Theoretiacal basis of elastoplasticity in FE software package

Theoretical basis of elastoplasticity in FE software package

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Abstract

2D elastoplastic FE program has been developed in 2002 (Hayashi, 2002). Improved FE program was applied to the Umaoi hills in Hokkaido, Japan to estimate plate convergence rate (Morishita and Hayashi, 2005; Hayashi and Morishita, 2006). This FE elastoplastic software is derived from 2D von Mises' plastic theory. The detail of elastoplastic theory and FE formulation are explained in this paper.

Introduction

Elasticity is the basic rheology in strain-stress relation. While considering the phenomena that occur in geological bodies, rock and soil show more complex property, i.e., plasticity where the relation of stress-strain is nonlinear and strain is no more recoverable but is permanent. When we assume rock property as plastic, we can treat a large deformation that would be performed during a geological time scale. There are many examples that have treated geological body as plastic material to analyze interesting problems as follows.

Willett(1992) treated the deformation of Coulomb wedge by means of plastic Finite Element Method (FEM), while the development of thrust faults in rectangular and wedge shape has been examined using plastic FEM by Makel and Waters (1993). Duplex structure was simulated using elastoplastic FEM by Erickson (1995). As more larger scale, subduction of elastoplastic plate was carried out and compared with the analog model of Shemenda (1993) by Hassani and Jongmans (1997). The deformation of the Variscan fold-and-thrust belt was simulated using elastic-plastic FEM by Vanbrabant et al. (1999) where the strain propagation and duplex structure were simulated with good agreement to the observation.

The foreland basin formation was simulated using elastoplastic FEM by Zhang and Bott (2000). The asymmetrical basins bounded by high angle reverse faults have been modeled to clarify the narrow and deep foreland basins of European mountain belts. The deformation of overriding Luzon Arc has modeled using elastoplastic FEM by Tang and Chemenda (2000) during continental margin subduction. Moisio et al. (2000) have cleared that with relatively low compressive stress levels the lower crust deforms in a plastic manner for a wet crustal rheology. This simulation was performed in the profile in south eastern Finland and in Estonia using elastoplastic FEM. Erickson et al. (2001) investigated the effects of fault geometry, fault friction, material properties and anisotropy on the initiation and reactivation of faults in the hanging wall of a thrust-fault ramp. The models use an elastic-plastic, frictional, dilatant, cohesion softening material, in which deformation may localize as shear bands by using not FEM but FDM. Subduction-induced crustal shortening is believed to be the primary cause of the Andean mountain building. Luo and Liu (2009) investigated how the cyclic trench coupling leads to long-term mountain building, which has been concentrated in the Subandes in the past few million years.

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Plastic theories

There are several plastic theories, for example, Tresca, von Mises, Mohr-Coulomb and Drucker-Prager (Hill, 1950; Fung, 1965; Zienkiewicz, 1977; Vermeer and Borst, 1984; Oettl et al., 1998; Gerbault et al., 1998; Guo and Li, 2008).

Here I am referring mainly the works of Yamada et al. (1968), Zienkiewicz et al. (1969) and Zienkiewicz (1977) to derive the plastic matrix D_p and parameter $\frac{1}{g}d\bar{\sigma}$ that is necessary to carry out the elastoplastic calculations as follows. Readers should refer to the table of Notations.

Notations

$d\varepsilon$	incremental total strain
$d\varepsilon_e$	incremental elastic strain
$d\varepsilon_p$	incremental plastic strain
$d\sigma$	incremental stress
D_e	elastic stress-strain matrix
f	plastic potential
$\bar{\sigma}$	equivalent stress
g	proportionality constant
df	total differential of f
D_p	plastic stress-strain matrix
$d\sigma_a$	incremental plastic stress
$d\bar{\sigma}$	incremental equivalent stress
$\overline{d\varepsilon_p}$	incremental equivalent plastic strain
H'	strain-hardening rate
dW_p	incremental plastic work

σ_x	σ_{11}
σ_y	σ_{22}
τ_{xy}	σ_{12}
σ'_x	deviatric stress of σ_x
σ'_y	deviatric stress of σ_y
$d\varepsilon_x$	$d\varepsilon_{11}$
$d\varepsilon_y$	$d\varepsilon_{22}$
$d\gamma_{xy}$	$d\varepsilon_{12}$

K	bulk modulus
G	modulus of rigidity
E	Young's modulus
ν	Poisson's ratio

$\{\ \}$	vector
$\{\ \}^T$	transpose vector
A^T	transpose matrix

Constitutive equation of plasticity

Total strain is composed of elastic and plastic parts in vector form as

$$\{d\varepsilon\} = \{d\varepsilon_e\} + \{d\varepsilon_p\} \tag{1}$$

then

$$\{d\varepsilon_e\} = \{d\varepsilon\} - \{d\varepsilon_p\} \tag{2}$$

The increment of stress is presented using the elastic matrix D_e as

$$\{d\sigma\} = D_e\{d\varepsilon\} - D_e\{d\varepsilon_p\} \tag{3}$$

We assume the plastic potential

$$f(\sigma_{ij}) = \bar{\sigma} \tag{4}$$

where f is the plastic potential and $\bar{\sigma}$ is the equivalent stress, and also assume the normality principle as

$$\{d\varepsilon_p\} = \frac{1}{g}\left\{\frac{\partial f}{\partial \sigma}\right\}df \tag{5}$$

Substituting (5) into (3), we have

$$\{d\sigma\} = D_e\{d\varepsilon\} - \frac{1}{g}D_e\left\{\frac{\partial f}{\partial \sigma}\right\}df \tag{6}$$

Since the total differential df is written as

$$df = \left\{\frac{\partial f}{\partial \sigma}\right\}^T\{d\sigma\} \tag{7}$$

substituting (7) into (6), we have

$$df = \left\{\frac{\partial f}{\partial \sigma}\right\}^T D_e\{d\varepsilon\} - \frac{1}{g}\left\{\frac{\partial f}{\partial \sigma}\right\}^T D_e\left\{\frac{\partial f}{\partial \sigma}\right\}df \tag{8}$$

Solving for df, we have

$$\frac{1}{g}df = \frac{\left\{\frac{\partial f}{\partial \sigma}\right\}^T D_e\{d\varepsilon\}}{g + \left\{\frac{\partial f}{\partial \sigma}\right\}^T D_e\left\{\frac{\partial f}{\partial \sigma}\right\}} \tag{9}$$

Then, the final form of $\{d\sigma\}$ is derived from (6)

$$\{d\sigma\} = D_e\{d\varepsilon\} - D_e\left\{\frac{\partial f}{\partial \sigma}\right\}\frac{\left\{\frac{\partial f}{\partial \sigma}\right\}^T D_e\{d\varepsilon\}}{g + \left\{\frac{\partial f}{\partial \sigma}\right\}^T D_e\left\{\frac{\partial f}{\partial \sigma}\right\}} \tag{10}$$

$$= \left[D_e - \frac{D_e \left\{ \frac{\partial f}{\partial \sigma} \right\} \left\{ \frac{\partial f}{\partial \sigma} \right\}^T D_e}{g + \left\{ \frac{\partial f}{\partial \sigma} \right\}^T D_e \left\{ \frac{\partial f}{\partial \sigma} \right\}} \right] \{d\varepsilon\} \quad (11)$$

$$= D_p \{d\varepsilon\} \quad (12)$$

$$= D_e \{d\varepsilon\} - \{d\sigma_a\} \quad (13)$$

Explicit form of g

The explicit form of proportionality constant g is defined by the hypothesis of strain hardening as follows.

When we set up the strain-hardening rate H' as

$$H' = \frac{d\bar{\sigma}}{d\bar{\varepsilon}_P} \quad (14)$$

and consider the normality principle (5)

$$\{d\varepsilon_p\} = \frac{1}{g} \left\{ \frac{\partial f}{\partial \sigma} \right\} df \quad (5)$$

$$= \frac{1}{g} \left\{ \frac{\partial f}{\partial \sigma} \right\} d\bar{\sigma} \quad (16)$$

$$= \frac{1}{g} \left\{ \frac{\partial f}{\partial \sigma} \right\} H' d\bar{\varepsilon}_P \quad (17)$$

Multiplying $\{\sigma\}^T$ to both sides, we have

$$\{\sigma\}^T \{d\varepsilon_p\} = dW_p \quad (18)$$

$$dW_p = \bar{\sigma} d\bar{\varepsilon}_P \quad (19)$$

$$\bar{\sigma} d\bar{\varepsilon}_P = \frac{1}{g} \{\sigma\}^T \left\{ \frac{\partial f}{\partial \sigma} \right\} H' d\bar{\varepsilon}_P \quad (20)$$

We have g from (20) as

$$g = \frac{1}{\bar{\sigma}} \{\sigma\}^T \left\{ \frac{\partial f}{\partial \sigma} \right\} H' \quad (21)$$

$$= \frac{1}{\bar{\sigma}} \{\sigma\}^T \left\{ \frac{\partial f}{\partial \sigma} \right\} \frac{d\bar{\sigma}}{d\bar{\varepsilon}_P} \quad (22)$$

Thus the plastic matrix D_p is rewritten by substituting (21) into (11) as

$$D_p = D_e - \frac{D_e \left\{ \frac{\partial f}{\partial \sigma} \right\} \left\{ \frac{\partial f}{\partial \sigma} \right\}^T D_e}{\frac{1}{\bar{\sigma}} \{\sigma\}^T \left\{ \frac{\partial f}{\partial \sigma} \right\} H' + \left\{ \frac{\partial f}{\partial \sigma} \right\}^T D_e \left\{ \frac{\partial f}{\partial \sigma} \right\}} \quad (23)$$

Another important parameter $\frac{1}{g} d\bar{\sigma}$ is derived from (22) as

$$\frac{1}{g} d\bar{\sigma} = \frac{\bar{\sigma} d\bar{\varepsilon}_P}{\{\sigma\}^T \left\{ \frac{\partial f}{\partial \sigma} \right\}} \quad (24)$$

$$= \frac{dW_p}{\{\sigma\}^T \left\{ \frac{\partial f}{\partial \sigma} \right\}} \quad (25)$$

The other form of $\frac{1}{g} d\bar{\sigma}$ is derived from (9) and (21) as

$$\frac{1}{g} d\bar{\sigma} = \frac{\left\{ \frac{\partial f}{\partial \sigma} \right\}^T D_e \{d\varepsilon\}}{\frac{1}{\bar{\sigma}} \{\sigma\}^T \left\{ \frac{\partial f}{\partial \sigma} \right\} H' + \left\{ \frac{\partial f}{\partial \sigma} \right\}^T D_e \left\{ \frac{\partial f}{\partial \sigma} \right\}} \quad (26)$$

Plastic matrix in plane strain

We are going to derive the plastic matrix D_p and parameter $\frac{1}{g} d\bar{\sigma}$ based on the 2D plastic theory of von Mises in this and next sections. The plastic potential f is defined as

$$f^2 = \sigma_x^{\ 2} - \sigma_x \sigma_y + \sigma_y^{\ 2} + 3\tau_{xy}^{\ 2} \quad (27)$$

The derivative form of f is

$$\left\{ \frac{\partial f}{\partial \sigma} \right\}^T = \frac{3}{2\bar{\sigma}} \{\sigma'_x \quad \sigma'_y \quad 2\tau_{xy}\} \quad (28)$$

The form of plastic matrix D_p is

$$D_p = D_e - \frac{1}{\frac{4\bar{\sigma}^2}{9} H' + (s_1 \sigma'_x + s_2 \sigma'_y + 2 s_6 \tau_{xy})} \begin{bmatrix} s_1^{\ 2} & & sym \\ s_1 s_2 & s_2^{\ 2} & \\ s_1 s_6 & s_2 s_6 & s_6^{\ 2} \end{bmatrix} \quad (29)$$

where s_1, s_2 and s_6 are

$$s_1 = \left(K + \frac{4}{3} G \right) \sigma'_x + \left(K - \frac{2}{3} G \right) \sigma'_y \quad (30)$$

$$s_2 = \left(K - \frac{2}{3} G \right) \sigma'_x + \left(K + \frac{4}{3} G \right) \sigma' \quad (31)$$

$$s_6 = 2G\tau_{xy} \quad (32)$$

The parameter $\frac{1}{g} d\bar{\sigma}$ is

$$\frac{1}{g} d\bar{\sigma} = \frac{s_1 d\varepsilon_x + s_2 d\varepsilon_y + s_6 d\gamma_{xy}}{\frac{2\bar{\sigma}}{3} H' + \frac{3}{2\bar{\sigma}} (s_1 \sigma'_x + s_2 \sigma'_y + 2 s_6 \tau_{xy})} \quad (33)$$

Plastic matrix in plane stress

The form of plastic matrix D_p and the parameter $\frac{1}{g} d\bar{\sigma}$ are same as in the plane strain case except for the definition of s_1, s_2 and s_6 as

$$s_1 = \frac{E}{1 - \nu^2} \sigma'_x + \frac{\nu E}{1 - \nu^2} \sigma'_y \quad (34)$$

$$s_2 = \frac{\nu E}{1 - \nu^2} \sigma'_x + \frac{E}{1 - \nu^2} \sigma'_y \quad (35)$$

$$s_6 = \frac{E}{1 + \nu} \tau_{xy} \quad (36)$$

Procedure of elastoplastic calculation

How to carry out the elastic and plastic calculations is written as follows.

(1) calculate strain and stress under elastic state to obtain a stable stress-strain condition
(2) calculate first (1^{st}) plastic potential using (27)
(3) judge whether elements are yielded to plastic state for each element using yield stress
(4) judge whether elements are recovered to elastic state for each element using (33)
(5) calculate 1^{st} plastic matrix D_p using (29)
(6) calculate 1^{st} increment of strain under 1^{st} incremental boundary condition
(7) calculate 1^{st} increment of stress using (12)
(8) calculate second (2^{nd}) plastic matrix D_p
(9) repeat from (2)

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