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Theory of Superconductivity on the Attractive Hubbard Model in the Strong Coupling Limit

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Abstract

The superconducting transition temperature is calculated on the simplest one-band Hubbard model with the negative U in the strong coupling region. Solving the self-consistent equation for renormalization factor λ which renormalizes the effect of the interaction in all orders of U, dependence of superconducting transition temperature on interaction in the strong coupling region is obtained. Resulting behavior of the transition temperature is consistent with physical intuition. T_c decreases in proportion to $t^2/|U|$ and $T_c \rightarrow 0$ at $|U| \rightarrow \infty$.

1. Introduction

The problem of the crossover from BCS superconductivity to Bose-Einstein condensation has been attracted much interest by many authers.[1-6] In the strong coupling region, electrons form bound pairs, for which Bose-Einstein condensation temperature T_c is identical to superconducting transition temperature. Temperatures T is in a region $T_c < T < T_U$, where T_U is a charactaristic temperature, electrons are bound into pairing state but not in the superconducting state.[7] This precursor pairing state is regarded as "pseudo-gap" state,[8] which is the hottest issue on the high temperature cuprate superconductors.[9] Therefore, it is very important to investigate the superconductivity in the strong attractive interaction limit. Then, we calculate the superconducting transition temperature of the attractive (negative-U) Hubbard model on the tightly binding lattice system in the strong attractive interaction limit. In such a case, the BCS type gap equation can not be used. We introduce the renormalization factor λ which renormalizes the effect of the interaction in all orders of U, and a virtual external field which induces the pairs of electrons above T_c. Then, we can define the pair-pair correlation function for electron pairs, which is corresponding to thermal susceptibility in some limiting case. Above T_c, this thermal susceptibility has a finite value. Lowering the temperature, however, this susceptibility diverges at T_c. This behavior is analogous to following thermal equation in case of the ferromagnet,

$$M = \chi H, \qquad \chi = \frac{M}{H}, \tag{1}$$

Here, M, χ, and H are magnetization, magnetic susceptibility, and magnetic external field which correspond to anomalous statistical average of electron pair, thermal susceptibility, and virtual external field, respectively. Above ferromagnetic transition temperature T_{Curie}, χ has a finite value for finite M and H, but at T_{Curie}, M spontaneously has finite value even though $H = 0$ because of divergence of χ. Therefore, in this paper, we obtain the superconducting transition temperature T_c as the temperature at which the thermal suscptibility diverge. Then, virtual external field breaks the $U(1)$ gauge symmetry of the system, and eventually, is set zero following the Bogoliubov's quasi-average method.[10]

2. Formulation

2.1 Definition and thermal quantities

2.1.1 Hamiltonian

We use the following Hubbard Hamiltonian,

$$\mathcal{H} = \mathsf{H} + \mathsf{H}_{\mathrm{ext}}, \tag{2}$$

$$\mathsf{H} = \mathsf{H}_0 + \mathsf{H}_1, \tag{3}$$

$$\mathsf{H}_0 = \sum_{i,j,s}(-t_{ij} - \mu\delta_{ij})a^\dagger_{is}a_{js}, \tag{4}$$

$$\mathsf{H}_1 = U\sum_i n_{i\uparrow}n_{i\downarrow}, \tag{5}$$

where $a^\dagger_{is}(a_{is})$ is the creation (annihilation) operator at i-th site with spin s and $n_{is} = a^\dagger_{is}a_{is}$ is the number operator. t_{ij}, μ and $U < 0$ are the electron hopping term between i-and j-th site, chemical potential and inter-electron attraction, respectively. $\mathsf{H}_{\mathrm{ext}}$ is the virtual external field term defined as follows:

$$\mathsf{H}_{\mathrm{ext}} = -\sum_i \left[\Delta_i a^\dagger_{i\uparrow}a^\dagger_{i\downarrow} + \Delta^*_i a_{i\downarrow}a_{i\uparrow}\right], \tag{6}$$

where Δ_i is the complex amplitude of the field at i-th site.

Hereafter, using the Nanbu representation, we rewrite the Hubbard Hamiltonian for convenience.

$$A_i = \begin{pmatrix} a_{i\uparrow} \\ a^\dagger_{i\downarrow} \end{pmatrix}, \qquad A^\dagger_i = \begin{pmatrix} a^\dagger_{i\uparrow} & a_{i\downarrow} \end{pmatrix}, \tag{7}$$

$$\begin{aligned}
\mathcal{H} &= \mathsf{H}_0 + \mathsf{H}_1 + \mathsf{H}_{\mathrm{ext}} \\
&= \sum_{i,j} A^\dagger_i \mathcal{E}^{(0)}_{ij} A_j + \mathcal{E}_0 + U\sum_i \left(A^\dagger_i \hat{s}_+ A_i\right)\left(A^\dagger_i \hat{s}_- A_i\right) \\
&- \sum_i \left[\Delta_i \left(A^\dagger_i \hat{s}_+ A_i\right) + \Delta^*_i \left(A^\dagger_i \hat{s}_- A_i\right)\right],
\end{aligned} \tag{8}$$

where

$$\mathcal{E}_{ij}^{(0)} = (-t_{ij} - \mu\delta_{ij})\hat{s}_3, \quad \mathcal{E}_0 = \sum_i (-t_{ii} - \mu), \tag{9}$$

and

$$\hat{s}_{\pm} \equiv \frac{1}{2}(\hat{s}_1 \pm i\hat{s}_2),$$

which is defined by the Pauli matrices

$$\hat{s}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hat{s}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \hat{s}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

2.1.2 Thermal quantities

Here, we define the partition function $\Xi(\{\Delta_i, \Delta_i^*\})$ and the free energy $\Phi(\{\Delta_i, \Delta_i^*\})$ under the Hamiltonian of the eq.(8),

$$\begin{aligned} \Xi(\{\Delta_i, \Delta_i^*\}) &\equiv \mathrm{Tr}e^{-\beta\mathcal{H}} \\ \Phi(\{\Delta_i, \Delta_i^*\}) &\equiv -\frac{1}{\beta}\log\Xi(\{\Delta_i, \Delta_i^*\}) \end{aligned} \tag{10}$$

where $\beta^{-1} = k_B T$, k_B the Boltzmann constant. Then, from this free energy we difine

$$D_i^* = -\frac{\partial\Phi}{\partial\Delta_i}, \quad D_i = -\frac{\partial\Phi}{\partial\Delta_i^*} \tag{11}$$

where

$$D_i \equiv \langle A_i^{\dagger}\hat{s}_- A_i\rangle = \langle a_{i\downarrow}a_{i\uparrow}\rangle \tag{12}$$

which is anomalous statistical average of the electron pair $\langle a_{i\downarrow}a_{i\uparrow}\rangle$. D_i is an order parameter of the superconducting state. Using the Legendre transformation,

$$\Omega(\{D_i, D_i^*\}) = \Phi(\{\Delta_i, \Delta_i^*\}) + \sum_i (\Delta_i D_i^* + \Delta_i^* D_i), \tag{13}$$

we replace the independent variables Δ_i, Δ_i^* by D_i, D_i^* and obtain

$$\frac{\partial\Omega}{\partial D_i} = \Delta_i^*, \quad \frac{\partial\Omega}{\partial D_i^*} = \Delta_i. \tag{14}$$

By making use of above equations, we obtain

$$\Phi(\{\Delta_i, \Delta_i^*\}) = \Phi_0(\{\Delta_i, \Delta_i^*\}) + \int_0^U dU' \left\langle \frac{\partial \mathrm{H}_1}{\partial U'} \right\rangle_{\Delta, U'} \tag{15}$$

$$\Omega(\{D_i, D_i^*\}) = \Omega_0(\{D_i, D_i^*\}) + \int_0^U dU' \left\langle \frac{\partial \mathrm{H}_1}{\partial U'} \right\rangle_{D, U'} \tag{16}$$

where Φ_0 and Ω_0 are free energies for $U=0$. Introducing the notations

$$\Delta_i = \Delta_i^- \quad , \quad \Delta_i^* = \Delta_i^+ ,$$
$$D_i = D_i^- \quad , \quad D_i^* = D_i^+ ,$$

we define the thermal susceptibility;

$$\chi_{ij}^{\rho\sigma} = \frac{\partial D_i^\rho}{\partial \Delta_j^{-\sigma}} = -\frac{\partial^2 \Phi}{\partial \Delta_i^{-\rho} \partial \Delta_j^{-\sigma}} \tag{17}$$

ehere ρ, $\sigma = \pm$.

2.1.3 Pair-pair correlation function

In this section, we define the pair correlation function, *i.e.*, the propagator of the electron pair by following equation.

$$\begin{aligned} \chi_U^{\rho\sigma}(i,j;\tau_1-\tau_2) \equiv & \left\langle T_\tau\{A_{iK}^\dagger(\tau_1)\hat{s}_\rho A_{iK}(\tau_1)\}\{A_{jK}^\dagger(\tau_2)\hat{s}_\sigma A_{jK}(\tau_2)\}\right\rangle \\ - & \left\langle A_i^\dagger \hat{s}_\rho A_i\right\rangle \left\langle A_j^\dagger \hat{s}_\sigma A_j\right\rangle \end{aligned} \tag{18}$$

where T_τ is the time-orderd product and $Q_K(\tau) = e^{\tau\mathcal{H}} Q e^{-\tau\mathcal{H}}$ is the Heisenberg picture of Q. Using the Fourier transformation of the correlation function with regard to $\tau (= \tau_1 - \tau_2)$ is given by

$$\chi_U^{\rho\sigma}(i,j;\tau) = \frac{1}{\beta}\sum_\nu \chi_U^{\rho\sigma}(i,j;\mathrm{i}\nu)e^{-\mathrm{i}\nu\tau}, \quad \left(\nu = \frac{2\pi}{\beta}n, \quad n = 0, \pm1, \pm2, \pm3, \cdots\right) \tag{19}$$

we obtain following relation;

$$\left\langle \frac{\partial \mathrm{H}_1}{\partial U}\right\rangle = \frac{1}{\beta}\sum_{i,\nu}\chi_U^{-+}(i,i;\mathrm{i}\nu)e^{+\mathrm{i}\nu 0_+} + \sum_i |D_i|^2 .$$

Then, eqs.(15) and (16) are rewritten as

$$\Phi(\{\Delta_i, \Delta_i^*\}) = \Phi_0(\{\Delta_i, \Delta_i^*\}) + \sum_i \int_0^U dU' |D_i|^2 + \frac{1}{\beta}\sum_{i,\nu}\int_0^U dU' \chi_{\Delta,U'}^{-+}(i,i;\mathrm{i}\nu)e^{+\mathrm{i}\nu 0_+}, \tag{20}$$

$$\Omega(\{D_i, D_i^*\}) = \Omega_0(\{D_i, D_i^*\}) + U\sum_i |D_i|^2 + \frac{1}{\beta}\sum_{i,\nu}\int_0^U dU' \chi_{D,U'}^{-+}(i,i;\mathrm{i}\nu)e^{+\mathrm{i}\nu 0_+}. \tag{21}$$

In the above, 0_+ is a infinitesimal positive constant. We note that D_i and D_i^* depend on U and that integration in eq.(20) can not be carried out. Neverthless, integration in eq.(21) is only multiplying by U because of D_i and D_i^* are the independent variables. Thus, we mainly treat the free energy $\Omega(\{D_i, D_i^*\})$.

2.2 Uniform system

From now on, we treat spatially uniform system with the translational symmetry. Therefore, assuming D_i and Δ_i real we rewrite

$$\begin{cases} D_i = D_i^* = D, \\ \Delta_i = \Delta_i^* = \Delta, \\ \chi_{ij}^{\rho\sigma} = \chi^{\rho\sigma}. \end{cases} \tag{22}$$

Then, eqs.(11), (13) and (14) become

$$\frac{\partial \Phi}{\partial \Delta} = -2ND, \tag{23}$$

$$\Omega(D) = \Phi(\Delta) + 2ND\Delta, \tag{24}$$

$$\frac{\partial \Omega}{\partial D} = 2N\Delta. \tag{25}$$

We define the matrix representation of the thermal susceptibility as

$$\underline{\chi} \equiv \begin{pmatrix} \chi^{-+} & \chi^{--} \\ \chi^{++} & \chi^{+-} \end{pmatrix}, \tag{26}$$

where the above matrix elements astisfy next relation

$$\chi^{-+} = \chi^{+-}, \quad \chi^{--} = \chi^{++}. \tag{27}$$

We difine macroscopic thermal susceptibility χ,

$$\chi \equiv \frac{\partial D}{\partial \Delta} = -\frac{1}{2N}\frac{\partial^2 \Phi}{\partial \Delta^2} \tag{28}$$

then, we can see the relation

$$\chi = \chi^{-+} + \chi^{--} = \chi^{+-} + \chi^{++}. \tag{29}$$

Furthermore, free energy eq.(21) is modified by

$$\Omega(D) = \Omega_0(D) + UND^2 + \frac{1}{\beta}\sum_{q,\nu}\int_0^U dU' \chi_{D,U'}^{-+}(q, i\nu)e^{+i\nu 0_+}. \tag{30}$$

2.2.1 Gap equation

In this section, we derive the gap equation for D using free energy Φ_0 for $U=0$ and some relation among thermal quantities. In this case, Hamiltonian is reduced as

$$\begin{aligned}\mathcal{H}^{(0)} &= \mathrm{H}_0 + \mathrm{H}_{\mathrm{ext}} \\ &= \sum_{i,j,s}(-t_{ij} - \mu\delta_{ij})a_{is}^{\dagger}a_{js} - \sum_{i}\left[\Delta_i a_{i\uparrow}^{\dagger}a_{i\downarrow}^{\dagger} + \Delta_i^{*}a_{i\downarrow}a_{i\uparrow}\right]. \end{aligned} \tag{31}$$

Taking account of the following Fourier transformation,

$$a_{is} = \frac{1}{\sqrt{N}}\sum_{k} a_{ks}e^{\mathrm{i}k\cdot R_i}, \quad a_{is}^{\dagger} = \frac{1}{\sqrt{N}}\sum_{k} a_{ks}^{\dagger}e^{-\mathrm{i}k\cdot R_i}, \tag{32}$$

Hamiltonian $\mathcal{H}^{(0)}$ is represented as

$$\mathcal{H}^{(0)} = \sum_{k,s}(\epsilon_k - \mu)a_{ks}^{\dagger}a_{ks} - \sum_{k}\left[\Delta a_{k\uparrow}^{\dagger}a_{-k\downarrow}^{\dagger} + \Delta^{*}a_{-k\downarrow}a_{k\uparrow}\right]. \tag{33}$$

The partition function and the free energy for $\mathcal{H}^{(0)}$ are given by

$$\begin{aligned}\Xi_0(\Delta,\Delta^{*}) &\equiv \mathrm{Tr}e^{-\beta\mathcal{H}^{(0)}}, \\ \Phi_0(\Delta,\Delta^{*}) &= -\frac{1}{\beta}\log\Xi_0(\Delta,\Delta^{*}). \end{aligned} \tag{34}$$

Then, we obtain

$$\begin{aligned}\frac{\partial\Phi_0}{\partial\Delta} &= -\frac{1}{\beta}\frac{1}{\Xi_0}\frac{\partial\Xi_0}{\Delta} \\ &= -\frac{1}{\beta}\sum_{k,\omega}\frac{\partial}{\partial\Delta}\log(\omega^2 + E^2). \end{aligned} \tag{35}$$

Integrating this equation with respect to Δ, we can obtain free enegy Φ_0 for $U=0$ as follows:

$$\Phi_0 = -\sum_{k}(E_k - \xi_k) - \frac{2}{\beta}\sum_{k}\log\left(1 + e^{\beta E_k}\right), \tag{36}$$

where $E_k = \sqrt{\xi_k^2 + |\Delta|^2}$ and $\xi_k = \epsilon_k - \mu$. From eq.(23) for above Φ_0, we can get

$$\begin{aligned} D &= -\frac{1}{2N}\frac{\partial\Phi_0}{\partial\Delta} \equiv \Lambda_0(\Delta) \\ &= \frac{1}{2N}\sum_{k}\frac{\Delta}{E_k}\tanh\frac{\beta E_k}{2} \end{aligned} \tag{37}$$

This equation is formally solved, so that we can get Δ_0 as function of D,

$$\Delta_0(D) \equiv \Delta(D, U=0) = \Lambda_0^{-1}(D). \tag{38}$$

Then eqs.(24) and (25) are modified as

$$\Omega_0(D) = \Phi_0(\Delta_0) + 2ND\Delta_0, \tag{39}$$

$$\frac{\partial \Omega_0}{\partial D} = 2N\Delta_0. \tag{40}$$

For simplicity, we define a quantity Ω_1 below,

$$\Omega_1 \equiv \frac{1}{\beta}\sum_{q,\nu}\int_0^U dU' \chi^{-+}_{D,U'}(q, i\nu)e^{+i\nu 0_+}, \tag{41}$$

$$\Omega(D) = \Omega_0(D) + UND^2 + \Omega_1.$$

Then, from eq.(25), we obtain

$$\Delta = \frac{1}{2N}\frac{\partial \Omega}{\partial D} = \Delta_0 + UD + \frac{1}{2N}\frac{\partial \Omega_1}{\partial D}. \tag{42}$$

Following the Bogoliubov's quasi-average method, we put $\Delta = 0$ and get the gap equation,

$$\Lambda_0^{-1}(D) = -UD - \frac{1}{2N}\frac{\partial \Omega_1}{\partial D},$$

$$D = \Lambda_0\left(-UD - \frac{1}{2N}\frac{\partial \Omega_1}{\partial D}\right). \tag{43}$$

Moreover, eq.(42) gives rise to an important thermodynamic equation,

$$\begin{aligned} \frac{\partial \Delta}{\partial D} &\equiv \frac{1}{\chi} = \frac{\partial \Delta_0}{\partial D} + U + \frac{1}{2N}\frac{\partial^2 \Omega_1}{\partial D^2}, \\ &= \frac{1}{\chi_0} + U + \frac{1}{2N}\frac{\partial^2 \Omega_1}{\partial D^2}, \qquad (44) \\ &= \left[\frac{\chi_0}{1 + U\chi_0 + \frac{\chi_0}{2N}\frac{\partial^2 \Omega_1}{\partial D^2}}\right]^{-1}, \qquad (45) \end{aligned}$$

where χ_0 is defined by

$$\chi_0 \equiv \frac{\partial D}{\partial \Delta_0} = \frac{\partial \Lambda_0(\Delta_0)}{\partial \Delta_0}. \tag{46}$$

3. Correlation function and renormalization

Here, using following notation

$$A_i^{\dagger}\hat{s}_{\rho}A_i \equiv B_i^{\rho}$$

we can express the virtual external field term $\mathsf{H}_{\mathrm{ext}}$ and pair-pair correlation function $\chi_U^{\rho\sigma}(i,j;\tau)$ as follows:

$$\mathsf{H}_{\mathrm{ext}} = -\sum_i \left[\Delta_i B_i^{+} + \Delta_i^{*} B_i^{-}\right].$$

$$\chi_U^{\rho\sigma}(i,j;\tau) = \langle T_\tau B_{iK}^{\rho}(\tau) B_{jK}^{\sigma}(0)\rangle - \langle B_i^{\rho}\rangle \langle B_j^{\sigma}\rangle. \quad (47)$$

3.1 Susceptivility

We define the "external field picture" instead of the usual interaction picture, that is,

$$Q(\tau) = e^{\tau \mathsf{H}} Q e^{-\tau \mathsf{H}},$$
$$\mathsf{H} = \mathsf{H}_0 + \mathsf{H}_1.$$

Under this picture, we can calculate the thermal susceptiility concretely. From eqs.(10) and (17), we get

$$\chi_{ij}^{\rho\sigma} = \frac{1}{\beta}\left[\frac{1}{\Xi}\frac{\partial^2 \Xi}{\partial \Delta_i^{-\rho}\partial \Delta_j^{-\sigma}} - \frac{1}{\Xi}\frac{\partial \Xi}{\partial \Delta_i^{-\rho}}\frac{1}{\Xi}\frac{\partial \Xi}{\partial \Delta_j^{-\sigma}}\right]. \quad (48)$$

Treating the $\mathsf{H}_{\mathrm{ext}}$ as a "perturbation", we can obtain

$$\begin{aligned}\frac{\partial \Xi}{\partial \Delta_i^{-\rho}} &= -\int_0^{\beta} d\tau \Xi \langle B_i^{\rho}\rangle = -\beta \Xi \langle B_i^{\rho}\rangle, \\ \frac{\partial^2 \Xi}{\partial \Delta_i^{-\rho}\partial \Delta_j^{-\sigma}} &= \Xi \int_0^{\beta} d\tau_1 \int_0^{\beta} d\tau_2 \langle T_\tau B_{iK}^{\rho}(\tau) B_{jK}^{\sigma}(0)\rangle.\end{aligned} \quad (49)$$

Substituing eq.(49) to eq.(48),

$$\chi_{ij}^{\rho\sigma} = \frac{1}{\beta}\int_0^{\beta} d\tau_1 \int_0^{\beta} d\tau_2 \left[\langle T_\tau B_{iK}^{\rho}(\tau) B_{jK}^{\sigma}(0)\rangle - \langle B_i^{\rho}\rangle\langle B_j^{\sigma}\rangle\right]. \quad (50)$$

Furthermore from eq.(47), we see that

$$\chi_{ij}^{\rho\sigma} = \frac{1}{\beta}\int_0^{\beta} d\tau_1 \int_0^{\beta} d\tau_2 \chi_U^{\rho\sigma}(i,j;\tau). \quad (51)$$

Using the Fourier transformation eq.(19),

$$\begin{aligned}\chi_{ij}^{\rho\sigma} &= \frac{1}{\beta^2}\sum_{\nu}\chi_U^{\rho\sigma}(i,j;\mathrm{i}\nu)\int_0^{\beta} d\tau_1 \int_0^{\beta} d\tau_2 e^{-\mathrm{i}\nu\tau} \\ &= \chi_U^{\rho\sigma}(i,j;\mathrm{i}\nu = 0)\end{aligned} \quad (52)$$

which is derived as a equality between pair-pair correlation function for $\nu = 0$ and thermal susceptibility.

3.2 Introduce of the renormalization factor

Under the usual interaction picture with Hamiltonian $\mathcal{H} = \mathcal{H}^{(0)} + \mathrm{H}_1$,

$$Q_I(\tau) = e^{\tau\mathcal{H}^{(0)}} Q e^{-\tau\mathcal{H}^{(0)}},$$

the Linked-cluster expansion in the first term of eq.(47) leads to

$$\langle T_\tau B^\rho_{iK}(\tau) B^\sigma_{jK}(0)\rangle = \langle T_\tau W(\beta) B^\rho_{iI}(\tau) B^\sigma_{jI}(0)\rangle_{0C} + \langle B^\rho_i\rangle \langle B^\sigma_j\rangle, \tag{53}$$

where

$$W(\beta) \equiv e^{\beta \mathrm{H}_1} e^{-\beta\mathcal{H}} = T_\tau exp\left\{-\int_0^\beta d\tau \mathrm{H}_{1I}(\tau)\right\}.$$

Subscripts $_0$ and $_C$ mean average for $U = 0$ and $W(\beta)$ has linked with both $B^\rho_{iI}(\tau)$ and $B^\sigma_{jI}(0)$ at the same time, respectively. Then. pair-pair correlation function become

$$\chi^{\rho\sigma}_U(i,j;\tau) = \langle T_\tau W(\beta) B^\rho_{iI}(\tau) B^\sigma_{jI}(0)\rangle_{0C}. \tag{54}$$

This 2-body Green function obeys following Dyson equation,

$$\underline{\chi}(q) = \underline{\chi}_*(q) - U\underline{\chi}_*(q)\cdot\underline{\chi}(q), \quad q \equiv (\boldsymbol{q}; \mathrm{i}\nu) \tag{55}$$

where $\underline{\chi}_*(q)$ is the irreducible part of $\underline{\chi}(q)$. The underline means the matrix form of the functions given by eq.(26). Here, we introduce the renormalization factor $\underline{\lambda}(q)$ following self-consistent renormalization (SCR) theory by Moriya and Kawabata,[11]

$$\underline{\lambda}(q) \equiv \underline{\chi}_0(q)\cdot\underline{\chi}_*^{-1}(q) - 1. \tag{56}$$

Then, the Dyson equation is modified as

$$\underline{\chi}^{-1}(q) = \underline{\chi}_0^{-1}(q)\cdot\left[1 + U\underline{\chi}_0(q) + \underline{\lambda}(q)\right]. \tag{57}$$

We define the matrix formed function $\underline{\Pi}(q)$ as

$$\underline{\Pi}(q) \equiv 1 + U\underline{\chi}_0(q) + \underline{\lambda}(q), \tag{58}$$

then $\underline{\chi}(q)$ can be expressed as

$$\begin{aligned} \underline{\chi}(q) &= \underline{\Pi}^{-1}(q)\cdot\underline{\chi}_0(q), \\ \underline{\Pi}(q) &= \underline{\chi}_0(q)\cdot\underline{\chi}^{-1}(q). \end{aligned} \tag{59}$$

Here, $\underline{\chi}_*(q)$ is unknown and includes all order of U. Above $\underline{\chi}(q)$ is rigorous and unambiguous, since all of unknowon parts and U-dependence of $\underline{\chi}_*(q)$'s are renormalized into the factor $\underline{\lambda}(q)$.

3.2.1 Thermodynamical relation

Under the translational symmetry, the following relation has been held between the thermal susceptibility and pair-pair correlation function,

$$\begin{aligned}\chi^{\rho\sigma} &= \frac{1}{N^2}\sum_{i,j}\sum_{q}\chi_U^{\rho\sigma}(\boldsymbol{q};\mathrm{i}\nu=0)e^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{R}_i-\boldsymbol{R}_j)}\\ &= \chi_U^{\rho\sigma}(\boldsymbol{q}=0;\mathrm{i}\nu=0)=\chi_U^{\rho\sigma}(0).\end{aligned} \tag{60}$$

Therefore, we can derive thermodynamic relation and define thermodynamic renormalization factor λ. From eq,(27), for $q=0$, we find

$$\begin{aligned}&\chi_U^{-+}(0)=\chi_U^{+-}(0),\chi_U^{--}(0)=\chi_U^{++}(0),\chi_0^{-+}(0)=\chi_0^{+-}(0)\\ &\chi_0^{--}(0)=\chi_0^{++}(0),\Pi^{-+}(0)=\Pi^{+-}(0),\Pi^{--}(0)=\Pi^{++}(0)\end{aligned} \tag{61}$$

Substitution of eq.(59) into eq.(29) leads to

$$\begin{aligned}\chi &= \chi^{-+}+\chi^{--}=\chi_U^{-+}(0)+\chi_U^{--}(0)=\frac{\chi_0}{\Pi^{-+}(0)+\Pi^{--}(0)},\\ \chi_0 &\equiv \chi_0^{-+}(0)+\chi_0^{--}(0)=\chi_0^{+-}(0)+\chi_0^{++}(0).\end{aligned} \tag{62}$$

Taking account of the elements of eq.(58), we get

$$\Pi^{-+}(q)+\Pi^{--}(q)=1+U\chi_0(q)+\lambda^{-+}(q)+\lambda^{--}(q). \tag{63}$$

We define the thermal renormalization factor

$$\lambda\equiv\lambda^{-+}(0)+\lambda^{--}(0), \tag{64}$$

and then we obtain

$$\chi=\frac{\chi_0}{1+U\chi_0+\lambda}, \tag{65}$$

or

$$\frac{1}{\chi}=\frac{1}{\chi_0}+U+\frac{\lambda}{\chi_0}. \tag{66}$$

Comparing the above expression and eq.(44), determination equation for λ is obtained as follows,

$$\lambda=\frac{\chi_0}{2N}\frac{\partial^2\Omega_1}{\partial D^2}, \tag{67}$$

Moreover, we can determine the transition temperature T_c from the equation,

$$1+U\chi_0+\lambda=0, \tag{68}$$

because the thermal susceptibility χ diverges when the temperature aproaches to T_c.

4. Self consistent equation for λ

To obtain the self consistent equation for renormalization factor λ, we will make some approximation to the integral of eq.(41).

First, we rewrite

$$\begin{aligned}\Omega_1 &= \frac{1}{\beta}\sum_q \int_0^U dU' \chi^{-+}_{D,U'}(q) e^{+i\nu 0_+} \\ &= \frac{1}{\beta}\sum_q \int_0^U dU' \chi^{-+}_{D,0}(q) e^{+i\nu 0_+} + \frac{1}{\beta}\sum_q \int_0^U dU' \left[\chi^{-+}_{D,U'}(q) - \chi^{-+}_{D,0}(q)\right], \end{aligned} \tag{69}$$

and define

$$\Delta\Omega \equiv \frac{1}{\beta}\sum_q \int_0^U dU' \left[\chi^{-+}_{D,U'}(q) - \chi^{-+}_{D,0}(q)\right], \tag{70}$$

$$\Omega_2 \equiv \frac{U}{\beta}\sum_q \chi^{-+}_{D,0}(q) e^{+i\nu 0_+}. \tag{71}$$

Then, eq.(41) can be rewritten

$$\Omega_1 = \Omega_2 + \Delta\Omega. \tag{72}$$

Using eq.(A·3), $\Delta\Omega$ is expressed as

$$\begin{aligned}\Delta\Omega &= \frac{1}{\beta}\sum_q \int_0^U dU' \frac{1}{2}\left[\left\{\chi^{-+}_{D,U'}(q) - \chi^{-+}_{D,0}(q)\right\} + \left\{\chi^{-+}_{D,U'}(-q) - \chi^{-+}_{D,0}(-q)\right\}\right] \\ &= \frac{1}{2\beta}\sum_q \int_0^U dU' \left[\left\{\chi^{-+}_{D,U'}(q) + \chi^{+-}_{D,U'}(q)\right\} - \left\{\chi^{-+}_{D,0}(q) + \chi^{+-}_{D,0}(q)\right\}\right] \\ &= \frac{1}{2\beta}\sum_q \int_0^U dU' \mathrm{tr}\left[\underline{\chi}_{D,U'}(q) - \underline{\chi}_{D,0}(q)\right]\end{aligned} \tag{73}$$

where tr{ } means

$$\mathrm{tr}\left\{\underline{\chi}_{D,U}(q)\right\} = \chi^{-+}_{D,U}(q) + \chi^{+-}_{D,U}(q). \tag{74}$$

Differentiating $\underline{\Pi}(q)$ of eq.(58) by U we get

$$\frac{\partial}{\partial U}\underline{\Pi}(q) = \underline{\chi}_{D,0}(q) + \frac{\partial}{\partial U}\underline{\lambda}(q).$$

If assume the approximation,

$$\underline{\chi}_{D,0} \gg \frac{\partial}{\partial U}\underline{\lambda}(q), \tag{75}$$

we obtain

$$\frac{\partial}{\partial U}\underline{\Pi}(q) \simeq \underline{\chi}_{D,0}(q), \tag{76}$$

and

$$\frac{\partial}{\partial U}|\underline{\Pi}(q)| = \Pi^{-+}\chi_{D,0}^{+-} + \Pi^{+-}\chi_{D,0}^{-+} - \Pi^{--}\chi_{D,0}^{++} - \Pi^{++}\chi_{D,0}^{--}. \tag{77}$$

On the other hand, eqs.(59) and (74) leads to

$$\mathrm{tr}\left\{\underline{\chi}_{D,U}(q)\right\} = \frac{1}{|\underline{\Pi}(q)|}\left[\Pi^{-+}\chi_{D,0}^{+-} + \Pi^{+-}\chi_{D,0}^{-+} - \Pi^{--}\chi_{D,0}^{++} - \Pi^{++}\chi_{D,0}^{--}\right] \tag{78}$$

Consequently, comparing eqs.(77) and (78), we find

$$\frac{\partial}{\partial U}|\underline{\Pi}(q)| \simeq |\underline{\Pi}(q)|\mathrm{tr}\left\{\underline{\chi}_{D,U}(q)\right\}. \tag{79}$$

For $U=0$ in eq.(58) it is shown that

$$\begin{aligned}\underline{\chi}_{D,U=0}(q) &= \underline{\chi}_{D,0}(q)\\ &= \underline{\Pi}(q)|_{U=0}\cdot\underline{\chi}_{D,0}(q),\end{aligned}$$

therefore

$$\underline{\Pi}(q)|_{U=0} = 1. \tag{80}$$

Integrating the both sides of eq.(79) by U

$$\int_0^U dU'\mathrm{tr}\left\{\underline{\chi}_{D,U'}(q)\right\} \simeq \log|\underline{\Pi}(q)|. \tag{81}$$

Substituting the above equation into eq.(73), $\Delta\Omega$ is modified as

$$\Delta\Omega \simeq \frac{1}{2\beta}\sum_q\left[\log|\underline{\Pi}(q)| - U\mathrm{tr}\left\{\underline{\chi}_{D,0}(q)\right\}\right]. \tag{82}$$

Next, suppose that

$$U\frac{\partial\underline{\chi}_{D,0}(q)}{\partial D} \gg \frac{\partial\underline{\lambda}(q)}{\partial D}, \tag{83}$$

utilizing this, we differentiate $\underline{\Pi}(q)$ by D

$$\begin{aligned}\frac{\partial}{\partial D}\underline{\Pi}(q) &= U\frac{\partial\underline{\chi}_{D,0}(q)}{\partial D} + \frac{\partial\underline{\lambda}(q)}{\partial D}\\ &\simeq U\frac{\partial\underline{\chi}_{D,0}(q)}{\partial D}.\end{aligned} \tag{84}$$

Therefore, for the determinant of $|\underline{\Pi}(q)|$,

$$\frac{\partial}{\partial D}|\underline{\Pi}(q)| \simeq U\left(\Pi^{-+}\frac{\partial\chi_{D,0}^{+-}}{\partial D}+\Pi^{+-}\frac{\partial\chi_{D,0}^{-+}}{\partial D}-\Pi^{--}\frac{\partial\chi_{D,0}^{++}}{\partial D}-\Pi^{++}\frac{\partial\chi_{D,0}^{--}}{\partial D}\right). \qquad (85)$$

From eq.(59), we can rewrite the above as

$$\frac{\partial}{\partial D}|\underline{\Pi}(q)| \simeq U|\underline{\Pi}(q)|\mathrm{tr}\left\{\underline{\Pi}^{-1}(q)\cdot\frac{\partial\underline{\chi}_{D,0}(q)}{\partial D}\right\}, \qquad (86)$$

thus, we can obtain

$$\frac{\partial}{\partial D}\log|\underline{\Pi}(q)| \simeq U\mathrm{tr}\left\{\underline{\Pi}^{-1}(q)\cdot\frac{\partial\underline{\chi}_{D,0}(q)}{\partial D}\right\}, \qquad (87)$$

and

$$\frac{\partial^2}{\partial D^2}\log|\underline{\Pi}(q)| \simeq -U\mathrm{tr}\left[U\left\{\underline{\Pi}^{-1}(q)\cdot\frac{\partial\underline{\chi}_{D,0}(q)}{\partial D}\right\}^2-\underline{\Pi}^{-1}(q)\cdot\frac{\partial^2\underline{\chi}_{D,0}(q)}{\partial D^2}\right]. \qquad (88)$$

Using the above equations, we have

$$\frac{\partial^2\Delta\Omega}{\partial D^2} \simeq -\frac{U}{2\beta}\sum_q\mathrm{tr}\left[U\left\{\underline{\Pi}^{-1}(q)\cdot\frac{\partial\underline{\chi}_{D,0}(q)}{\partial D}\right\}^2-\{\underline{\Pi}^{-1}(q)-1\}\cdot\frac{\partial^2\underline{\chi}_{D,0}(q)}{\partial D^2}\right]. \qquad (89)$$

Here, we consider the system is in the normal state, that is, $T > T_C$ thus, $D = 0$ and $\Delta = 0$ (see eq.(37)). We define the quantity,

$$\kappa(q) \equiv \left.\frac{\partial^2\chi_{D0}^{-+}(q)}{\partial D^2}\right|_{D=0}, \qquad (90)$$

therefore, applying the eq.(B·10) and eq.(A·9) to eq.(89), we obtain following equations;

$$\left.\frac{\partial^2\Delta\Omega}{\partial D^2}\right|_{D=0} \simeq \frac{U}{2\beta}\sum_q\left[\left\{\frac{1}{\Pi^{-+}(q)}-1\right\}\kappa(q)+\left\{\frac{1}{\Pi^{+-}(q)}-1\right\}\kappa(-q)\right],$$

$$\begin{aligned}\left.\frac{\partial^2\Delta\Omega}{\partial D^2}\right|_{D=0} &\simeq \frac{U}{\beta}\sum_q\left[\frac{1}{1+U\chi_{D,0}^{-+}(q)+\lambda^{-+}(q)}-1\right]\kappa(q), \qquad (91)\\ \Pi^{-+}(q) &= 1+U\chi_{D,0}^{-+}(q)+\lambda^{-+}(q),\end{aligned}$$

$$\left.\frac{\partial^2\Omega_2}{\partial D^2}\right|_{D=0} = \frac{U}{\beta}\sum_q\kappa(q). \qquad (92)$$

Substituting above equations into eq.(72), we get

$$\left.\frac{\partial^2\Omega_1}{\partial D^2}\right|_{D=0} \simeq \frac{U}{\beta}\sum_q\frac{\kappa(q)}{1+U\chi_{D,0}^{-+}(q)+\lambda^{-+}(q)}. \qquad (93)$$

As a results, when $T > T_c$, from eq.(67), we find the selr-consistent equation for λ;

$$\begin{aligned}\lambda &= \frac{\chi_0}{2N}\frac{\partial^2\Omega_1}{\partial D^2}\bigg|_{D=0}, \\ &\simeq \frac{U\chi_0(0)}{2\beta N}\sum_q \frac{\kappa(q)}{1+U\chi_{D,0}^{-+}(q)+\lambda^{-+}(q)}. \end{aligned} \tag{94}$$

In the above, λ on left hand side is thermodynamic quantity, neverthless, $\lambda^{-+}(q)$ is defined from pair-pair correlation function. Then, we make approximation

$$\lambda^{-+}(q) \simeq \lambda^{-+}(0,0), \tag{95}$$

and since it has been kept the relation eq.(A·7) and eq.(64),

$$\lambda = \lambda^{-+}(0,0),$$

consequently we introduce following approximation

$$\lambda^{-+}(q) \simeq \lambda \tag{96}$$

therefore we obtain the self-consistent equation for thermodynamic λ,

$$\lambda \simeq \frac{U\chi_0(0)}{2\beta N}\sum_q \frac{\kappa(q)}{1+U\chi_{00}^{-+}(q)+\lambda}. \tag{97}$$

In the above, since main contoributed regions to summation of q and ν are $q \sim 0$ and $\nu \sim 0$, we consider the system is in the strong coupling region $U \to \infty$. Eq.(97) is approximately modified as

$$\lambda \simeq \frac{\kappa_0}{2\beta NA}\sum_{q,\nu} \frac{1}{\mathrm{i}\nu - \theta q^2 - \eta}, \tag{98}$$

where

$$\eta \equiv -\frac{1+\lambda+U\chi_0(0)}{U\chi_0(0)A} \tag{99}$$

and in this case,

$$\begin{aligned}&\mu \sim \frac{U}{2},\quad A \simeq -\frac{1}{2\mu},\quad \kappa_0 \simeq \frac{4}{\mu}, \\ &\theta \simeq -\frac{t^2a^2}{\mu},\quad \chi_0(0) \simeq -\frac{1}{2\mu}.\end{aligned} \tag{100}$$

a is the lattice constant. We assume $T \sim T_c$, thus, $\eta \sim 0$ since $1+\lambda+U\chi_0(0) = 0$ at $T = T_c$. Under conditions eq.(100) and $\beta\eta \ll 1$, carrying out the summation of q and ν, we attain the approximated selr-consistent equation as follows:

$$\lambda \simeq \frac{v_0}{\pi c_0}\frac{1}{(\beta\theta)^{3/2}}\left(1 - c_0\sqrt{\beta\eta}\right), \tag{101}$$

$$\eta \simeq -\frac{4\mu^2}{U}(1+\lambda) + 2\mu, \tag{102}$$

where

$$v_0 = \frac{V}{N}, \quad c_0 = \frac{\pi}{\zeta(3/2)\Gamma(3/2)}. \tag{103}$$

N is the number of lattice sites and V is the volume of the system.

5. Superconducting transition temperature

From the second equation of eq.(41), we can derive the total number of electrons of the system.

$$N_e = -\left(\frac{\partial\Omega}{\partial\mu}\right)_D = -\left(\frac{\partial\Omega_0}{\partial\mu}\right)_D - \left(\frac{\partial\Omega_1}{\partial\mu}\right)_D. \tag{104}$$

We can show the following results in the present case, *i.e.*, $T > T_c$, $D = \Delta = 0$, using eqs.(23) and (39);

$$\begin{aligned}\left(\frac{\partial\Omega_0}{\partial\mu}\right)_D &= \left(\frac{\partial\Phi_0}{\partial\mu}\right)_{\Delta_0} + \left(\frac{\partial\Phi_0}{\partial\Delta_0} + 2ND\right)\frac{\partial\Delta_0}{\partial\mu}, \\ &= \left(\frac{\partial\Phi_0}{\partial\mu}\right)_{\Delta_0} \simeq 0, \end{aligned} \tag{105}$$

and

$$N_e \simeq -\left(\frac{\partial\Omega_1}{\partial\mu}\right)_D. \tag{106}$$

Carrying out the above differentiation in similar way to the last section, we obtain

$$\frac{\partial\Omega_1}{\partial\mu} \simeq -\frac{Nv_0}{2\pi c_0}\frac{1}{(\beta\theta)^{3/2}}\left[1 + \frac{2\mu^2}{U}\frac{\partial\lambda}{\partial\mu}\right]\left(1 - c_0\sqrt{\beta\eta}\right), \tag{107}$$

in the limit of $U \to \infty$ and in $T > T_c$. The electron number per site is obtained;

$$n \equiv \frac{N_e}{N} \simeq \frac{v_0}{2\pi c_0}\frac{1}{(\beta\theta)^{3/2}}\left[1 + \frac{2\mu^2}{U}\frac{\partial\lambda}{\partial\mu}\right]\left(1 - c_0\sqrt{\beta\eta}\right). \tag{108}$$

At the T_c, thermal susceptibility χ in eq.(65) diverge and the system transit to the superconducting state, thereby η in eq.(99) becomes zero and we have

$$\lambda \simeq \frac{v_0}{\pi c_0}\frac{1}{(\beta\theta)^{3/2}}, \tag{109}$$

$$n \simeq \frac{v_0}{2\pi c_0}\frac{1}{(\beta\theta)^{3/2}}\left[1 + \frac{2\mu^2}{U}\frac{\partial\lambda}{\partial\mu}\right]. \tag{110}$$

Using eq.(100) and a constant $g \equiv v_0/(\pi c_0 a^3)$,

$$\lambda \simeq g\left(\frac{|\mu|}{\beta t^2}\right)^{3/2}, \tag{111}$$

$$\begin{aligned} n &\simeq \frac{g}{2}\left(\frac{|\mu|}{\beta t^2}\right)^{3/2}\left(1 + \frac{2\mu^2}{U}\frac{\partial\lambda}{\partial\mu}\right), \\ &= \frac{\lambda}{2}\left(1 + \frac{2\mu^2}{U}\frac{\partial\lambda}{\partial\mu}\right). \end{aligned} \tag{112}$$

Straightforwardly from eq.(111) and eq.(112),

$$\frac{\partial \lambda}{\partial \mu} \simeq -\frac{3g}{2}\left(\frac{|\mu|}{\beta^3 t^6}\right)^{1/2}, \tag{113}$$

and

$$\begin{aligned} n &\simeq \frac{\lambda}{2}\left\{1+\frac{3g}{|U|}\left(\frac{|\mu|^5}{\beta^3 t^6}\right)^{1/2}\right\}, \\ &= \frac{\lambda}{2}\left\{1+3g\frac{\beta t^2}{|U|}\left(\frac{\lambda}{g}\right)^{5/3}\right\}. \end{aligned} \tag{114}$$

Here, we define parameters

$$X \equiv \left(\frac{|\mu|}{\beta t^2}\right)^{1/2}, \quad Y \equiv \left(\frac{\beta t^2}{|U|}\right)^{1/2},$$

and obtain

$$3gY^2X^8 + X^3 - \frac{2n}{g} = 0. \tag{115}$$

Basically, we have to solve the above equation and will obtain a relation between μ and U. However, it can hardly be solved the equation analytically. Then, we assume following relation

$$\mu = \frac{U}{2\mathrm{f}_c}. \tag{116}$$

Using this,

$$\begin{aligned} n &\simeq \frac{\lambda}{2}\left(1+3\frac{|\mu|}{|U|}\lambda\right) \\ &= \frac{\lambda}{2}\left(1+\frac{3\lambda}{2\mathrm{f}_c}\right), \end{aligned} \tag{117}$$

and we have

$$\lambda^2 + \frac{2\mathrm{f}_c}{3}\lambda - \frac{4n\mathrm{f}_c}{3} = 0, \tag{118}$$

$$\lambda = \frac{1}{3}\left(\sqrt{\mathrm{f}_c{}^2 + 12n\mathrm{f}_c} - \mathrm{f}_c\right). \tag{119}$$

Substitute μ in eq.(111) into $\chi_0(0)$ in eq.(100), the determination equation eq.(68) is modified as

$$1+\lambda-\frac{|U|}{2\beta t^2(\lambda/g)^{2/3}} = 0. \tag{120}$$

When $T = T_c$ in the above equation, we attain the superconducting transition temperature in the strong coupling limit,

$$T_c = \frac{2t^2}{k_B|U|}(1+\lambda)\left(\frac{\lambda}{g}\right)^{2/3}. \tag{121}$$

On the other hand, replacing the T by T_c directly in eq.(111),

$$\begin{aligned} T_c &= \frac{t^2}{k_B|\mu|}\left(\frac{\lambda}{g}\right)^{2/3}, \\ &= \frac{2t^2}{k_B|U|}\mathrm{f}_c\left(\frac{\lambda}{g}\right)^{2/3}. \end{aligned} \tag{122}$$

Since this T_c must be identical to that of eq.(121), we obtain

$$1+\lambda = \mathrm{f}_c. \tag{123}$$

Accordingly, from eq.(119),

$$\mathrm{f}_c = \frac{1}{5}\left\{2(n+2)+\sqrt{4n^2+16n+1}\right\}, \tag{124}$$

$$\begin{aligned} \lambda &= \mathrm{f}_c - 1 \\ &= \frac{1}{5}\left(2n-1+\sqrt{4n^2+16n+1}\right). \end{aligned} \tag{125}$$

When electron number per site n is zero, $\lambda = 0$ then $T_c = 0$. It is just natural since the system has no electrons. Also of cause $T_c \to 0$ in the limit $U \to \infty$, because a electron pair is completly bounded on a lattice site by the strong attraction and can not move, consequently, the system can never transit to the superconducting state. As a results, our T_c obtained here is consistent with physical intuition. Moreover, the chemical potential is expressed by λ as

$$\mu = \frac{U}{2(1+\lambda)}. \tag{126}$$

6. Conclusions and discussions

In this paper, we have derived the superconducting transition temperature on the attractive Hubbard model in the strong coupling limit where the BCS theory is invalid. Then, we have adopted the procedure of SCR theory[11] to renormalize the strong attraction U and have made a superconducting theory beyond the BCS theory. There are some problems. First, T_c vanishes at $n = 0$ but does not at $n = 2$. This is inconsistent, since the system has electron-hole symmetry in the present case. Next, bases of approximations of eqs.(75) and (83) are ambiguous. These problems must be cleared in near future.

On the other hand, the present theory has ability to treat the pseudo-gap state of the high-T_c superconductor because of the strong coupling state of electrons above T_c is the precursor pairing state. In this state, there exist electron pairs but the system is the normal state, then, excitation spectrum has a gap. This is similar situation with the pseudo-gap state.[9] In the future, we will be able to obtain the T_c as a function of U in any strength, because of the present procedure can treat the all region of $|U|$ rigorously. Therefore, the extensions of the present theory to middle and weak $|U|$ and to anisotropic superconductors like d-wave, spin triplet p-wave and f-wave superconductor are next tasks.

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Appendix A: Some propaties of the pair-pair correlation function

The definition of the pair-pair correlation function is written down again,

$$\begin{aligned}\chi_U^{\rho\sigma}(i,j;\tau) &= \left\langle T_\tau\{A_{iK}^\dagger(\tau)\hat{s}_\rho A_{iK}(\tau)\}\{A_{jK}^\dagger(0)\hat{s}_\sigma A_{jK}(0)\}\right\rangle \\ &- \left\langle A_i^\dagger\hat{s}_\rho A_i\right\rangle\left\langle A_j^\dagger\hat{s}_\sigma A_j\right\rangle, \end{aligned} \tag{A·1}$$

and obviously from this definition, we can see

$$\chi_U^{\rho\sigma}(i,j;\tau) = \chi_U^{\sigma\rho}(j,i;-\tau). \tag{A·2}$$

We apply the Fourie transformation to the both sides of the above equation, we obtain

$$\chi_U^{\rho\sigma}(q) = \chi_U^{\sigma\rho}(-q). \tag{A·3}$$

When $U=0$,

$$\chi_0^{\rho\sigma}(q) = \chi_0^{\sigma\rho}(-q). \tag{A·4}$$

Now, we consider $T > T_c$, the system is in the normal state. Therefore, χ_U^{--} and χ_U^{++} which contains anomalous statistical average, disappear. Then, the matrix form of the correlation functions are

$$\underline{\chi}(q) = \begin{pmatrix} \chi_U^{-+}(q) & 0 \\ 0 & \chi_U^{+-}(q) \end{pmatrix}, \quad \underline{\chi}_0(q) = \begin{pmatrix} \chi_0^{-+}(q) & 0 \\ 0 & \chi_0^{+-}(q) \end{pmatrix}. \tag{A·5}$$

Substituting the above matrices to eq.(59), we obtain

$$\underline{\Pi}(q) = \frac{1}{|\underline{\chi}(q)|}\begin{pmatrix} \chi_0^{-+}(q)\chi_U^{+-}(q) & 0 \\ 0 & \chi_0^{+-}(q)\chi_U^{-+}(q) \end{pmatrix} = \begin{pmatrix} \Pi^{-+}(q) & 0 \\ 0 & \Pi^{+-}(q) \end{pmatrix}, \tag{A·6}$$

and

$$\Pi^{--}(q) = \Pi^{++}(q) = 0, \quad \lambda^{--}(q) = \lambda^{++}(q) = 0. \tag{A·7}$$

Utilizing eq.(A·3), we can show

$$\underline{\chi}^{-1}(-q) = \frac{1}{|\underline{\chi}(q)|}\underline{\chi}(q),$$

thus, we have

$$\underline{\Pi}(-q) = \frac{1}{|\underline{\chi}(q)|}\begin{pmatrix} \chi_0^{+-}(q)\chi_U^{-+}(q) & 0 \\ 0 & \chi_0^{-+}(q)\chi_U^{+-}(q) \end{pmatrix}. \tag{A·8}$$

Therefore, we obtain the relations

$$\Pi^{\rho\sigma}(q) = \Pi^{\sigma\rho}(-q), \quad \lambda^{\rho\sigma}(q) = \lambda^{\sigma\rho}(-q). \tag{A·9}$$

Appendix B: 0-th order correlation function

Green function for Hamiltonian $\mathcal{H}^0$ given by eq,(33) can be obtained as follows;

$$\begin{aligned}\hat{G}^{(0)}(k) &= \begin{pmatrix} G_{11}^{(0)}(k) & G_{12}^{(0)}(k) \\ G_{21}^{(0)}(k) & G_{22}^{(0)}(k) \end{pmatrix} \\ &= -\frac{1}{\omega^2 + E_k^2}\begin{pmatrix} \mathrm{i}\omega + \xi_k & -\Delta \\ -\Delta^* & \mathrm{i}\omega - \xi_k \end{pmatrix},\end{aligned} \tag{B·1}$$

where $k = (\boldsymbol{k}; \mathrm{i}\omega)$ and $E_k = \sqrt{\xi_k^2 + |\Delta|^2}$. We obtain following results for $\Delta = 0$:

$$\hat{G}^{(00)}(k) = \begin{pmatrix} g_-(k) & 0 \\ 0 & g_+(k) \end{pmatrix}, \tag{B·2}$$

$$g_\pm(k) \equiv \frac{1}{\mathrm{i}\omega \pm \xi_k}, \tag{B·3}$$

$$\left.\frac{\partial \hat{G}^{(0)}(k)}{\partial \Delta}\right|_{\Delta=0} = -g_+(k)g_-(k)\hat{s}_1, \tag{B·4}$$

$$\left.\frac{\partial^2 \hat{G}^{(0)}(k)}{\partial \Delta^2}\right|_{\Delta=0} = 2g_+(k)g_-(k)\hat{G}^{(00)}(k). \tag{B·5}$$

0-th order pair-pair correlation function is derived from Fourie transformation of eq.(54) for $U = 0$ as follows

$$\chi_{\Delta 0}^{\rho\sigma}(q) = -\frac{1}{\beta N}\sum_k \mathrm{tr}\left\{\hat{s}_\rho \hat{G}^{(0)}(k+q)\hat{s}_\sigma \hat{G}^{(0)}(k)\right\}, \tag{B·6}$$

where subscript $_\Delta$ means that $\chi^{pp}_{\Delta 0}(q)$ has independent variable Δ through $\hat{G}^{(0)}(k)$ in eq.(B·1). We can show

$$\left.\frac{\partial \chi^{-+}_{\Delta 0}(q)}{\partial \Delta}\right|_{\Delta=0} = 0, \tag{B·7}$$

$$\left.\frac{\partial^2 \chi^{+-}_{\Delta 0}(-q)}{\partial \Delta^2}\right|_{\Delta=0} = \left.\frac{\partial^2 \chi^{-+}_{\Delta 0}(q)}{\partial \Delta^2}\right|_{\Delta=0}. \tag{B·8}$$

If we treat D as independent variable through the relation given by eq.(38),

$$\begin{aligned}\frac{\partial \chi^{-+}_{D0}(q)}{\partial D} &= \frac{\partial}{\partial D}\chi^{-+}_{\Delta_0(D),0}(q) = \frac{\partial \chi^{-+}_{\Delta_0 0}(q)}{\partial \Delta_0}\frac{\partial \Delta_0}{\partial D} \\ &= \frac{1}{\chi_0(D)}\frac{\partial \chi^{-+}_{\Delta 0}(q)}{\partial \Delta}.\end{aligned} \tag{B·9}$$

Therefore, we obtain

$$\left.\frac{\partial \chi^{-+}_{D0}(q)}{\partial D}\right|_{D=0} = 0. \tag{B·10}$$

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