Remarks on the universal covering space of the space of rational functions

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REMARKS ON THE UNIVERSAL COVERING SPACE OF THE SPACE OF RATIONAL FUNCTIONS

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1. Introduction

Let Rat_k denote the space of based holomorphic maps of degree k from the Riemannian sphere S^2 to itself. The basepoint condition we assume is that $f(\infty) = 1$. Such holomorphic maps are given by rational functions. Thus:

(1.1)

$$\begin{aligned}\mathrm{Rat}_k = \{(p(z), q(z)) : &\ p(z) \text{ and } q(z) \text{ are monic, degree-}k \text{ polynomials} \\ &\text{and such that there are no roots common to } p(z) \text{ and } q(z)\}.\end{aligned}$$

The study of the topology of Rat_k originates in [5]. The result is that the inclusion $i_k : \mathrm{Rat}_k \to \Omega^2_k S^2$ is a homotopy equivalence up to dimension k, that is, $i_{k*} : \pi_i(\mathrm{Rat}_k) \to \pi_i(\Omega^2_k S^2)$ is bijective for $i < k$ and surjective for $i = k$. In particular, we have $\pi_1(\mathrm{Rat}_k) \cong \mathbf{Z}$. Later the global homology of Rat_k was described in [2] and [3] in terms of the homology of Artin's braid groups. In particular, their result tells us that $i_{k*} : H_*(\mathrm{Rat}_k; K) \to H_*(\Omega^2_k S^2; K)$ is injective, where K is a field.

Let V_k denote the universal covering space of Rat_k, and $\tilde{\Omega}^2 S^3$ denote the universal covering space of $\Omega^2_k S^2 \simeq \Omega^2 S^3$. Let $\tilde{\imath}_k : V_k \to \tilde{\Omega}^2 S^3$ be a lift of i_k. Thus we have the following commutative diagram:

$$\begin{array}{ccc} V_k & \xrightarrow{\tilde{\imath}_k} & \tilde{\Omega}^2 S^3 \\ \downarrow & & \downarrow \\ \mathrm{Rat}_k & \xrightarrow{i_k} & \Omega^2 S^3. \end{array}$$

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If we take the results of [2], [3] and [5] into account, we naturally encounter the following question: Is $\tilde{\imath}_{k*} : H_*(V_k; K) \to H_*(\tilde{\Omega}^2 S^3; K)$ injective?

It is elementary to prove that $\tilde{H}_*(\tilde{\Omega}^2 S^3; \mathbf{C}) = 0$. Hence if the above question is true, then we have in particular that $\tilde{H}_*(V_k; \mathbf{C}) = 0$.

In fact we have a negative answer to the above question for $k > 1$ by calculating the Euler characteristic $\chi(V_k)$:

Theorem A. *We have* $\chi(V_k) = k$.

Theorem A tells us that $\tilde{H}_*(V_k; \mathbf{C}) \neq 0$ for $k > 1$. Hence we have a negative answer to the above question for $k > 1$ and $K = \mathbf{C}$.

It is well known that Rat_1 is homeomorphic to $\mathbf{C} \times \mathbf{C}^*$. Hence V_1 is homeomorphic to $\mathbf{C}^2$. Thus the topology of V_k is non-trivial for $k > 1$. Then about V_2 and V_3, we have the following:

Theorem B. V_2 *is homotopically equivalent to* S^2.

Theorem C. *For* $K = \mathbf{Z}_p$ *(p: a prime* > 3*) or* $K = \mathbf{C}$, *we have*

$$H_*(V_3; K) \cong H_*(S^4 \vee S^4; K).$$

Remark 1.2. As $\Omega^2 S^3$ is an H-space, we see that the Serre local system of the fibration $\tilde{\Omega}^2 S^3 \to \Omega^2 S^3 \to S^1$ is simple. On the other hand, Theorem A and the fact that $H_*(\mathrm{Rat}_k; \mathbf{C}) \cong H_*(S^1; \mathbf{C})$ $(k \geq 1)$ (see [2] and [3]) tell us that the Serre local system of the fibration $V_k \to \mathrm{Rat}_k \to S^1$ is not simple for $k > 1$. However it is known that Rat_k is a nilpotent space up to dimension n [5]. In relation to this, recall that we have a loop sum $\mathrm{Rat}_k \times \mathrm{Rat}_l \to \mathrm{Rat}_{k+l}$ which is compatible with the loop sum in $\Omega^2 S^3$ [1].

Note that we can realize V_k as a smooth complex affine algebraic hypersurface in $\mathbf{C}^{2k}$. (See Proposition 2.1.) Hence our results can be understood as the topology of such a hypersurface. In particular by a well-known theorem about smooth complex affine algebraic varieties, we have $H_i(V_k; \mathbf{Z}) = 0$ for $i > 2k - 1$. On the other hand, by the result of [5] we have $H_i(V_k; \mathbf{C}) = 0$ for $i < k$.

Complete details of this paper will appear elsewhere.

2. Filtration of V_k

We explain the crucial steps for the proof of Theorem A. First we realize V_k homotopically as a smooth complex affine algebraic hypersurface in $\mathbf{C}^{2k}$. We define

$$D_k : \mathrm{Rat}_k \to \mathbf{C}^*$$

by

$$D_k(p(z), q(z)) = \prod_{i,j=1}^{k} (\alpha_i - \beta_j),$$

where $p(z) = \prod_{i=1}^{k}(z - \alpha_i)$ and $q(z) = \prod_{j=1}^{k}(z - \beta_j)$. It is easy to see that D_k is a fiber bundle with fiber $D_k^{-1}(1)$. As $\pi_1(\mathrm{Rat}_k) \cong \mathbf{Z}$ (see [5]), we have $\pi_1(D_k^{-1}(1)) = 0$. Hence we can regard $D_k^{-1}(1)$ as the universal covering space of Rat_k. So we have the following:

Proposition 2.1. *V_k is homotopically equivalent to*

$$D_k^{-1}(1) = \{(p(z), q(z)) : p(z) \text{ and } q(z) \text{ are monic, degree-}k \text{ polynomials and such that } D_k(p(z), q(z)) = 1\}.$$

Let us filter V_k by closed subspaces:

$$V_k = F_{k,k} \supset F_{k,k-1} \supset \cdots \supset F_{k,i} \supset \cdots \supset F_{k,1} \supset F_{k,0} = \emptyset,$$

where

(2.2) $F_{k,i} = \{(p(z), q(z)) \in V_k : p(z) \text{ has at most } i \text{ distinct zeros}\}.$

Then we can prove that

Proposition 2.3.

$$\chi(F_{k,i} - F_{k,i-1}) = \begin{cases} k & i = 1 \\ 0 & 2 \le i \le k. \end{cases}$$

Then Theorem A follows from the long exact sequence of cohomology with compact supports of the pair $(F_{k,i}, F_{k,i-1})$.

If we calculate the homology of $F_{k,i}$ in detail, then by the same argument as in the proof of Theorem A, we can prove Theorems B and C.

Remark 2.4. By the same argument as in the proofs of Theorems A, B and C, we can prove that $\tilde{H}_q(V_4; \mathbf{C}) = 0$ for $q \le 3$ or $q \ge 7$.

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