A Note on Quotient Graphs

A Note on Quotient Graphs

Hiroshi MAEHARA*

(Received August 20, 1984)

Abstract

Two kinds of quotient graphs: reduced graphs and modified graphs, are considered and enumerated.

1. Introduction

Our objects are two kinds of quotient graphs. One is the reduced graph introduced by Roberts to characterize the indifference graphs [4]. The other is the modified graph used to describe bounds on a certain combinatorial dimension of a forest [2]. The definitions of these two quotient graphs are very similar and there is a simple relation between them.

In this note we will show some simple results concerning these quotient graphs and enumerate them by a typical application of Robinson's composition theorem [5]. The labeled case is also treated.

2. Some simple results

Throughout, a graph means a finite simple graph. The vertex set of a graph G is denoted by $V(G)$. For a vertex x of G, $N(x)$ and $N[x]$ denote the neighborhood of x and the closed neighborhood of x, respectively, that is, $N(x)$ is the set of vertices adjacent to x and $N[x] = N(x) \cup \{x\}$. Define two binary relation μ and ρ on $V(G)$ by

$$x \ \mu \ y \quad \longleftrightarrow \quad N(x) = N(y),$$

$$x \ \rho \ y \quad \longleftrightarrow \quad N[x] = N(y).$$

These are clearly equivalence relations on $V(G)$.

The *reduction* G^* of a graph G is a graph obtained from G by cancelling out the equivalence relation ρ, i.e. the vertices of G^* are the equivalence classes and adjacency holds between equivalence classes if and only if it holds between their representatives. See Figure 1. The *modification* G° of G is similarly defined using the equivalence relation μ instead of ρ. A graph G is *reduced* if $G^* = G$ and is *modified* if $G^\circ = G$.

It is obvious that $G^{**} = G^*$ and $G^{\circ\circ} = G^\circ$. Hence G^* is a reduced graph and G° is a modified graph. Moreover, G^* and G° are embeddable in G as induced subgraphs, as easily seen.

Since $x \ \mu \ y$ in G if and only if $x \ \rho \ y$ in the complement $\overline{G}$ of G, we have the following theorem.

This research was partially supported by Grant-in-Aid for Scientific Research (No. 59540049), Ministry of Education.

* College of Education, Ryukyu University.

Theorem 1. $\overline{G^\circ} = (\overline{G})^*$.

Let $\mathscr{R}$ be the set of all reduced graphs and $\mathscr{M}$ be the set of all modified graphs.

Corollary 2. *The correspondence* $G \rightarrow \overline{G}$ *is a bijection from* $\mathscr{R}$ *to* $\mathscr{M}$.

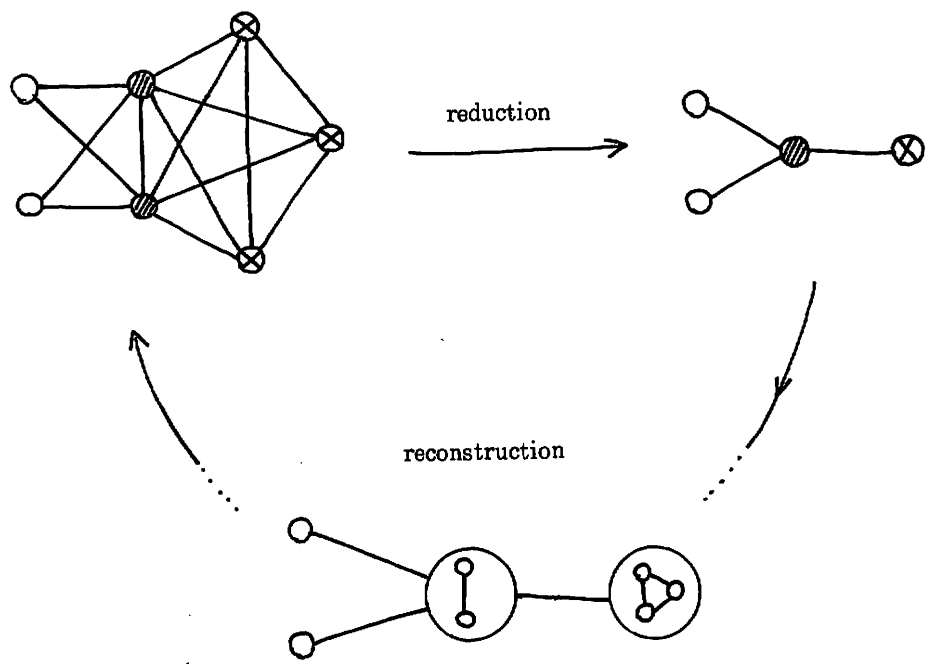

Figure 1

Reduced graphs and modified graphs are considered as basic patterns of graphs in the following sense.

Theorem 3. *Any graph* G *is obtained from* G^* *(or* G° *) by "substituting" for each vertex of* G^* *(or* G°*) an appropriate* K_n *(or* $\overline{K_n}$ *).*

For the proof, see Figure 1.

It is intuitively clear that the reduction does not raise the connectivity of a graph. What about the modification? Denote the connectivity of G by $\kappa(G)$.

Theorem 4. $\kappa(G^\circ) \leqslant \kappa(G)$ *and* $\kappa(G^*) \leqslant \kappa(G)$.

Proof. For $X \subset V(G)$ and $x \varepsilon V(G)$, we denote by X° and x°, the images of X and x under the natural projection $V(G) \rightarrow V(G^\circ)$. Let $k = \kappa(G)$. Then there is a subset $U \subset V(G)$ of size k such that $G-U$ (the removal of U from G) is a disconnected graph or a trivial graph. If $G-U$ is a trivial graph then $G^\circ - U^\circ$ is a trivial graph or empty. Since the size of U° should be at most k, we have $\kappa(G^\circ) \leqslant k$ in this case. Suppose now $G-U$ is disconnected and let x, y be two vertices from two different components of $G-U$. If $x^\circ \neq y^\circ$ then x° and y° must belong to different components of $G^\circ - U^\circ$. Hence $\kappa(G^\circ) \leqslant k$. If $x^\circ = y^\circ$ then $N(x) = N(y) \subset U$ in G. Hence $N(x^\circ) \subset U^\circ$ in G° and the degree of x° in G° is at most k. Hence $\kappa(G^\circ) \leqslant k$, too. The reduction case is easy and omitted.

3. Unlabeled enumeration

By Corollary 2, the number r_n of reduced graphs of order n is equal to the number of modified graphs of order n. To find r_n, we want the cycle index sum $Z(\mathcal{R}_n)$ for the set $\mathcal{R}_n$ of all reduced graphs of order n. Once $Z(\mathcal{R}_n)$ is computed, the number r_n is found by summing up the coefficients. We denote by $\mathcal{G}_n$ the set of all graphs of order n.

Theorem 5. *Using the notation in* [1],

$$(*) \quad \sum_{n=1}^{\infty} Z(\mathcal{R}_n) \left[\sum_{j=1}^{\infty} Z(S_j) \right] = \sum_{n=1}^{\infty} Z(\mathcal{G}_n).$$

This follows easily from Theorem 3 and Robinson's composition theorem [5] (see also [1, p. 182]).

Comparing the terms of order n in both sides of (*), we see that $Z(\mathcal{R}_n)$ consists of the terms of order n in

$$(**) \quad Z(\mathcal{G}_n) - \sum_{k=1}^{n-1} Z(\mathcal{R}_k) \left[\sum_{j=1}^{n-k} Z(S_j) \right].$$

Since $Z(\mathcal{G}_n) = Z(S_n^{(2)}\ ;\ s_k, 2)$ (see [1, p. 166]), we may regard $Z(\mathcal{G}_n)$ as known. Hence starting from $Z(\mathcal{R}_1) = s_1$, we can derive inductively $Z(\mathcal{R}_2)$, $Z(\mathcal{R}_3)$,....

The explicit formula of $Z(\mathcal{G}_n)$ for $3 \leqslant n \leqslant 8$ is easily obtained from the formulas of $Z(S_n)$ and $Z(S_n^{(2)})$ in [1, Appendix III]: We have only to replace the ceofficient of each term of $Z(S_n)$ by the 'corresponding' term of $Z(S_n^{(2)})$ in which, however, every variable s_k should be replaced by 2.

For example, from

$$Z(S_4) = \frac{1}{4\,!} (s_1^4 + 6 s_1^2 s_2 + 8 s_1 s_3 + 3 s_2^2 + 6 s_4)$$

and

$$Z(S_4^{(2)}) = \frac{1}{4\,!} (s_1^6 + 6 s_1^2 s_2^2 + 8 s_3^2 + 3 s_1^2 s_2^2 + 6 s_2 s_4),$$

we have

$$Z(\mathcal{G}_4) = \frac{1}{4\,!} (2^6 s_1^4 + 6 \cdot 2^2 \cdot 2^2 s_1^2 s_2 + 8 \cdot 2^2 s_1 s_3 + 3 \cdot 2^2 \cdot 2^2 s_2^2 + 6 \cdot 2 \cdot 2 s_4).$$

After some calculations I obtained the following results for $Z(\mathcal{R}_n)$.

$$Z(\mathcal{R}_1) = s_1$$

$$Z(\mathcal{R}_2) = \frac{1}{2} s_1^2 + \frac{1}{2} s_2$$

$$Z(\mathcal{R}_3) = \frac{2}{3}s_1^3 + s_1s_2 + \frac{1}{3}s_3$$

$$Z(\mathcal{R}_4) = \frac{4}{3}s_1^4 + \frac{3}{2}s_1^2s_2 + \frac{2}{3}s_1s_3 + s_2^2 + \frac{1}{2}s_4$$

$$Z(\mathcal{R}_5) = \frac{49}{10}s_1^5 + \frac{11}{3}s_1^3s_2 + s_1^2s_3 + \frac{9}{2}s_1s_2^2 + s_1s_4 + \frac{1}{3}s_2s_3 + \frac{3}{5}s_5$$

$$Z(\mathcal{R}_6) = \frac{5369}{180}s_1^6 + \frac{179}{12}s_1^4s_2 + \frac{22}{9}s_1^3s_3 + \frac{67}{4}s_1^2s_2^2 + \frac{3}{2}s_1^2s_4 + \frac{4}{3}s_1s_2s_3 + \frac{6}{5}s_1s_5$$
$$+ \frac{77}{12}s_2^3 + \frac{3}{2}s_2s_4 + \frac{23}{18}s_3^2 + \frac{5}{6}s_6$$

From these results the values of r_n , $n \leqslant 6$ in Table 1 follow. To calculate $Z(\mathcal{R}_7)$ by hand is a hard task, but its coefficient sum is rather easily obtained.

4. Labeled enumeration

Let $\mathcal{F}$ be a family of graphs and $Z(\mathcal{F})$ be the cycle index sum of graphs in $\mathcal{F}$. Let $F(x)$ be the exponential generating function (egf) for labeled enumeration of graphs in $\mathcal{F}$, that is,

$$F(x) = \sum_{n=1}^{\infty} F_n x^n/n!$$

where F_n is the number of different labeled graphs of order n obtained when we label each graph of $\mathcal{F}$ in as many ways as possible. We will follow the notation in [1].

Lemma. $F(x) = Z(\mathcal{F}; x, 0, 0, \ldots, 0)$

Proof. Let G be a graph of order n in $\mathcal{F}$ and $\Gamma(G)$ be the automorphism group of G. Then the number of ways of labeling G is

$$\ell(G) = n!/|\Gamma(G)|,$$

where $|\Gamma(G)|$ is the cardinality of $\Gamma(G)$, see [1, p. 4]. For a permutation $\alpha \in \Gamma(G)$, let $j(\alpha, k)$ be the number of cycles of length k in the disjoint cycle decomposition of α. Then α is the identity element of $\Gamma(G)$ if and only if $j(\alpha, k) = 0$ for all $k \geqslant 2$. Now since the cycle index of $\Gamma(G)$ is

$$Z(G) = |\Gamma(G)|^{-1} \sum_{\alpha \in \Gamma(G)} \prod_{k=1}^{n} s_k^{\,j(\alpha,k)}$$

we have

$$Z(G\,;x,0,0,\ldots,0) = |\Gamma(G)|^{-1}x^n = \ell(G)x^n/n!.$$

Therefore

$$Z(\mathscr{F};x,0,0,\ldots,0) = \sum_{G\in\mathscr{F}} Z(G\,;x,0,\ldots,0) = \sum_{G\in\mathscr{F}} \ell(G)x^n/n! = F(x).$$

Now let $\mathscr{D}, \mathscr{E}, \mathscr{F}$ be three families of graphs, $Z(\mathscr{D}), Z(\mathscr{E}), Z(\mathscr{F})$ be their cycle index sums, and $D(x), E(x), F(x)$ be the egf's for labeled enumeration of graphs in $\mathscr{D}$, in $\mathscr{E}$, in $\mathscr{F}$.

Corollary 6. *If* $Z(\mathscr{D})[Z(\mathscr{E})] = Z(\mathscr{F})$ *then* $D(E(x)) = F(x)$.

Theorem 7. *Let* R_n *be the number of labeled reduced graphs of order* n. *Then*

$$R_n = \sum_{m=1}^{n} s(n,m)\, 2^{\binom{m}{2}},$$

where $s(n,m)$ is the Stirling number of the first kind.

Proof. Let $\mathscr{K}$ be the set of all complete graphs and $\mathscr{G}$ be the set of all graphs. Let $R(x)$, $K(x)$, $G(x)$ be the corresponding egf's for labeled enumeration. Then by Theorem 3, $Z(\mathscr{R})[Z(\mathscr{K})] = Z(\mathscr{G})$, and hence $R(K(x)) = G(x)$ by Corollary 6. Since $K(x) = e^x - 1$ and

$$G(x) = \sum_{k=1}^{\infty} 2^{\binom{k}{2}} x^k/k!,$$

we have

$$\sum_{n=1}^{\infty} R_n (e^x - 1)^n/n! = \sum_{k=1}^{\infty} 2^{\binom{k}{2}} x^k/k!.$$

Note here that $(e^x-1)^n/n!$ is the egf of $S(*,n)$, the Stirling number of the second kind (see [3, p. 43]), that is,

$$(e^x-1)^n/n! = \sum_{k=1}^{\infty} S(k,n)x^k/k!.$$

So we have

$$\sum_{k=1}^{\infty}\Big(\sum_{n=1}^{\infty} R_n S(k,n)\Big)x^k/k! = \sum_{k=1}^{\infty} 2^{\binom{k}{2}} x^k/k!$$

and hence

$$\sum_{n=1}^{\infty} R_n S(k,n) = 2^{\binom{k}{2}},\quad k = 1, 2, 3, \ldots.$$

Now inverting this relation by the Stirling number of the first kind, we have the theorem.

Table 1. Number of reduced graphs.

n	1	2	3	4	5	6	7
r_n	1	1	2	5	16	78	588
R_n	1	1	4	32	588	21476	1551368

References

[1] F. Harary and E. M. Palmer, *Graphical Enumeration*, Academic Press, New York 1973.

[2] S. Poljak and A. Pulter, On the dimension of trees, Discrete Math. 34 (1981) 165–171.

[3] J. Riordan, *An Introduction to Combinatorial Analysis*, Wiley, New York 1958.

[4] F. S. Roberts, Indifference graphs, in *Proof Techniques in Graph Theory* (F. Harary, ed.) pp. 139–146, Academic Press, New York 1969.

[5] R. W. Robinson, Enumeration on non-separable graphs, J. Combinatorial Theory 9 (1970) 327–356.