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A NOTE ON MONOMORPHISMS OF IRRATIONAL ROTATION C*-ALGEBRAS INTO THEIR MATRIX ALGEBRAS

KAZUNORI KODAKA

1. INTRODUCTION

In this note we will show that there is a difference between the irrational rotation C*-algebras corresponding to quadratic irrational numbers and the irrational rotation C*-algebras corresponding to non-quadratic ones by studying monomorphisms of them into their matrix algebras.

Throughout this note we will say *monomorphism* when we mean *unital monomorphism*.

2. MONOMORPHISMS OF A_θ INTO $M_n(A_\theta)$

Let θ be an irrational number and A_θ the corresponding irrational rotation C*-algebra. Let τ be the unique tracial state on A_θ and for any $n \in \mathbf{N}$ let $M_n(A_\theta)$ be the $n \times n$-matrix algebra over A_θ. We extend the unique tracial state τ to the unnormalized finite trace on $M_n(A_\theta)$. We also denote it by τ. Let ϕ be a monomorphism of A_θ into $M_n(A_\theta)$ and E a conditional expectation of $M_n(A_\theta)$ onto $\phi(A_\theta)$. We suppose that E is of index-finite type. Since A_θ is simple, by Watatani [8, Proposition 2.7.3] $\phi(A_\theta)' \cap M_n(A_\theta)$ is finite dimensional where

$$\phi(A_\theta)' \cap M_n(A_\theta) = \{x \in M_n(A_\theta) | x\phi(a) = \phi(a)x \quad \text{for any} \quad a \in A_\theta\}.$$

PROPOSITION 1. *With the above assumptions we suppose that θ is non-quadratic. Then for any non-zero projection $q \in \phi(A_\theta)' \cap M_n(A_\theta)$, $\tau(q)$ is a positive integer.*

PROOF: Since $q \in \phi(A_\theta)' \cap M_n(A_\theta)$, there are integers k, l such that $\tau(q) = k + l\theta > 0$. Let ψ be the monomorphism of A_θ to $qM_n(A_\theta)q$

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defined for any $x \in A_\theta$ by $\psi(x) = q\phi(x)$. Let τ_1 be a tracial state on A_θ defined for any $x \in A_\theta$ by

$$\tau_1(x) = \frac{1}{k + l\theta}\tau \circ \psi(x).$$

By the uniqueness of the tracial state on A_θ, we can see that $\tau_1 = \tau$. Let p be a projection in A_θ with $\tau(p) = \theta$. Then

$$\tau_1(p) = \frac{1}{k + l\theta}\tau \circ \psi(p).$$

Since $\psi(p) \in qM_n(A_\theta)q \subset M_n(A_\theta)$, there are integers s, t such that $\tau(\psi(p)) = s + t\theta$. Thus $\tau_1(p) = \dfrac{s + t\theta}{k + l\theta}$. Hence since $\tau_1(p) = \tau(p) = \theta$,

$$l\theta^2 + (k - t)\theta - s = 0.$$

Since θ is non-quadratic, $l = 0$. Therefore $\tau(q) = k > 0$. Hence we obtain the conclusion. Q.E.D.

For any quadratic irrational number θ we will show that there are a positive integer n and a monomorphism ϕ of A_θ into $M_n(A_\theta)$ satisfying the following properties:
(1) There is a conditional expectation E of index-finite type of $M_n(A_\theta)$ onto $\phi(A_\theta)$,
(2) There is a non-zero projection $q \in \phi(A_\theta)' \cap M_n(A_\theta)$ such that $\tau(q) \notin \mathbf{N}$.

First we will give definitions and well-known facts on quadratic irrational numbers (see Lang [5]).

Let $GL(2, \mathbf{Z})$ be the group of all 2×2-matrices over $\mathbf{Z}$ with determinant ± 1 and $SL(2, \mathbf{Z})$ the group of all 2×2-matrices over $\mathbf{Z}$ with determinant 1. Let $g = \begin{bmatrix} k & l \\ m & n \end{bmatrix} \in GL(2, \mathbf{Z})$ and θ an irrational number. We define $g\theta = \dfrac{m + n\theta}{k + l\theta}$ and we call g a *fractional transformation*. Furthermore if $g \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, then we say that g is *non-trivial*.

From now on we suppose that θ is a quadratic irrational number. By Lang [5, Chap. I, §1, Theorems 1, 2, Corollary 1, and Chap.

IV, §1, Theorems 2, 3] there is a non-trivial fractional transformation $h = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in GL(2, \mathbf{Z})$ such that

$$\theta = \frac{c_1 + d_1\theta}{a_1 + b_1\theta}, \quad 0 < a_1 + b_1\theta < 1.$$

If we consider $h^2 \in SL(2, \mathbf{Z})$, by [4, Lemma 3] we can see that there is a non-trivial fractional transformation $g = h^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbf{Z})$ such that

$$\theta = \frac{c + d\theta}{a + b\theta}, \quad 0 < a + b\theta < 1.$$

The quadratic equation for θ can be written in the form

$$k\theta^2 + l\theta + m = 0$$

where k, l, m are relatively prime integers and $k > 0$. Let $D = l^2 - 4km > 0$ be the discriminant of θ. Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbf{Z})$ be a non-trivial fractional transformation such that

$$\theta = \frac{c + d\theta}{a + b\theta}, \quad 0 < a + b\theta < 1.$$

Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be written in the following form:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{t+ls}{2} & ks \\ -ms & \frac{t-ls}{2} \end{bmatrix}$$

where s, t are integers satisfying $t^2 - Ds^2 = 4$ since $ad - bc = 1$.

Lemma 2. *With the above notations let $r = a + b\theta$. Then there is a positive integer n such that $r + r^{-1} = n$.*

Proof: Since $k\theta^2 + l\theta + m = 0$ and D is the discriminant of θ, $\theta = \dfrac{-l \pm \sqrt{D}}{2k}$. And since $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{t+ls}{2} & ks \\ -ms & \frac{t-ls}{2} \end{bmatrix}$,

$$r = a + b\theta = \frac{t + ls}{2} + ks\theta = \frac{t + ls}{2} + \frac{-ls \pm s\sqrt{D}}{2} = \frac{t \pm s\sqrt{D}}{2}.$$

Hence $s^2D = 4r^2 - 4tr + t^2$. Since $t^2 - Ds^2 = 4$, we obtain that $r^2 - tr + 1 = 0$. Thus $r^2 + 1 = tr$. Since $r > 0$, $t > 0$. Let $n = t$. Then $r^2 + 1 = nr$. Therefore we obtain that $r + r^{-1} = n$. Q.E.D.

Let q_1 be a projection in $M_n(A_\theta)$ with $\tau(q_1) = r = a + b\theta$ and $q_2 = I_n - q_1$ where I_n is the unit element in $M_n(A_\theta)$.

Lemma 3. *With the above notations for $j = 1,2$ $q_jM_n(A_\theta)q_j$ is isomorphic to A_θ.*

Proof: Since $g\theta = \dfrac{c + d\theta}{a + b\theta} = \theta$ and $\tau(q_1) = a + b\theta$, in the same way as in [3, Lemma 7](see also [2, the proof of Theorem 5]) we see that $q_1M_n(A_\theta)q_1 \cong A_\theta$. Next we will show that $q_2M_n(A_\theta)q_2 \cong A_\theta$. Since $q_2 = I_n - q_1$, $\tau(q_2) = n - (a + b\theta) = n - r$. Hence by Lemma 2 $\tau(q_2) = r^{-1}$. Since $\theta = \dfrac{c + d\theta}{a + b\theta}$, $r^{-1} = d - b\theta$. Thus $g^{-1}\theta = \dfrac{-c + a\theta}{d - b\theta} = \theta$. Therefore in the same way as above $q_2M_n(A_\theta)q_2 \cong A_\theta$. Q.E.D.

For $j = 1,2$ let ϕ_j be an isomorphism of A_θ onto $q_jM_n(A_\theta)q_j$. Let ϕ be a monomorphism of A_θ into $M_n(A_\theta)$ defined by $\phi = \phi_1 + \phi_2$. Let E_1 be the linear map of $M_n(A_\theta)$ onto $\phi(A_\theta)$ defined by

$$E_1(x) = q_1xq_1 + \phi_2(\phi_1^{-1}(q_1xq_1))$$

for any $x \in A_\theta$ and let E_2 be the linear map of $M_n(A_\theta)$ onto $\phi(A_\theta)$ defined by

$$E_2(x) = q_2xq_2 + \phi_1(\phi_2^{-1}(q_2xq_2))$$

for any $x \in A_\theta$. By easy computation we can see that E_1 and E_2 are conditional expectations of $M_n(A_\theta)$ onto $\phi(A_\theta)$. Furthermore let $E = \dfrac{1}{2}(E_1 + E_2)$. Then by easy computation we see that E is a faithful conditional expectation of $M_n(A_\theta)$ onto $\phi(A_\theta)$.

We will find a quasi-basis for E in order to show that E is of index-finite type.

Let k be a positive integer such that $r^{-1} - kr > 0$ and $r^{-1} - (k + 1)r < 0$. We will find an orthogonal family $\{\tilde{q}_{1j}\}_{j=1}^k$ of projections in $M_n(A_\theta)$ and a family $\{w_j\}_{j=1}^k$ of unitary elements in $M_n(A_\theta)$ such that $\tilde{q}_{1j} \le q_2$ and $\tilde{q}_{1j} = w_j^*q_1w_j$ for $j = 1, 2, \ldots, k$.

For any unital C*-algebra B we denote by $Proj(B)$ the set of all projections in B.

LEMMA 4. *For $j = 0, 1, 2, \ldots, k-1$ let f_j be a projection in $M_n(A_\theta)$ with $\tau(f_j) = r^{-1} - jr$. Then there is a projection $\tilde{f}_j$ in $f_j M_n(A_\theta) f_j$ such that $\tau(\tilde{f}_j) = r$.*

PROOF: Since $f_j M_n(A_\theta) f_j$ is strongly Morita equivalent to A_θ, by Rieffel [7, Corollary 2.6] there are a positive integer h, a real number η and $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in SL(2, \mathbf{Z})$ such that

$$f_j M_n(A_\theta) f_j \cong M_h(A_\eta), \quad r^{-1} - jr = h(a_1 + b_1\theta), \quad \eta = \frac{c_1 + d_1\theta}{a_1 + b_1\theta}.$$

Hence by Rieffel [6, Proposition 1.3] we see that

$$\tau(f_j)^{-1}\tau(Proj(f_j M_n(A_\theta) f_j)) = h^{-1}\mathbf{Z} + h^{-1}\mathbf{Z}\eta \cap [0, \quad 1].$$

Thus in order to see that there is a projection $\tilde{f}_j$ in $f_j M_n(A_\theta) f_j$ it suffices to show that there are integers l, m such that

$$h^{-1}(l + m\eta)(r^{-1} - jr) = r, \quad 0 < l + m\eta < h.$$

However since $r^{-1} - jr = h(a_1 + b_1\theta)$, $\eta = \dfrac{c_1 + d_1\theta}{a_1 + d_1\theta}$ and $r = a + b\theta$, it is sufficient to show that there are integers l, m such that

$$(a_1 l + c_1 m) + (b_1 l + d_1 m)\theta = a + b\theta, \quad 0 < m\frac{c_1 + d_1\theta}{a_1 + b_1\theta} < h.$$

Let $l = ad_1 + bc_1$ and $m = -ab_1 + ba_1$. Then by direct calculation l and m satisfy the above relations. Therefore we obtain the conclusion. Q.E.D.

LEMMA 5. *There are an orthogonal family $\{\tilde{q}_{1j}\}_{j=1}^{k}$ of projections in $M_n(A_\theta)$ and a family $\{w_j\}_{j=1}^{k}$ of unitary elements in $M_n(A_\theta)$ such that for $j = 1, 2, \ldots, k$*

$$\tilde{q}_{1j} \le q_2, \quad \tilde{q}_{1j} = w_j^* q_1 w_j.$$

PROOF: By Lemma 4 for a projection q_2 there is a projection $\tilde{q}_{11}$ in $q_2 M_n(A_\theta) q_2$ such that $\tau(\tilde{q}_{11}) = r$. We suppose that there is an orthogonal family $\{\tilde{q}_{1j}\}_{j=1}^{m}$ $(1 \le m \le k-1)$ of projections in $q_2 M_n(A_\theta) q_2$

such that $\tau(\tilde{q}_{1j}) = r$ for $j = 1, 2, \ldots, m$. Let $f_m = q_2 - \sum_{j=1}^{m} \tilde{q}_{1j}$. Then $\tau(f_m) = r^{-1} - mr$. Hence by Lemma 4 there is a projection $\tilde{q}_{1m+1}$ in $f_m M_n(A_\theta) f_m$ such that $\tau(q_{1m+1}) = r$. Thus by induction we can see that there is an orthogonal family $\{\tilde{q}_{1j}\}_{j=1}^{k}$ of projections in $M_n(A_\theta)$ such that $\tilde{q}_{1j} \leq q_2$, $\tau(\tilde{q}_{1j}) = r$ for $j = 1, 2, \ldots, k$. Since $M_n(A_\theta)$ has cancellation property, for $j = 1, 2, \ldots, k$ there is a unitary element w_j in $M_n(A_\theta)$ such that $\tilde{q}_{1j} = w_j^* q_1 w_j$. Therefore we obtain the conclusion. Q.E.D.

Let $\tilde{q} = \sum_{j=1}^{k} \tilde{q}_{1j}$. We note that $\tilde{q} \leq q_2$.

Lemma 6. *With the above notations there are a projection $\bar{q}$ and a unitary element z in $M_n(A_\theta)$ such that*

$$\bar{q} \leq q_1, \quad z\bar{q}z^* = q_2 - \tilde{q}.$$

Proof: Since $q_1 M_n(A_\theta) q_1$ is isomorphic to A_θ, by Rieffel [6, Theorem 1] we see that

$$\tau(Proj(q_1 M_n(A_\theta) q_1)) = r(\mathbf{Z} + \mathbf{Z}\theta) \cap [0, \quad r].$$

In order to see that there is a projection $\bar{q}$ in $q_1 M_n(A_\theta) q_1$ such that $\tau(\bar{q}) = \tau(q_2) - \tau(\tilde{q})$ it suffices to show that there are integers l, m such that

$$r(l + m\theta) = n - r - kr, \quad 0 < l + m\theta < 1.$$

Since $r = a + b\theta$ and $r + r^{-1} = n$, we obtain that

$$l + m\theta = (n^2 - an - k - 1) - bn\theta, \quad 0 < l + m\theta < 1.$$

Let $l = n^2 - an - k - 1$, $m = -bn$. Then by direct calculation we see that l and m satisfy the above relations. Therefore we obtain the conclusion. Q.E.D.

Proposition 7. *With the above notations a family*

$$\left\{ \begin{array}{l} (2q_1, \quad q_1), \quad (2q_1 w_1 q_2, \quad q_2 w_1^*) \\ (2w_j^* q_1, \quad q_1 w_j) \quad j = 1, 2, \ldots, k \\ (2z\bar{q}, \quad q_1 z^*) \end{array} \right\}$$

is a quasi-basis for a conditional expectation E defined in this section.

PROOF: We will show that for any $x \in M_n(A_\theta)$

$$x = 2q_1 E(q_1 x) + 2q_2 E(q_2 x) + 2q_1 w_1 q_2 E(q_2 w_1^* x) + 2\sum_{j=1}^{k} w_j^* q_1 E(q_1 w_j x) + 2z\bar{q}E(q_1 z^* x).$$

By easy computation $2q_j E(q_j x) = q_j x q_j$ for $j = 1, 2$ and since $q_2 \geq w_1^* q_1 w_1$ by Lemma 5, $2q_1 w_1 q_2 E(q_2 w_1^* x) = q_1 w_1 q_2 w_1^* x q_2 = q_1 x q_2$. Since $\sum_{j=1}^{k} w_j^* q_1 w_j = \sum_{j=1}^{k} \tilde{q}_{1j} = \tilde{q}$ by Lemma 5,

$$2\sum_{j=1}^{k} w_j^* q_1 E(q_1 w_j x) = \sum_{j=1}^{k} w_j^* q_1 w_j x q_1 = \tilde{q} x q_1.$$

Furthermore since $\bar{q} \leq q_1$ and $z\bar{q}z^* = q_2 - \tilde{q}$ by Lemma 6,

$$2z\bar{q}E(q_1 z^* x) = z\bar{q}z^* x q_1 = (q_2 - \tilde{q}) x q_1.$$

Hence

$$2q_1 E(q_1 x) + 2q_2 E(q_2 x) + 2q_1 w_1 q_2 E(q_2 w_1^* x) + 2\sum_{j=1}^{k} w_j^* q_1 E(q_1 w_j x) + 2z\bar{q}E(q_1 z^* x)$$

$$= q_1 x q_1 + q_2 x q_2 + q_1 x q_2 + \tilde{q} x q_1 + (q_2 - \tilde{q}) x q_1 = x.$$

Similarly we see that

$$x = 2E(xq_1)q_1 + 2E(xq_2)q_2 + 2E(xq_1 w_1 q_2)q_2 w_j^* + 2\sum_{j=1}^{k} E(x w_j^* q_1) q_1 w_j + 2E(xz\bar{q}) q_1 z^*.$$

Therefore we obtain the conclusion. Q.E.D.

Remark. By the above proposition E is of index-finite type. And by direct computation we can see that $Index E = 4I_n$. Furthermore by the definition of ϕ projections q_1 and q_2 are in $\phi(A_\theta)' \cap M_n(A_\theta)$ and $\tau(q_1) = a + b\theta$, $\tau(q_2) = \dfrac{1}{a+b\theta}$.

Theorem 8. *If θ is a quadratic irrational number, there are a positive integer n and a monomorphism ϕ of A_θ into $M_n(A_\theta)$ satisfying the following properties:*
(1) There is a conditional expectation E of index-finite type of $M_n(A_\theta)$ onto $\phi(A_\theta)$,
(2) There is a non-zero projection $q \in \phi(A_\theta)' \cap M_n(A_\theta)$ such that $\tau(q) \notin \mathbf{N}$.

Proof: This is immediate by Proposition 7 and the above remark. Q.E.D.

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Department of Mathematics
College of Science
University of the Ryukyus
Nishihara-cho, Okinawa 903-01
JAPAN