$$\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} = 1 \qquad (\alpha \in D,\ \lambda \in \Lambda).$$

Obviously, if $\mathcal{A}$ is stochastic, then Conditions (A-2) and (A-3) are automatically satisfied.

A sequence $\{f_m\}_{m\in\mathbb{N}_0}$ of elements in E is said to be $\mathcal{A}$-summable to f if

$$\lim_\alpha \Big\|\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} f_m - f\Big\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda, \tag{9}$$

where it is assumed that the series in (9) converges for each $\alpha \in D$ and $\lambda \in \Lambda$.

Concerning the relation between the regularity of $\mathcal{A}$ and $\mathcal{A}$-summability, $\mathcal{A}$ is regular if and only if every convergent sequence of elements in E is $\mathcal{A}$-summable to its limit (cf. [1], [14]).

As the following examples with $D = \mathbb{N}_0$ show, there are a wide variety of families $\mathcal{A}$ and their particular cases cover many important summability methods:

(1°) Given an infinite matrix $A = (a_{nm})_{n,m\in\mathbb{N}_0}$, if $a_{n,m}^{(\lambda)} = a_{nm}$ for all $n, m \in \mathbb{N}_0$ and all $\lambda \in \Lambda$, then we obtain the usual matrix summability by A.

(2°) If $\Lambda = \mathbb{N}_0$, then we obtain the summation method introduced by Petersen [24] (cf. [1]). In particular, if

$$a_{n,m}^{(\lambda)} = \begin{cases} \frac{1}{n+1} & \text{if } \lambda \le m \le \lambda + n, \\ 0 & \text{otherwise}, \end{cases}$$

then we obtain the notion of almost convergent method (F-summability) introduced by Lorentz [10].

(3°) Let $Q = \{q^{(\lambda)} : \lambda \in \Lambda\}$ be a familiy of sequences $q^{(\lambda)} = \{q_n^{(\lambda)}\}_{n\in\mathbb{N}_0}$ of nonnegative real numbers such that

$$Q_n^{(\lambda)} := q_0^{(\lambda)} + q_1^{(\lambda)} + \cdots + q_n^{(\lambda)} > 0 \qquad (n \in \mathbb{N}_0,\ \lambda \in \Lambda).$$

We define

$$a_{n,m}^{(\lambda)} = \begin{cases} \frac{q_{n-m}^{(\lambda)}}{Q_n^{(\lambda)}} & \text{if } \quad m \le n, \\ 0 & \text{if } m > n. \end{cases}$$

Then $\mathcal{A}$-summability is called a (N, Q)-summability, and this kind of summability is called the Nörlund summability in the case where $q^{(\lambda)} = \{q_n\}_{n\in\mathbb{N}_0}$ is a fixed sequence of nonnegative real numbers satisfying $q_0 > 0$. The special case of interest is the following: Let $\Lambda \subseteq [0, \infty)$, $\beta > 0$ and

$$q_n^{(\lambda)} = C_n^{(\lambda+\beta-1)} \qquad (\lambda \in \Lambda,\ n \in \mathbb{N}_0),$$

where

$$C_0^{(\nu)} = 1, \quad C_n^{(\nu)} = \binom{n+\nu}{n} = \frac{(\nu+1)(\nu+2)\cdots(\nu+n)}{n!}$$

$$(\nu > -1,\ n \in \mathbb{N}).$$

In particular, if $\Lambda = \{0\}$, then we have the Cesàro summability of order β.

(4°) Let $\Lambda \subseteq (0, \infty), \beta > -1$ and define

$$a_{n,m}^{(\lambda)} = \begin{cases} C_{n-m}^{(\lambda-1)} C_m^{(\beta)} / C_n^{(\beta+\lambda)} & \text{if } m \le n, \\ 0 & \text{if } m > n, \end{cases}$$

(*Cesàro type*).

(5°) Let $\Lambda \subseteq [0,1]$ and define

$$a_{n,m}^{(\lambda)} = \begin{cases} \binom{n}{m}\lambda^m(1-\lambda)^{n-m} & \text{if } m \leq n, \\ 0 & \text{if } m > n, \end{cases}$$

(*Euler-Knopp-Bernstein type*).

Note that this can be a particular case of the generalized Lototsky matrix defined as follows (cf. [6], [7], [26]): Let $\{h_i\}_{i\in\mathbb{N}}$ be a sequence of functions of $[0,1]$ into itself and define

$$a_{0,0}^{(\lambda)} = 1, \quad a_{n,m}^{(\lambda)} = 0 \qquad (m > n),$$

$$\prod_{i=1}^{n}(xh_i(\lambda) + 1 - h_i(\lambda)) = \sum_{m=0}^{n} a_{n,m}^{(\lambda)} x^m.$$

(6°) Let $\Lambda \subseteq [0,1)$ and define

$$a_{n,m}^{(\lambda)} = \binom{n+m}{m}\lambda^m(1-\lambda)^{n+1},$$

(*Meyer-König-Vermes-Zeller type*).

(7°) Let $\Lambda \subseteq [0,\infty)$ and define

$$a_{n,m}^{(\lambda)} = \exp(-n\lambda)\frac{(n\lambda)^m}{m!},$$

(*Borel-Szász type*).

(8°) Let $\Lambda \subseteq [0,\infty)$ and define

$$a_{n,m}^{(\lambda)} = \binom{n+m-1}{m}\lambda^m(1+\lambda)^{-n-m},$$

(*Baskakov type*).

This can be generalized as follows (cf. [4], [12]): Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued functions on $[0,\infty)$ which possess the following properties:

(φ-1) Each function φ_n is expanded in Taylor's series on $[0,\infty)$;

(φ-2) $\qquad \varphi_n(0) = 1 \qquad (n \in \mathbb{N})$;

(φ-3) Each function φ_n is completely monotone, i.e.,

$$(-1)^m\varphi_n^{(m)}(t) \geq 0 \qquad (t \in [0,\infty), n \in \mathbb{N}, m \in \mathbb{N}_0);$$

(φ-4) There exists a strictly monotone increasing sequence $\{\ell_n\}_{n\in\mathbb{N}}$ of positive integers and a sequence $\{\alpha_{n,m}\}_{n,m\in\mathbb{N}}$ of real-valued functions on $[0,\infty)$ such that

$$\varphi_n^{(m)}(t) = -n\varphi_{\ell_n}^{(m-1)}(1+\alpha_{n,m}(t)) \qquad (t \in [0,\infty)).$$

Now we define

$$a_{0,0}^{(\lambda)} = 1, \quad a_{0,m}^{(\lambda)} = 0 \qquad (m \in \mathbb{N}),$$

$$a_{n,m}^{(\lambda)} = (-1)^m \frac{\varphi_n^{(m)}(\lambda)}{m!}\lambda^m \qquad (n \in \mathbb{N}, m \in \mathbb{N}_0).$$

Note that all the families $\mathcal{A}$ of the generic entries $a_{n,m}^{(\lambda)}$ given in the above Examples (2°)-(8°) are stochastic and all the families $\mathcal{A}$ of the generic entries $a_{n,m}^{(\lambda)}$ given in the above Examples (4°)-(8°) are regular for any finite interval Λ.

A sequence $\{K_n\}_{n\in\mathbb{N}_0}$ of operators of $BC(X,E)$ into $B(X_0,E)$ is called an equi-uniform $\mathcal{A}$-summation process on $BC(X,E)$ if the famly $\mathfrak{K} = \{K_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ is an equi-uniform approximation process on $BC(X,E)$, where each $K_{\alpha,\lambda}$ is defined by

$$(10) \qquad K_{\alpha,\lambda}(F)(x) = \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} K_m(F)(x) \qquad (F \in BC(X,E)),$$

which is assumed to be convergent. A family $\{L_n(x) : n \in \mathbb{N}_0, x \in X\}$ of mappings of E_0 into E is called an equi-uniform $\mathfrak{T}$-$\mathcal{A}$-summation process on E_0 if the family $\mathfrak{L} = \{L_{\alpha,\lambda}(x) : \alpha \in D, \lambda \in \Lambda, x \in X\}$ is an equi-uniform $\mathfrak{T}$-approximation process on E_0, where each $L_{\alpha,\lambda}(x)$ is defined by

$$(11) \qquad L_{\alpha,\lambda}(x)(f) = \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} L_m(x)(f) \qquad (f \in E_0),$$

which is assumed to be convergent.

Now, let $\{Y_n\}_{n\in\mathbb{N}_0}$ be a sequence of separable topological spaces with a Borel measure μ_n on Y_n, and let $\{\xi_n\}_{n\in\mathbb{N}_0}$ be a sequence of continuous mappings of Y_n into X and let $\mathfrak{B} = \{\chi_n(x;\cdot) : n \in \mathbb{N}_0, x \in X\}$ be a family of functions in $L^1(Y_n,\mu_n)$ such that

$$b_{\alpha,\lambda}(x) := \sum_{m=0}^{\infty} \int_{Y_m} |a_{\alpha,m}^{(\lambda)}\chi_m(x;y)|\, d\mu_m(y) < \infty \quad (\alpha \in D, \lambda \in \Lambda, x \in X).$$

We define

$$(12) \quad K_n(F)(x) = \int_{Y_n} \chi_n(x;y) F(\xi_n(y))\, d\mu_n(y) \qquad (F \in BC(X,E)),$$

$$L_n(x)(f) = \int_{Y_n} \chi_n(x;y) T(\xi_n(y))(f)\, d\mu_n(y) \qquad (f \in E_0) \tag{13}$$

and let $K_{\alpha,\lambda}$ and $L_{\alpha,\lambda}$ be defined by (10) and (11), respectively. Suppose that $\chi_n(x;\cdot)\Phi_i^{s_i}(x,\xi_n(\cdot)) \in L^1(Y_n,\mu_n)$ for all $n \in \mathbb{N}_0, x \in X_0$ and for $i = 1,2,\ldots,r$ and we define

$$\mu_{n,i}(x;s_i) = \|\chi_n(x;\cdot)\Phi_i^{s_i}(x,\xi_n(\cdot))\|_1.$$

For any $\alpha \in D, \lambda \in \Lambda$ and $x \in X_0$, we define

$$\tau_{\alpha,\lambda}(x) = \Big|\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} \int_{Y_m} \chi_m(x;y)\, d\mu_m(y) - 1\Big|$$

and

$$\theta_{\alpha,\lambda,i}(x;s_i) = \sum_{m=0}^{\infty} |a_{\alpha,m}^{(\lambda)}| \mu_{m,i}(x;s_i) \qquad (i = 1,2,\ldots,r).$$

Theorem 2. *For all $\alpha \in D, \lambda \in \Lambda, F \in BC(X,E)$ and all $x \in X_0$,*

$$\|K_{\alpha,\lambda}(F)(x) - F(x)\| \le \|F(x)\|\tau_{\alpha,\lambda}(x) + \zeta_{\alpha,\lambda}(x)\omega(F,\epsilon_\alpha), \tag{14}$$

where

$$\zeta_{\alpha,\lambda}(x) = Cb_{\alpha,\lambda}(x) + K\eta_{\alpha,\lambda}(x)$$

and

$$\eta_{\alpha,\lambda}(x) = \min\Big\{L\epsilon_\alpha^{-p}\sum_{i=1}^{r}\theta_{\alpha,\lambda,i}(x;s_i), L^{1/p}\epsilon_\alpha^{-1}\Big(\sum_{i=1}^{r}\theta_{\alpha,\lambda,i}(x;s_i)\Big)^{1/p} b_{\alpha,\lambda}(x)^{1-1/p}\Big\}.$$

In particular, if $\mathfrak{B}$ is positive and normal and if $\mathcal{A}$ is stochastic, then (14) reduces to

$$\|K_{\alpha,\lambda}(F)(x) - F(x)\| \le (C + K\eta_{\alpha,\lambda}(x))\omega(F,\epsilon_\alpha),$$

and

$$\eta_{\alpha,\lambda}(x) = \min\Big\{L\epsilon_\alpha^{-p}\sum_{i=1}^{r}\theta_{\alpha,\lambda,i}(x;s_i),\ L^{1/p}\epsilon_\alpha^{-1}\Big(\sum_{i=1}^{r}\theta_{\alpha,\lambda,i}(x;s_i)\Big)^{1/p}\Big\}.$$

Proof. We have

$$\|K_{\alpha,\lambda}(F)(x) - F(x)\| \tag{15}$$

$$\le \sum_{m=0}^{\infty} |a_{\alpha,m}^{(\lambda)}| \Big\|\int_{Y_m} \chi_m(x;y)(F(\xi_m(y)) - F(x))\, d\mu_m(y)\Big\|$$

$$+\left|\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} \int_{Y_m} \chi_m(x;y)\, d\mu_m(y) - 1\right| \|F(x)\| = I_{\alpha,\lambda}^{(1)}(x) + I_{\alpha,\lambda}^{(2)}(x),$$

say. We have $I_{\alpha,\lambda}^{(2)}(x) = \|F(x)\|\tau_{\alpha,\lambda}(x)$. Taking $(Y,\mu) = (Y_m,\mu_m)$, $\chi(x;\cdot) = \chi_m(x;\cdot)$ and $\tau = \xi_m$ in Lemma 4, we get

$$(16) \qquad I_{\alpha,\lambda}^{(1)}(x) \le \Big(Cb_{\alpha,\lambda}(x) + K\sum_{m=0}^{\infty} |a_{\alpha,m}^{(\lambda)}| c_m(x;p,\delta)\Big)\omega(F,\delta),$$

where

$$c_m(x;p,\delta) = \min\Big\{\delta^{-p}\|\chi_m(x;\cdot)d^p(x,\xi_m(\cdot))\|_1,$$
$$\delta^{-1}\|\chi_m(x;\cdot)\|_1^{1-1/p}\|\chi_m(x;\cdot)d^p(x,\xi_m(\cdot))\|_1^{1/p}\Big\}.$$

Now, if $p > 1$, then by Hölder's inequality we have

$$\sum_{m=0}^{\infty} |a_{\alpha,m}^{(\lambda)}| \|\chi_m(x;\cdot)\|_1^{1-1/p} \|\chi_m(x;\cdot)d^p(x;\xi_m(\cdot))\|_1^{1/p}$$

$$\le \Big(\sum_{m=0}^{\infty} |a_{\alpha,m}^{(\lambda)}| \|\chi_m(x;\cdot)\|_1\Big)^{1-1/p} \Big(\sum_{m=0}^{\infty} |a_{\alpha,m}^{(\lambda)}| \|\chi_m(x;\cdot)d^p(x;\xi_m(\cdot))\|_1\Big)^{1/p},$$

which clearly holds for $p = 1$. Also, by (4) we have

$$\|\chi_m(x;\cdot)d^p(x,\xi_m(\cdot))\|_1 \le L\sum_{i=1}^{r} \mu_{m,i}(x;s_i) \qquad (m \in \mathbb{N}_0).$$

Therefore, we obtain

$$\sum_{m=0}^{\infty} |a_{\alpha,m}^{(\lambda)}| c_m(x;p,\delta) \le \min\Big\{\delta^{-p}L\sum_{i=1}^{r}\theta_{\alpha,\lambda,i}(x;s_i),$$
$$\delta^{-1}b_{\alpha,\lambda}(x)^{1-1/p}L^{1/p}\Big(\sum_{i=1}^{r}\theta_{\alpha,\lambda,i}(x;s_i)\Big)^{1/p}\Big\}\omega(F,\delta),$$

and so putting $\delta = \epsilon_\alpha$ in the above inequality, (15) and (16) yield the desired estimate (14).

Corollary 2. *For all $\alpha \in D, \lambda \in \Lambda, f \in E_0$ and all $x \in X_0$,*

$$\|L_{\alpha,\lambda}(x)(f) - T(x)(f)\| \le \|T(x)(f)\|\tau_{\alpha,\lambda}(x) + \zeta_{\alpha,\lambda}(x)\omega_{\mathfrak{T}}(f,\epsilon_\alpha).$$

In pariticular, if $\mathfrak{B}$ is positive and normal and if $\mathcal{A}$ is stochastic, then

$$\|L_{\alpha,\lambda}(x)(f) - T(x)(f)\| \le (C + K\eta_{\alpha,\lambda}(x))\omega_{\mathfrak{T}}(f,\epsilon_\alpha)$$

and

$$\eta_{\alpha,\lambda}(x) = \min\Big\{L\epsilon_\alpha^{-p}\sum_{i=1}^{r}\theta_{\alpha,\lambda,i}(x;s_i),\ L^{1/p}\epsilon_\alpha^{-1}\Big(\sum_{i=1}^{r}\theta_{\alpha,\lambda,i}(x;s_i)\Big)^{1/p}\Big\}.$$

5. Interpolation type operators

Let $\{Y_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of finite sets. Then the integral operators given by (1) and (7) reduce to

$$(17)\quad K_{\alpha,\lambda}(F)(x) = \sum_{y\in Y_{\alpha,\lambda}} \chi_{\alpha,\lambda}(x;y)F(\xi_{\alpha,\lambda}(y)) \qquad (F \in BC(X,E))$$

and

$$(18)\qquad L_{\alpha,\lambda}(x)(f) = \sum_{y\in Y_{\alpha,\lambda}} \chi_{\alpha,\lambda}(x;y)T(\xi_{\alpha,\lambda}(y))(f) \qquad (f \in E_0),$$

respectively. These are called interpolation type operators with the interpolation system $\{\chi_{\alpha,\lambda}(\cdot;y) : y \in Y_{\alpha,\lambda}\}$ and nodes $\{\xi_{\alpha,\lambda}(y) : y \in Y_{\alpha,\lambda}\}$. Here we restrict ourselves to the following situation:

Let $1 \le s \le \infty$ be fixed and let $\mathbb{R}^r$ denote the metric linear space of all r-tuples of real numbers, equipped with the usual metric

$$d(x,t) = d_s(x,t) := \begin{cases} \Big(\sum\limits_{i=1}^{r} |x_i - t_i|^s\Big)^{1/s} & (1 \le s < \infty) \\ \max\{|x_i - t_i| : 1 \le i \le r\} & (s = \infty), \end{cases}$$

where $x = (x_1, x_2, \dots, x_r), t = (t_1, t_2, \dots, t_r) \in \mathbb{R}^r$. Now, let X be a convex subset of $\mathbb{R}^r$. Therefore, by Lemma 1 (b), (2) holds with $C = K = 1$. For $i = 1, 2, \dots, r$, p_i denotes the ith coordinate function on $\mathbb{R}^r$ defined by $p_i(x) = x_i$ for all $x = (x_1, x_2, \dots, x_r) \in \mathbb{R}^r$. Then we have

$$d_s^p(x,t) \le c(p,r,s)\sum_{i=1}^{r} |p_i(x) - p_i(t)|^p \qquad (x,t \in \mathbb{R}^r,\ p > 0),$$

where

$$c(p,r,s) = \begin{cases} r^{p/s} & (1 \le s < \infty,\ s \ne p) \\ 1 & (1 \le s < \infty,\ s = p) \\ 1 & (s = \infty). \end{cases}$$

Therefore, (4) holds with

$$s_i = p \geq 1, \quad \Phi_i(x,t) = |p_i(x) - p_i(t)|, \quad L = c(p,r,s),$$

and so by Theorem 1 and Corollary 1, we have the following result which can be more convenient for later applications to the concrete examples of interpolation type operators.

Theorem 3. *Let $p \geq 1$ and suppose that $\mathfrak{A}$ is positive and normal. Then the following statements hold:*

(a) For all $\alpha \in D, \lambda \in \Lambda, F \in BC(X,E)$ and all $x \in X_0$,

$$\|K_{\alpha,\lambda}(F)(x) - F(x)\| \leq (1 + c_{\alpha,\lambda}(x))\omega(F, \epsilon_\alpha),$$

where

$$c_{\alpha,\lambda}(x)$$

$$= \min\Big\{c(p,r,s)\epsilon_\alpha^{-p}\sum_{i=1}^{r}\mu_{\alpha,\lambda,i}(x;p),\ c(p,r,s)^{1/p}\epsilon_\alpha^{-1}\Big(\sum_{i=1}^{r}\mu_{\alpha,\lambda,i}(x;p)\Big)^{1/p}\Big\}$$

and

$$\mu_{\alpha,\lambda,i}(x;p) = \sum_{y \in Y_{\alpha,\lambda}} \chi_{\alpha,\lambda}(x;y)|p_i(x) - p_i(\xi_{\alpha,\lambda}(y))|^p.$$

(b) For all $\alpha \in D, \lambda \in \Lambda, f \in E_0$ and all $x \in X_0$,

$$\|L_{\alpha,\lambda}(x)(f) - T(x)(f)\| \leq (1 + c_{\alpha,\lambda}(x))\omega_{\mathfrak{T}}(f, \epsilon_\alpha).$$

Let $X = [0,\infty)^r$ be the first hyperquadrant and let

$$m_{\alpha,i} : \Lambda \to \mathbb{N}, \quad a_{\alpha,i} : \Lambda \to (0,\infty) \qquad (\alpha \in D,\ i = 1,2,\ldots,r)$$

and

$$I_{\alpha,\lambda} := \{k = (k_1,k_2,\ldots,k_r) \in \mathbb{N}_0^r : 0 \leq k_i \leq m_{\alpha,i}(\lambda), i = 1,2,\ldots,r\}.$$

We define

$$\chi_{\alpha,\lambda}(x;k) = \prod_{i=1}^{r}\binom{m_{\alpha,i}(\lambda)}{k_i}x_i^{k_i}(1-x_i)^{m_{\alpha,i}(\lambda)-k_i} \qquad (x \in X,\ k \in I_{\alpha,\lambda})$$

and

$$\xi_{\alpha,\lambda}(k) = (a_{\alpha,1}(\lambda)k_1, a_{\alpha,2}(\lambda)k_2, \ldots, a_{\alpha,r}(\lambda)k_r) \qquad (k \in I_{\alpha,\lambda}).$$

Then the interpolation type operators (17) and (18) become

$$K_{\alpha,\lambda}(F)(x) = \sum_{k \in I_{\alpha,\lambda}} \chi_{\alpha,\lambda}(x;k)F(\xi_{\alpha,\lambda}(k)) \qquad (F \in BC(X,E))$$

and

$$L_{\alpha,\lambda}(x)(f) = \sum_{k \in I_{\alpha,\lambda}} \chi_{\alpha,\lambda}(x;k) T(\xi_{\alpha,\lambda}(k))(f) \qquad (f \in E_0),$$

respectively. These generalize the r-dimensional Bernstein operators, which are defined as follows (cf. [11], [17]): Let $\mathbb{I}_r = [0,1]^r$ be the unit r-cube and let $\{\nu_{n,i}\}_{n \in \mathbb{N}}, i = 1, 2, \ldots, r$, be strictly monotone increasing sequences of positive integers. Then we define

$$B_n(F)(x) = \sum_{k_1=0}^{\nu_{n,1}} \sum_{k_2=0}^{\nu_{n,2}} \cdots \sum_{k_r=0}^{\nu_{n,r}} F\left(\frac{k_1}{\nu_{n,1}}, \frac{k_2}{\nu_{n,2}}, \ldots, \frac{k_r}{\nu_{n,r}}\right) \times \prod_{j=1}^{r} \binom{\nu_{n,j}}{k_j} x_j^{k_j} (1 - x_j)^{\nu_{n,j} - k_j}$$

$$(F \in C(\mathbb{I}_r, E), \ x \in \mathbb{I}_r)$$

and

$$C_n(x)(f) = \sum_{k_1=0}^{\nu_{n,1}} \sum_{k_2=0}^{\nu_{n,2}} \cdots \sum_{k_r=0}^{\nu_{n,r}} T\left(\frac{k_1}{\nu_{n,1}}, \frac{k_2}{\nu_{n,2}}, \ldots, \frac{k_r}{\nu_{n,r}}\right)(f) \times \prod_{j=1}^{r} \binom{\nu_{n,j}}{k_j} x_j^{k_j} (1 - x_j)^{\nu_{n,j} - k_j}$$

$$(f \in E_0, \ x \in \mathbb{I}_r).$$

Now, let X_0 be a subset of $\mathbb{I}_r$ and let $x \in X_0$. Then we have

$\mu_{\alpha,\lambda,i}(x;2) = (a_{\alpha,i}(\lambda) m_{\alpha,i}(\lambda) - 1)^2 p_i^2(x) + a_{\alpha,i}^2(\lambda) m_{\alpha,i}(\lambda) p_i(x)(1 - p_i(x))$

and

$$c(r,s) := c(2,r,s) = \begin{cases} r^{2/s} & (1 \le s < \infty, \ s \neq 2) \\ 1 & (s = 2, \infty). \end{cases}$$

Therefore, Theorem 3 can be applied for $p = 2$. In particular, if

$$m_{\alpha,i}(\lambda) a_{\alpha,i}(\lambda) = 1$$

for all $\alpha \in D, \lambda \in \Lambda$ and for $i = 1, 2, \ldots, r$, then

$$\mu_{\alpha,\lambda,i}(x;2) = \frac{1}{m_{\alpha,i}(\lambda)} (p_i(x) - p_i^2(x)).$$

Therefore, for the Bernstein operators, we have

$$\|B_n(F)(x) - F(x)\| \le (1 + c_n(r,s)) \omega(F, \epsilon_n \theta_n(x)) \tag{19}$$

and

(20) $$\|C_n(x)(f) - T(x)(f)\| \leq (1 + c_n(r,s))\omega_{\mathfrak{T}}(f, \epsilon_n\theta_n(x)),$$

where $\{\epsilon_n\}_{n\in\mathbb{N}}$ is a sequence of positive integers,

$$c_n(r,s) := \min\{\sqrt{c(r,s)}\epsilon_n^{-1}, c(r,s)\epsilon_n^{-2}\}$$

and

$$\theta_n(x) := \Big(\sum_{i=1}^{r} \frac{1}{\nu_{n,i}}(p_i(x) - p_i^2(x))\Big)^{1/2}.$$

In paritucular, (19) and (20) yield the following estimates for all $x \in \mathbb{I}_r$:

(21) $$\|B_n(F)(x) - F(x)\| \leq \theta_n(r,s)\omega\Big(F, \epsilon_n\sqrt{\sum_{i=1}^{r}\frac{1}{\nu_{n,i}}}\Big);$$

(22) $$\|C_n(x)(f) - T(x)(f)\| \leq \theta_n(r,s)\omega_{\mathfrak{T}}\Big(f, \epsilon_n\sqrt{\sum_{i=1}^{r}\frac{1}{\nu_{n,i}}}\Big).$$

Here

$$\theta_n(r,s) := 1 + \min\Big\{\frac{\sqrt{c(r,s)}}{2\epsilon_n}, \frac{c(r,s)}{4\epsilon_n^2}\Big\}.$$

Let $\{T_j(t) : t \in [0,1], j = 1,2,\ldots,r\}$ be a family of strongly continuous mappings of E_0 into itself such that for every $t,u \in [0,1]$, $tT_j(u)$ commutes with $(1-t)I$, where I is the identity operator on E and $T_j(v)^n = T_j(nv)$ whenever $v \in [0,1], n \in \mathbb{N}_0$ and $nv \in [0,1]$. If

$$T(x) = \prod_{j=1}^{r} T_j(x_j)$$

for all $x = (x_1, x_2, \ldots, x_r) \in \mathbb{I}_r$, then

$$C_n(x)(f) = \prod_{j=1}^{r}\Big((1-x_j)I + x_jT_j(\frac{1}{\nu_{n,j}})\Big)^{\nu_{n,j}}(f)$$

$$= \prod_{j=1}^{r}\Big(I + x_j(T_j(\frac{1}{\nu_{n,j}}) - I)\Big)^{\nu_{n,j}}(f).$$

Therefore, the inequality (22) estimates the rate of convergence in [16, Theorem 5] for $r = 1$, which improves the estimate in [3, Proposition 1.2.9] and furthermore, it generalizes and improves the convergence rate in [5, Theorem 1.1]. Also, we note that in view of Lemma 3, other results of [5] can be generalized and improved by the same argument as the above manner.

Suppose that $\mathcal{A}$ is stochastic, and let $\{\ell_n\}_{n\in\mathbb{N}_0}$ be a sequence of positive integers. We define

$$K_{\alpha,\lambda}(F)(x) = \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} B_{\ell_m}(F)(x) \qquad (F \in C(\mathbb{I}_r, E),\ x \in \mathbb{I}_r)$$

and

$$L_{\alpha,\lambda}(x)(f) = \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} C_{\ell_m}(x)(f) \qquad (f \in E_0,\ x \in \mathbb{I}_r).$$

Then Theorem 2 and Corollary 2 establish the following estimates:

$$\|K_{\alpha,\lambda}(F)(x) - F(x)\| \le (1 + c_\alpha(r,s))\omega(F, \epsilon_\alpha \vartheta_{\alpha,\lambda}(x));$$

$$\|L_{\alpha,\lambda}(x)(f) - T(x)(f)\| \le (1 + c_\alpha(r,s))\omega_{\mathfrak{T}}(f, \epsilon_\alpha \vartheta_{\alpha,\lambda}(x)).$$

Here

$$c_\alpha(r,s) := \min\{\sqrt{c(r,s)}\epsilon_\alpha^{-1},\ c(r,s)\epsilon_\alpha^{-2}\}$$

and

$$\vartheta_{\alpha,\lambda}(x) := \Big(\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} \theta_{\ell_m}^2(x)\Big)^{1/2}.$$

Let $\Lambda = D = \mathbb{N}_0$. Let $\nu_{m,i} = m$ for all $m \in \mathbb{N}$ and for $i = 1, 2, \ldots, r$. Let $\ell_m = m+1$ for all $m \in \mathbb{N}_0$. Then concerning the method of almost convergence (see, Sec. 4, Example (2°)), we have

$$\vartheta_{\alpha,\lambda}(x) \le \Big(\sum_{i=1}^{r}(p_i(x) - p_i^2(x))\Big)^{1/2}\Big(\frac{1+\log(\alpha+1)}{\alpha+1}\Big)^{1/2}$$

$$\le \frac{\sqrt{r}}{2}\sqrt{\frac{1+\log(\alpha+1)}{\alpha+1}} \qquad (\alpha,\ \lambda \in \mathbb{N}_0)$$

(cf. [13] for $r = 1$).

The statements analogous to the above results hold for the following settings: Let X_0 be a closed subset of Δ_r, where

$$\Delta_r := \Big\{x = (x_1, x_2, \ldots, x_r) \in \mathbb{R}^r : x_i \ge 0, i = 1, 2, \ldots, r, \sum_{i=1}^{r} x_i \le 1\Big\}$$

is the standard r-simplex; Let $m_\alpha : \Lambda \to \mathbb{N}$.

$$J_{\alpha,\lambda} := \{k = (k_1, k_2, \ldots, k_r) \in \mathbb{N}_0^r : k_1 + k_2 + \cdots + k_r \le m_\alpha(\lambda)\};$$

$$\chi_{\alpha,\lambda}(x;k) := \binom{m_\alpha(\lambda)}{k} \prod_{i=1}^{r} x_i^{k_i} \left(1 - \sum_{j=1}^{r} x_j\right)^{m_\alpha(\lambda) - \sum_{j=1}^{r} k_j}$$
$$(x \in X,\ k \in J_{\alpha,\lambda}),$$

where

$$\binom{m_\alpha(\lambda)}{k} := \frac{m_\alpha(\lambda)!}{k_1!k_2!\cdots k_r!(m_\alpha(\lambda) - k_1 - k_2 - \cdots - k_r)!};$$

$$\xi_{\alpha,\lambda}(k) := (a_{\alpha,1}(\lambda)k_1, a_{\alpha,2}(\lambda)k_2, \ldots, a_{\alpha,r}(\lambda)k_r) \qquad (k \in J_{\alpha,\lambda}).$$

Next, let $X = X_r := [-1,1]^r$, and let X_0 be a closed subset of X_r. Let $Q_n(t) = \cos(n \arccos t)$ be the Chebyshev polynomial of degree n, and let $t_{n,j}, j = 1, 2, \ldots, n$, be zeros of $Q_n(t)$, i.e.,

$$t_{n,j} = \cos\left(\frac{2j-1}{2n}\pi\right), \qquad (j = 1, 2, \ldots, n).$$

Let

$$m_{\alpha,i} : \Lambda \to \mathbb{N}, \quad a_{\alpha,i} : \Lambda \to [-1,1] \qquad (\alpha \in D,\ i = 1, 2, \ldots, r)$$

and let

$$N_{\alpha,\lambda} := \{k = (k_1, k_2, \ldots, k_r) \in \mathbb{N}^r : 1 \le k_i \le m_{\alpha,i}(\lambda), i = 1, 2, \ldots, r\}.$$

We define

$$\chi_{\alpha,\lambda}(x;k) = \prod_{i=1}^{r} \chi_{m_{\alpha,i}(\lambda)}(x_i;k_i) \qquad (x \in X_r,\ k \in N_{\alpha,\lambda}),$$

where

$$\chi_{m_{\alpha,i}(\lambda)}(x_i;k_i) = (1 - x_i t_{m_{\alpha,i}(\lambda),k_i}) \left\{ \frac{Q_{m_{\alpha,i}(\lambda)}(x_i)}{m_{\alpha,i}(\lambda)(x_i - t_{m_{\alpha,i}(\lambda),k_i})} \right\}^2$$

and

$$\xi_{\alpha,\lambda}(k) = (a_{\alpha,1}(\lambda) t_{m_{\alpha,1}(\lambda),k_1}, \ldots, a_{\alpha,r}(\lambda) t_{m_{\alpha,r}(\lambda),k_r}) \qquad (k \in N_{\alpha,\lambda}).$$

Then the interpolation type operators (17) and (18) become

$$K_{\alpha,\lambda}(F)(x) = \sum_{k \in N_{\alpha,\lambda}} \chi_{\alpha,\lambda}(x;k) F(\xi_{\alpha,\lambda}(k)) \qquad (F \in C(X_r, E))$$

and

$$L_{\alpha,\lambda}(x)(f) = \sum_{k \in N_{\alpha,\lambda}} \chi_{\alpha,\lambda}(x;k) T(\xi_{\alpha,\lambda}(k))(f) \qquad (f \in E_0),$$

respectively. These generalize the r-dimensinal Hermite-Fejér operators, which are defined as follows (cf. [9], [17]):

We define

$$H_n(F)(x) = \sum_{k_1=1}^{\nu_{n,1}} \sum_{k_2=1}^{\nu_{n,2}} \cdots \sum_{k_r=1}^{\nu_{n,r}} F(t_{\nu_{n,1},k_1}, t_{\nu_{n,2},k_2}, \dots, t_{\nu_{n,r},k_r})$$

$$\times \prod_{j=1}^{r} (1 - x_j t_{\nu_{n,j},k_j}) \Big\{ \frac{Q_{\nu_{n,j}}(x_j)}{\nu_{n,j}(x_j - t_{\nu_{n,j},k_j})} \Big\}^2$$

$$(F \in C(X_r, E),\ x \in X_r)$$

and

$$G_n(x)(f) = \sum_{k_1=1}^{\nu_{n,1}} \sum_{k_2=1}^{\nu_{n,2}} \cdots \sum_{k_r=1}^{\nu_{n,r}} T(t_{\nu_{n,1},k_1}, t_{\nu_{n,2},k_2}, \dots, t_{\nu_{n,r},k_r})(f)$$

$$\times \prod_{j=1}^{r} (1 - x_j t_{\nu_{n,j},k_j}) \Big\{ \frac{Q_{\nu_{n,j}}(x_j)}{\nu_{n,j}(x_j - t_{\nu_{n,j},k_j})} \Big\}^2$$

$$(f \in E_0,\ x \in X_r).$$

Now, let $x \in X_0$. Then we have

$$\mu_{\alpha,\lambda,i}(x;2) = \frac{Q^2_{m_{\alpha,i}(\lambda)}(x_i)}{m_{\alpha,i}(\lambda)} - 2x_i(a_{\alpha,i}(\lambda) - 1) \sum_{k_i=1}^{m_{\alpha,i}(\lambda)} t_{m_{\alpha,i}(\lambda),k_i} \chi_{m_{\alpha,i}(\lambda)}(x_i; k_i)$$

$$+ (a^2_{\alpha,i}(\lambda) - 1) \sum_{k_i=1}^{m_{\alpha,i}(\lambda)} {t_{m_{\alpha,i}(\lambda),k_i}}^2 \chi_{m_{\alpha,i}(\lambda)}(x_i; k_i).$$

Therefore, Theorem 3 can be applied for $p = 2$. In particular, if

$$a_{\alpha,i}(\lambda) = 1$$

for all $\alpha \in D, \lambda \in \Lambda$ and for $i = 1, 2, \dots, r$, then

$$\mu_{\alpha,\lambda,i}(x;2) = \frac{(Q_{m_{\alpha,i}(\lambda)} \circ p_i)^2(x)}{m_{\alpha,i}(\lambda)}.$$

Therefore, for the Hermite-Fejér operators, we have

(23) $$\|H_n(F)(x) - F(x)\| \le (1 + c_n(r,s))\omega, (F, \epsilon_n \tau_n(x))$$

and

(24) $$\|G_n(x)(f) - T(x)(f)\| \le (1 + c_n(r,s))\omega_{\mathfrak{T}}(f, \epsilon_n, \tau_n(x)),$$

where $\{\epsilon_n\}_{n\in\mathbb{N}}$ is a sequence of positive integers and

$$\tau_n(x) := \Big(\sum_{i=1}^{r} \frac{(Q_{\nu_{n,i}} \circ p_i)^2(x)}{\nu_{n,i}}\Big)^{1/2}.$$

In particular, (23) and (24) establish the following estimates for all $x \in X_r$:

$$\|H_n(F)(x) - F(x)\| \leq (1 + c_n(r,s))\omega\Big(F, \epsilon_n\sqrt{\sum_{i=1}^{r} \frac{1}{\nu_{n,i}}}\Big);$$

$$\|G_n(x)(f) - T(x)(f)\| \leq (1 + c_n(r,s))\omega_{\mathfrak{T}}\Big(f, \epsilon_n\sqrt{\sum_{i=1}^{r} \frac{1}{\nu_{n,i}}}\Big).$$

Suppose again that $\mathcal{A}$ is stochastic, and we define

$$K_{\alpha,\lambda}(F)(x) = \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} H_{\ell_m}(F)(x) \qquad (F \in C(X_r, E),\ x \in X_r)$$

and

$$L_{\alpha,\lambda}(x)(f) = \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} G_{\ell_m}(x)(f) \qquad (f \in E_0,\ x \in X_r).$$

Then Theorem 2 and Corollary 2 establish the following estimates:

$$\|K_{\alpha,\lambda}(F)(x) - F(x)\| \leq (1 + c_\alpha(r,s))\omega(F, \epsilon_\alpha \delta_{\alpha,\lambda}(x));$$

$$\|L_{\alpha,\lambda}(x)(f) - T(x)(f)\| \leq (1 + c_\alpha(r,s))\omega_{\mathfrak{T}}(f, \epsilon_\alpha \delta_{\alpha,\lambda}(x)).$$

Here

$$\delta_{\alpha,\lambda}(x) := \Big(\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} \tau_{\ell_m}^2(x)\Big)^{1/2}.$$

6. Convolution type operators

In this section, we treat the equi-uniform $\mathcal{A}$-summation processes of convolution type operators on $BC(\mathbb{R}^r, E)$. For this end, in (12) and 13) we especially take

$$Y_n = X = \mathbb{R}^r,$$

$$\mu_n(y) = dy, \quad \xi_n(y) = y$$

and

$$\chi_n(x;y) = \prod_{i=1}^{r}(h_n \circ p_i)(x-y),$$

where $\{h_n\}_{n\in\mathbb{N}_0}$ be a sequence of nonnegative Lebesgue integrable functions on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} h_n(t)\,dt = 1 \tag{25}$$

for all $n \in \mathbb{N}_0$. In this case, the integral operators K_n and L_n defined by (12) and (13) are called convolution type operators.

Suppose that $\mathcal{A}$ is stochastic, and let $\{K_{\alpha,\lambda}\}$ and $\{L_{\alpha,\lambda}\}$ be defined as in (10) and (11), respecively.

Theorem 4. *The following statements holds:*

(a) For all $\alpha \in D, \lambda \in \Lambda, F \in BC(X,E)$ and all $x \in X_0$,

$$\|K_{\alpha,\lambda}(F)(x) - F(x)\| \le (1 + c_\alpha(p,r,s))\omega(F, \epsilon_\alpha\theta_{\alpha,\lambda}(p)),$$

where

$$c_\alpha(p,r,s) = \min\{(rc(p,r,s))^{1/p}\epsilon_\alpha^{-1},\ rc(p,r,s)\epsilon_\alpha^{-p}\}$$

and

$$\theta_{\alpha,\lambda}(p) = \Big(\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} \int_{\mathbb{R}} |t|^p h_m(t)\,dt\Big)^{1/p} < \infty.$$

(b) For all $\alpha \in D, \lambda \in \Lambda, f \in E_0$ and all $x \in X_0$,

$$\|L_{\alpha,\lambda}(x)(f) - T(x)(f)\| \le (1 + c_\alpha(p,r,s))\omega_{\mathfrak{T}}(f, \epsilon_\alpha\theta_{\alpha,\lambda}(p)).$$

Proof. We have

$$\int_X \chi_n(x;y)\,dy = 1 \qquad (n \in \mathbb{N}_0,\ x \in X),$$

and so $\tau_{\alpha,\lambda}(x) = 0$ and $b_{\alpha,\lambda}(x) = 1$. Also, we have

$$\sum_{i=1}^{r} \theta_{\alpha,\lambda,i}(x;p) = \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} \sum_{i=1}^{r} \int_{\mathbb{R}} h_m(x_i - y_i)|x_i - y_i|^p\,dy_i$$

$$= r\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} \int_{\mathbb{R}} h_m(t)|t|^p\,dt = r\theta_{\alpha,\lambda}(p)^p.$$

Thus, by putting $\epsilon_\alpha\theta_{\alpha,\lambda}(p)$ instead of ϵ_α, the desired result follows from Theorem 2.

Let $\{k_n\}_{n\in\mathbb{N}_0}$ be a sequence of nonnegatve, even 2π-periodic, Lebesgue integrable functions on $\mathbb{R}$ having Fourier series expansions

$$k_n(t) \quad \sim \quad \sum_{j=-\infty}^{\infty} \hat{k_n}(j)e^{ijt}, \qquad \hat{k_n}(j) := \frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(t)e^{-ijt}\,dt$$

with $\hat{k_n}(0) = 1$, and we define

$$h_n(t) = \begin{cases} \frac{1}{2\pi}k_n(t) & (|t| \le \pi) \\ 0 & (|t| > \pi). \end{cases}$$

Corollary 3. *(a) For all $\alpha \in D, \lambda \in \Lambda, F \in BC(X, E)$ and all $x \in X_0$,*

$$\|K_{\alpha,\lambda}(F)(x) - F(x)\| \le (1 + \pi_\alpha(r, s))\omega(F, \epsilon_\alpha \varrho_{\alpha,\lambda}),$$

where

$$\pi_\alpha(r, s) = \min\Big\{\pi\sqrt{\frac{rc(r,s)}{2}}\epsilon_\alpha^{-1},\ \frac{\pi^2 rc(r,s)}{2}\epsilon_\alpha^{-2}\Big\}$$

and

$$\varrho_{\alpha,\lambda} = \Big(\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)}(1 - \hat{k_m}(1))\Big)^{1/2}.$$

(b) For all $\alpha \in D, \lambda \in \Lambda, f \in E_0$ and all $x \in X_0$,

$$\|L_{\alpha,\lambda}(x)(f) - T(x)(f)\| \le (1 + \pi_\alpha(r, s))\omega_{\mathfrak{T}}(f, \epsilon_\alpha \varrho_{\alpha,\lambda}).$$

Indeed, by the inequality

$$\frac{2}{\pi}t \le \sin t \le t \qquad (0 \le t \le \frac{\pi}{2}), \tag{26}$$

for all $m \in \mathbb{N}_0$, we have

$$\int_{-\pi}^{\pi} t^2 k_m(t)\,dt \le \pi^2 \int_{-\pi}^{\pi} k_m(t)\sin^2\frac{t}{2}\,dt = \frac{\pi^2}{2}\int_{-\pi}^{\pi}(1 - \cos t)k_m(t)\,dt.$$

Thus, we have $\theta_{\alpha,\lambda}(2) \le (\pi/\sqrt{2})\varrho_{\alpha,\lambda}$, and so putting $(\pi/\sqrt{2})^{-1}\epsilon_\alpha$ instead of ϵ_α the desired result follows from Theorem 4.

Let $(\lambda_n(j))$ $(n, j = 1, 2, \dots)$ be a lower triangular infinite matrix of real numbers and we define

$$k_0(t) = 1, \quad k_n(t) = 1 + 2\sum_{j=1}^{n}\lambda_n(j)\cos jt \qquad (n \in \mathbb{N},\ t \in \mathbb{R}).$$

Then applying Abel's transformation twice to the function $k_n(t)$, we have

$$k_n(t) = \sum_{j=0}^{n-1}(j+1)F_j(t)\Delta^2\lambda_n(j) + (n+1)\lambda_n(n)F_n(t), \quad \lambda_n(0) = 1,$$

where $F_m(t)$ is the mth Fejér kernel and

$$\Delta^2\lambda_n(j) = \lambda_n(j) - 2\lambda_n(j+1) + \lambda_n(j+2).$$

Therefore, if $\lambda_n(n) \geq 0$ and $\{\lambda_n(j)\}_{j\in\mathbb{N}_0}$ is convex, i.e., $\Delta^2\lambda_n(j) \geq 0$ for all $j \in \mathbb{N}_0$, then $k_n(t)$ is a nonnegative, even trigonometric polynomial of degree at most n and so Corollary 3 (a) and (b) hold with

$$\varrho_{\alpha,\lambda} = \Big(\sup\Big\{\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)}(1-\lambda_m(1))\Big)^{1/2}.$$

Several examples of $\lambda_n(j)$ produce important positive summability kernels mentioned as follows:

(1°) *Fejér*:

$$\lambda_n(j) = \begin{cases} 1 - \frac{j}{n+1} & (1 \leq j \leq n), \\ 0 & (j > n). \end{cases}$$

(2°) *de la Vallée-Poussin*:

$$\lambda_n(j) = \begin{cases} \frac{(n!)^2}{(n-j)!(n+j)!} & (1 \leq j \leq n), \\ 0 & (j > n). \end{cases}$$

(3°) *Fejér-Korovkin*:

$$\lambda_n(j) = \begin{cases} A_n \sum_{m=0}^{n-j} a_m a_{j+m} & (1 \leq j \leq n), \\ 0 & (j > n), \end{cases}$$

where

$$a_m = \sin\Big(\frac{m+1}{n+2}\Big)\pi \qquad (m = 0, 1, \ldots, n), \quad A_n = \Big(\sum_{m=0}^{n} a_m^2\Big)^{-1}.$$

In this case, we have

$$k_n(t) = A_n\Big|\sum_{m=0}^{n} a_m e^{imt}\Big|^2, \quad \lambda_n(1) = \cos\Big(\frac{\pi}{n+2}\Big).$$

(4°) *Nörlund*:

$$\lambda_n(j) = \begin{cases} \frac{Q_{n-j}}{Q_n} & (1 \le j \le n), \\ 0 & (j > n), \end{cases}$$

where

$$0 < q_0 \le q_n \le q_{n+1}, \quad Q_n = \sum_{m=0}^{n} q_m \qquad (n \in \mathbb{N}_0).$$

Obviously, if $q_n = 1$ for all $n \in \mathbb{N}_0$, then the Nörlund kernel reduces to the Fejér kernel.

(5°) *Cesàro*:

$$\lambda_n(j) = \begin{cases} \frac{C_{n-j}^{(\beta)}}{C_n^{(\beta)}} & (1 \le j \le n), \\ 0 & (j > n), \end{cases} \qquad (\beta \ge 1)$$

where $C_n^{(\nu)}$ $(n \in \mathbb{N}_0, \nu > -1)$ is defined as in Example (3°) of Section 4. Note that if $q_n = C_n^{(\beta-1)}$ for all $n \in \mathbb{N}_0$, then the Nörlund kernel reduces to the Cesàro kernel. Also, if $\beta = 1$, then the Cesàro kernel turns out to be the Fejér kernel.

Other important examples of nonnegative, even trigonometric polynomials are the following:

(6°) *Jackson*: Let $\nu \in \mathbb{N}$ and

$$k_n(t) = k_{n,\nu}(t) = c_{n,\nu}\left\{\frac{\sin\frac{1}{2}(n+1)t}{\sin\frac{1}{2}t}\right\}^{2\nu},$$

where the positive constant $c_{n,\nu}$ is chosen so that

$$\frac{1}{\pi}\int_0^{\pi} k_{n,\nu}(t)\,dt = 1.$$

Since $k_{n,\nu}(t) = c_{n,\nu}(n+1)^{\nu}F_n(t)^{\nu}$, we have $c_{n,1} = 1/(n+1)$ and $k_{n,1}(t)$ becomes the nth Fejér kernel. Also, for $\nu = 2$ we have

$$c_{n,2} = \frac{3}{(n+1)(2(n+1)^2+1)}, \quad \hat{k_n}(1) = \frac{2n(n+2)}{2(n+1)^2+1},$$

and so, we have

$$\varrho_{\alpha,\lambda} = \sqrt{3}\Big(\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} \frac{1}{2(m+1)^2+1}\Big)^{1/2}$$

$$\le \sqrt{\frac{3}{2}}\Big(\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} \frac{1}{(m+1)^2}\Big)^{1/2}.$$

Therefore, by selecting $\epsilon_\alpha = (3/2)^{-1/2}$ Corollary 3 yields the following estimates:

$$\|K_{\alpha,\lambda}(F)(x) - F(x)\| \le \Big(1 + \frac{\pi}{2}\sqrt{3rc(r,s)}\Big)\omega(F, \upsilon_{\alpha,\lambda});$$

$$\|L_{\alpha,\lambda}(x)(f) - T(x)(f)\| \le \Big(1 + \frac{\pi}{2}\sqrt{3rc(r,s)}\Big)\omega_{\mathfrak{T}}(f, \upsilon_{\alpha,\lambda}).$$

Here

$$\upsilon_{\alpha,\lambda} = \Big(\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} \frac{1}{(m+1)^2}\Big)^{1/2}.$$

Furthermore, making use of the inequality (26) we have for $\nu \ge 3$,

$$\Big(\frac{\pi}{2}\Big)^{1-2\nu} \frac{2\nu - 1}{2\nu}(n+1)^{1-2\nu} < c_{n,\nu} \le \Big(\frac{\pi}{2}\Big)^{2\nu}(n+1)^{1-2\nu}$$

and

$$1 - \hat{k_{n,\nu}}(1) < \Big(\frac{\pi}{2}\Big)^{2(2\nu-1)} \frac{8\nu}{3\pi(2\nu-3)}(n+1)^{-2}.$$

Therefore, we have

$$\varrho_{\alpha,\lambda} \le \Big(\frac{\pi}{2}\Big)^{2\nu-1} \sqrt{\frac{8\nu}{3\pi(2\nu-3)}}\,\upsilon_{\alpha,\lambda},$$

and so by selecting

$$\epsilon_\alpha = \Big(\frac{\pi}{2}\Big)^{1-2\nu}\Big(\frac{8\nu}{3\pi(2\nu-3)}\Big)^{-1/2},$$

Corollary 3 again yields the following estimates:

$$\|K_{\alpha,\lambda}(F)(x) - F(x)\| \le (1 + \theta(p,r,s))\omega(F, \upsilon_{\alpha,\lambda});$$

$$\|L_{\alpha,\lambda}(x)(f) - T(x)(f)\| \le (1 + \theta(p,r,s))\omega_{\mathfrak{T}}(f, \upsilon_{\alpha,\lambda}).$$

Here

$$\theta(\nu,r,s) = \min\Big\{\frac{4}{\sqrt{\pi}}\Big(\frac{\pi}{2}\Big)^{2\nu}\sqrt{\frac{\nu rc(r,s)}{3(2\nu-3)}},\ \frac{16}{\pi}\Big(\frac{\pi}{2}\Big)^{4\nu}\frac{\nu rc(r,s)}{3(2\nu-3)}\Big\}.$$

(7°) *Abel-Poisson*:

$$k_n(t) = 1 + 2\sum_{m=1}^{\infty} r_n^m \cos mt \qquad (n \in \mathbb{N}_0,\ t \in \mathbb{R}),$$

where $\{r_n\}_{n\in\mathbb{N}_0}$ is a sequence of real numbers converging to one such that $0 \le r_n < 1$ for all $n \in \mathbb{N}_0$. Since

$$k_n(t) = \frac{1-r_n^2}{1-2r_n\cos t + r_n^2} = \frac{1-r_n^2}{(1-r_n)^2 + 4r_n\sin^2(t/2)} \ge 0,$$

Corollary 3 (a) and (b) hold with

$$\varrho_{\alpha,\lambda} = \Big(\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)}(1-r_m)\Big)^{1/2}.$$

(8°) *Gauss-Weierstrass*:

$$k_n(t) = \sqrt{\frac{\pi}{\lambda_n}}\sum_{m=-\infty}^{\infty}\exp\Big\{-\frac{(t-2\pi m)^2}{4\lambda_n}\Big\} \qquad (n\in\mathbb{N}_0,\ t\in\mathbb{R}),$$

where $\{\lambda_n\}_{n\in\mathbb{N}_0}$ is a sequence of positive real numbers converging to zero. We can rewrite $k_n(t)$ as

$$k_n(t) = \sum_{m=-\infty}^{\infty} e^{-\lambda_n m^2}e^{imt} = 1 + 2\sum_{m=1}^{\infty} e^{-\lambda_n m^2}\cos mt.$$

Therefore, Corollary 3 (a) and (b) hold with

$$\varrho_{\alpha,\lambda} = \Big(\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)}(1-e^{-\lambda_m})\Big)^{1/2}.$$

Next, we give several examples of nonperiodic, nonnegative functions $h_n(t)$ satisfying (25) for which Theorem 4 can be applied, from a probabilistic point of view. These can be induced by various probability density functions mentioned as follows:

Let $\{\alpha_n\}_{n\in\mathbb{N}_0}, \{\beta_n\}_{n\in\mathbb{N}_0}$ and $\{\gamma_n\}_{n\in\mathbb{N}_0}$ be sequences of positive real numbers. We define

$$\mu_n(p) = \mu(h_n;p) := \int_{\mathbb{R}} |t|^p h_n(t)\,dt < \infty,$$

which is called the pth absolute moment of h_n.

(9°) *Burr type distribution*:

$$h_n(t) := \begin{cases} \alpha_n\beta_n\gamma_n^{\alpha_n}\frac{t^{\beta_n-1}}{\gamma_n+t_n^{\beta})^{\alpha_n+1}} & (t>0), \\ 0 & (t\le 0). \end{cases}$$

Then we have

$$\mu_n(p) = \frac{p\gamma_n^{p/\beta_n} \Gamma\left(\alpha_n - \frac{p}{\beta_n}\right)\Gamma\left(\frac{p}{\beta_b}\right)}{\beta_n \Gamma(\alpha_n)},$$

and so

$$\mu_n(1) = \frac{\gamma_n^{1/\beta_n} \Gamma\left(\alpha_n - \frac{1}{\beta_n}\right)\Gamma\left(\frac{1}{\beta_n}\right)}{\beta_n \Gamma(\alpha_n)}$$

and

$$\mu_n(2) = \frac{2\gamma_n^{2/\beta_n} \Gamma\left(\alpha_n - \frac{2}{\beta_n}\right)\Gamma\left(\frac{2}{\beta_n}\right)}{\beta_n \Gamma(\alpha_n)}.$$

(10°) *Gauss type distribution*:

$$h_n(t) := \sqrt{\frac{1}{\pi\alpha_n}} \exp\left(-\frac{t^2}{\alpha_n}\right) \qquad (t \in \mathbb{R}).$$

Then we have

$$\mu_n(p) = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{p+1}{2}\right)\alpha_n^{p/2},$$

where

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt \qquad (x > 0)$$

is the gamma function. In particular, we have

$$\mu_n(1) = \sqrt{\frac{\alpha_n}{\pi}}, \qquad \mu_n(2) = \frac{\alpha_n}{2}$$

and

$$\mu_n(2m) = \left(m - \frac{1}{2}\right)\left(m - \frac{3}{2}\right)\left(m - \frac{5}{2}\right)\cdots\frac{3}{2}\frac{1}{2}\alpha_n^m \qquad (m \in \mathbb{N}).$$

(11°) *Laplace type distribution*:

$$h_n(t) := \frac{1}{2\alpha_n} \exp\left(-\frac{|t|}{\alpha_n}\right) \qquad (t \in \mathbb{R}).$$

Then we have

$$\mu_n(p) = p\Gamma(p)\alpha_n^p.$$

In particular, we have

$$\mu_n(m) = m!\alpha_n^m \qquad (m \in \mathbb{N}),$$

and so

$$\mu_n(1) = \alpha_n, \qquad \mu_n(2) = 2\alpha_n^2.$$

(12°) *Student (t) type distribution*:

$$h_n(t) := \sqrt{\frac{\alpha_n}{\pi}} \frac{\Gamma(\beta_n)}{\Gamma\left(\beta_n - \frac{1}{2}\right)} (1 + \alpha_n t^2)^{-\beta_n} \qquad (t \in \mathbb{R}).$$

Then we have

$$\mu_n(p) = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}} \left(\frac{1}{\sqrt{\alpha_n}}\right)^p \frac{\Gamma\left(\beta_n - \frac{p+1}{2}\right)}{\Gamma\left(\beta_n - \frac{1}{2}\right)},$$

and so

$$\mu_n(1) = \frac{1}{\sqrt{\pi\alpha_n}} \frac{\Gamma(\beta_n - 1)}{\Gamma\left(\beta_n - \frac{1}{2}\right)}, \qquad \mu_n(2) = \frac{1}{\alpha_n(2\beta_n - 3)}.$$

(13°) *Gamma type distribution*:

$$h_n(t) := \begin{cases} \frac{\beta_n^{\alpha_n}}{\Gamma(\alpha_n)} t^{\alpha_n - 1} e^{-\beta_n t} & (t > 0), \\ 0 & (t \leq 0). \end{cases}$$

Then we have

$$\mu_n(p) = \frac{1}{\beta_n^p} \frac{\Gamma(p + \alpha_n)}{\Gamma(\alpha_n)}.$$

In particular, we have

$$\mu_n(m) = \frac{1}{\beta_n^m} \prod_{i=0}^{m-1} (\alpha_n + i) \qquad (m \in \mathbb{N}),$$

and so

$$\mu_n(1) = \frac{\alpha_n}{\beta_n}, \qquad \mu_n(2) = \frac{\alpha_n(\alpha_n + 1)}{\beta_n^2}.$$

(14°) *Beta type distribution*:

$$h_n(t) := \begin{cases} \frac{1}{B(\alpha_n, \beta_n)} (t/\delta)^{\alpha_n \gamma_n} (\gamma_n / t) \left(1 - \left(t/\delta\right)^{\gamma_n}\right)^{\beta_n - 1} & (0 < t < \delta), \\ 0 & (t \leq 0 \text{ or } \ \delta \leq t), \end{cases}$$

where

$$B(x, y) := \int_0^1 t^{x-1} (1 - t)^{y-1} \, dt \qquad (x,\ y > 0)$$

is the beta functiion and δ is any fixed positive real numbers. Then we have

$$\mu_n(p) = \delta^p \frac{B\left(\alpha_n + \frac{p}{\gamma_n}, \beta_n\right)}{B(\alpha_n, \beta_n)} = \delta^p \frac{\Gamma(\alpha_n + \beta_n)}{\Gamma(\alpha_n)} \frac{\Gamma\left(\alpha_n + \frac{p}{\gamma_n}\right)}{\Gamma\left(\alpha_n + \beta_n + \frac{p}{\gamma_n}\right)},$$

and so

$$\mu_n(1) = \delta \frac{\Gamma(\alpha_n + \beta_n)}{\Gamma(\alpha_n)} \frac{\Gamma\Big(\alpha_n + \frac{1}{\gamma_n}\Big)}{\Gamma\Big(\alpha_n + \beta_n + \frac{1}{\gamma_n}\Big)}$$

and

$$\mu_n(2) = \delta^2 \frac{\Gamma(\alpha_n + \beta_n)}{\Gamma(\alpha_n)} \frac{\Gamma\Big(\alpha_n + \frac{2}{\gamma_n}\Big)}{\Gamma\Big(\alpha_n + \beta_n + \frac{2}{\gamma_n}\Big)}.$$

In particular,if $\delta = 1$ and $\gamma_n = 1$ for all $n \in \mathbb{N}_0$, then we have

$$\mu_n(m) = \prod_{i=0}^{m-1} \frac{\alpha_n + i}{\alpha_n + \beta_n + i} \qquad (m \in \mathbb{N}),$$

and so

$$\mu_n(1) = \frac{\alpha_n}{\alpha_n + \beta_n}, \qquad \mu_n(2) = \frac{\alpha_n(\alpha_n + 1)}{(\alpha_n + \beta_n)(\alpha_n + \beta_n + 1)}.$$

(15°) *Landau type distribution*:

$$h_n(t) := \begin{cases} \frac{\alpha_n}{2B(1/\alpha_n, \beta_n)}(1 - |t|^{\alpha_n})^{\beta_n - 1} & (|t| \leq 1), \\ 0 & (|t| > 1). \end{cases}$$

Then we have

$$\mu_n(p) = \frac{\Gamma\big(\frac{p+1}{\alpha_n}\big)}{\Gamma\big(\frac{1}{\alpha_n}\big)} \frac{\Gamma\big(\beta_n + \frac{1}{\alpha_n}\big)}{\Gamma\big(\beta_n + \frac{p+1}{\alpha_n}\big)},$$

and so

$$\mu_n(1) = \frac{\Gamma\big(\frac{2}{\alpha_n}\big)}{\Gamma\big(\frac{1}{\alpha_n}\big)} \frac{\Gamma\big(\beta_n + \frac{1}{\alpha_n}\big)}{\Gamma\big(\beta_n + \frac{2}{\alpha_n}\big)}, \qquad \mu_n(2) = \frac{\Gamma\big(\frac{3}{\alpha_n}\big)}{\Gamma\big(\frac{1}{\alpha_n}\big)} \frac{\Gamma\big(\beta_n + \frac{1}{\alpha_n}\big)}{\Gamma\big(\beta_n + \frac{3}{\alpha_n}\big)}.$$

In particular, if $\alpha_n = 2$, then

$$\mu_n(p) = \frac{\Gamma\big(\frac{p+1}{2}\big)}{\sqrt{\pi}} \frac{\Gamma\big(\beta_n + \frac{1}{2}\big)}{\Gamma\big(\beta_n + \frac{p+1}{2}\big)},$$

and so

$$\mu_n(1) = \frac{1}{\sqrt{\pi}} \frac{\Gamma\big(\beta_n + \frac{1}{2}\big)}{\beta_n \Gamma(\beta_n)}, \qquad \mu_n(2) = \frac{1}{2\beta_n + 1}.$$

Furthermore, if $\beta_n = n + 1$, then

$$\mu_n(p) = \Gamma\Big(\frac{p+1}{2}\Big) \frac{\big(n + \frac{1}{2}\big)\big(n - \frac{1}{2}\big)\big(n - \frac{3}{2}\big)\big(n - \frac{5}{2}\big) \cdots \frac{3}{2}\frac{1}{2}}{\Gamma\big(n + \frac{p+3}{2}\big)},$$

and so

$$\mu_n(1) = \frac{(n+\frac{1}{2})(n-\frac{1}{2})(n-\frac{3}{2})(n-\frac{5}{2})\cdots\frac{3}{2}\frac{1}{2}}{(n+1)!}, \qquad \mu_n(2) = \frac{1}{2n+3}.$$

Also, if $\ell_n := 1/\alpha_n \in \mathbb{N}$, then

$$\mu_n(p) = \frac{\Gamma(\ell_n(p+1))}{\Gamma(\ell_n)} \frac{\Gamma(\beta_n+\ell_n)}{\Gamma(\beta_n+\ell_n(p+1))},$$

and so

$$\mu_n(1) = \prod_{i=\ell_n}^{2\ell_n-1} \frac{i}{\beta_n+i}, \qquad \mu_n(2) = \prod_{i=\ell_n}^{3\ell_n-1} \frac{i}{\beta_n+i}.$$

(16°) *Weibull type distribution*:

$$h_n(t) := \begin{cases} \frac{\beta_n}{\alpha_n\gamma_n^{\beta_n}} t^{\beta_n-1} \exp\left(-\frac{t^{\beta_n}}{\alpha_n\gamma_n^{\beta_n}}\right) & (t>0), \\ 0 & (t\le 0). \end{cases}$$

Then we have

$$\mu_n(p) = \frac{p\alpha_n^{p/\beta_n}\gamma_n^p}{\beta_n}\Gamma\left(\frac{p}{\beta_n}\right),$$

and so

$$\mu_n(1) = \frac{\alpha_n^{1/\beta_n}\gamma_n}{\beta_n}\Gamma\left(\frac{1}{\beta_n}\right), \qquad \mu_n(2) = \frac{2\alpha_n^{2/\beta_n}\gamma_n^2}{\beta_n}\Gamma\left(\frac{2}{\beta_n}\right).$$

(17°) *Paretto type distribution*:

$$h_n(t) := \begin{cases} \frac{\gamma_n^{\alpha_n}}{B(\alpha_n,\beta_n)} \frac{t^{\beta_n-1}}{(t+\gamma_n)^{\alpha_n+\beta_n}} & (t>0), \\ 0 & (t\le 0). \end{cases}$$

Then we have

$$\mu_n(p) = \frac{\Gamma(\alpha_n-p)\Gamma(\beta_n+p)}{\Gamma(\alpha_n)\Gamma(\beta_n)}\gamma_n^p,$$

and so

$$\mu_n(1) = \frac{\beta_n\gamma_n}{\alpha_n-1}, \qquad \mu_n(2) = \frac{\beta_n(\beta_n+1)\gamma_n^2}{(\alpha_n-1)(\alpha_n-2)}.$$

Finally, we remark that the estimates given in [21, Theorem 6.1 and Corollary 6.2] (cf. [2]) can be improved, and we omit details.

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