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AN ELEMENTARY PROOF OF A THEOREM OF T. F. HAVEL

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1. Introduction

We consider the configuration space of the planar equilateral pentagon linkage. More precisely, we define M by

$$M = \{(x_1, x_2, x_3) \in \mathbf{R}^6; \| x_i - x_{i+1} \|= 1, i = 1, 2, \ldots 5\}$$

,where x_4 and x_5 are fixed vectors in $\mathbf{R}^2$ and we regard x_6 as x_1.

Note that the freedom of independent parameters of M equals to 2. Then it is natural to ask whether M is a manifold and, if in this case, what kind of manifold. In [2], T. F. Havel answers this question and the result is as follows.

Theorem 1. *M is a compact, connected and orientable two-dimensional manifold of genus 4.*

In order to prove this theorem, Havel considers the following steps. (1) First prove that M is a smooth manifold by showing local coordinates explicitly. (2) Next make a function $f : M \to \mathbf{R}$ by assigning a point of M to its directed area. Then prove that f is a Morse function and $\chi(M) = -6$ is obtained by the Morse theory, here $\chi(M)$ is the Euler number of M. (3) Finally prove that M is orientable.

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It will be natural that one hopes to prove Theorem 1 more directly without using the Morse theory. And the purpose of this paper is to execute this.

2. Geometric proof of Theorem 1

We write the coordinates of x_4 and x_5 by $x_4 = (-1/2, 0)$ and $x_5 = (1/2, 0)$ respectively and write the the clockwise angle from the vector $\overrightarrow{x_5 x_4}$ to $\overrightarrow{x_5 x_1}$ by α and the counterclockwise angle from the vector $\overrightarrow{x_4 x_5}$ to $\overrightarrow{x_4 x_3}$ by β respectively. It is clear that $x_1 = (1/2 - \cos\alpha, \sin\alpha)$, $x_3 = (-1/2 + \cos\beta, \sin\beta)$.

Fix x_1 and x_3 arbitrarily, then the freedom of x_2 will be given by the following:

(i) If $0 < \| x_1 - x_3 \| < 2$, then we can take x_2 at exactly 2 different points. In fact if x_2 is taken so that $\| x_1 - x_2 \| = 1, \| x_2 - x_3 \| = 1$, then the symmetric point x_2' of x_2 with respect to the segment $x_1 x_3$ also satisfies $\| x_1 - x_2' \| = 1, \| x_2' - x_3 \| = 1$.

(ii) If $\| x_1 - x_3 \| = 2$, then we can take x_2 at exactly one point. In fact in this case x_2 should be the middle point of the segment $x_1 x_3$.

(iii) If $\| x_1 - x_3 \| = 0$, then the freedom of x_2 is homeomorphic to S^1. In fact in this case x_2 can be taken at any point of the circle of radius 1 centered at $x_1 = x_3$.

(iv) If $2 < \| x_1 - x_3 \|$, then it is clear that we cannot take x_2 at any point.

Note that the case (iii) occurs if and only if $\alpha = \beta = \pi/3$ or $\alpha = \beta = 5\pi/3$.

Let R be the subspace of M consisting of points of the cases (i) or (ii) and let D be the subspace of T^2 consisting of (α, β) such that $0 <\| x_1 - x_3 \| \leq 2$, where T^2 is the 2 dimensional torus obtained from $[0, 2\pi] \times [0, 2\pi]$ by the identification $(\alpha, 0) \sim (\alpha, 2\pi)$ and $(0, \beta) \sim (2\pi, \beta)$. Note that the boundary of D, which will be denoted by ∂D, consists of points of the case (ii).

Thus R will be obtained by the following manner. Let $D^{(1)}$ and $D^{(2)}$ be two copies of D, $\partial D^{(1)}$ and $\partial D^{(2)}$ be boundary of $D^{(1)}$ and $D^{(2)}$ respectively and let $i : \partial D^{(1)} \to \partial D^{(2)}$ be the identity map. Then R is homeomorphic to $D^{(1)} \coprod D^{(2)} / \sim$, where $D^{(1)} \coprod D^{(2)}$ is the disjoint union of $D^{(1)}$ and $D^{(2)}$ and the identification $\sim$ is given by $x^{(1)} \sim x^{(2)}$ if and only if $x^{(1)} \in \partial D^{(1)}, x^{(2)} \in \partial D^{(2)}$ such that $x^{(2)} = ix^{(1)}$.

Because of the above observations, we first investigate the domain $\bar{D}$ which is defined by $\bar{D} = \{(\alpha, \beta) \in T^2; \| x_1 - x_3 \| \leq 2\}$. In order to do this, we shall see $\partial \bar{D}$, which is by definition $\{(\alpha, \beta) \in T^2; \| x_1 - x_3 \| = 2\}$.

Lemma 2.1. *$\partial \bar{D}$ is homeomorphic to S^1.*

Proof. This lemma seems clear from the definition of $\partial \bar{D}$. But for the completeness we shall give some details.

Note that

(2.2) $\partial \bar{D} = \{(\alpha, \beta) \in T^2; (1 - \cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\}$.

Once $\alpha \in [0, 2\pi]$ is fixed, then β will be given by

$$\beta = -\alpha/2 + \sin^{-1}\{(1/2 + \cos\alpha)/(-2\sin(\alpha/2))\} \tag{2.3}$$

,where $\sin^{-1} x = \{y \in (-\infty, \infty); \sin y = x\}$.

Note that (2.3) asserts that $-1 \leq (1/2+\cos\alpha)/(-2\sin(\alpha/2)) \leq 1$. Then we can easily show that α must satisfy $\pi/3 \leq \alpha \leq 5\pi/3$ such that if $\alpha = \pi/3$, then $\beta = 4\pi/3$ and $\alpha = 5\pi/3$, then $\beta = 2\pi/3$.

By using these results, we can easily prove that $\partial \bar{D}$ is homeomorphic to S^1. □

By using Lemma 2.1, we can show that $\bar{D}$ is homeomorphic to $T^2 - e^2$, where e^2 is a small open disk contained in T^2. Hence D is homeomorphic to $T^2 - \{e^2 \cup p_1 \cup p_2\}$, where p_1 corresponds to $\alpha = \beta = \pi/3$ and p_2 corresponds to $\alpha = \beta = 5\pi/3$.

Recall that R is homeomorphic to $D^{(1)} \coprod D^{(2)}/\sim$. Hence we have the following:

Proposition 2.4. *R is homeomorphic to $\Sigma_2 - \{p_1^{(1)}, p_1^{(2)}, p_2^{(1)}, p_2^{(2)}\}$, where Σ_2 is the Riemannian surface of genus 2 and $\{p_1^{(1)}, p_1^{(2)}\}$ are copies of p_1, $\{p_2^{(1)}, p_2^{(2)}\}$ copies of p_2.*

Next we shall investigate the case of (iii), i.e. the situation around $p_1^{(1)}, p_1^{(2)}, p_2^{(1)}$ and $p_2^{(2)}$ in R. We think of a small closed neighborhood of $p_1^{(1)}$ as $CS^1 - \{p_1^{(1)}\}$, where CS^1 is the cone of S^1 and the vertex corresponds to $p_1^{(1)}$. We also consider a small closed neighborhood of $p_1^{(2)}$ in the same manner. Then by the insights (i) and (iii), it is clear that the topology around $p_1^{(1)}$ and $p_1^{(2)}$ is given by the following: First consider $CS^1 \vee CS^1$ (= one point union of two CS^1's attached by the verteces). Then replace the vertex by S^1.

Note that $CS^1 \vee CS^1$ changes into $S^1 \times [0,1]$ by this operation. Hence we have proved that the topology around $p_1^{(1)}$ and $p_1^{(2)}$ is $S^1 \times [0,1]$.

If we consider the situation around $p_2^{(1)}$ and $p_2^{(2)}$ in the same manner, then we have the following:

Proposition 2.5. *Let M' be Σ_2 attached with two $S^1 \times [0,1]$'s in some manner. Then M is homeomorphic to M'.*

Finally we prove that M is orientable. In order to do this, we shall investigate how two $S^1 \times [0,1]$'s are attached to Σ_2.

We cut off a small open neighborhood of the vertex in CS^1 and write the remaining subspace of CS^1 by $S^1 \times [0, 1-\epsilon]$, where $\epsilon > 0$ is small enough. Note that the $S^1 \times [0,1]$ around $p_1^{(1)}$ and $p_1^{(2)}$ is obtained by $S^1 \times [0, 1-\epsilon] \coprod S^1 \times [0, 1-\epsilon] / \simeq$, where $\simeq$ is induced by a homeomorphism $g : S^1 \times \{1-\epsilon\} \to S^1 \times \{1-\epsilon\}$. (c.f. the identification of R with $D^{(1)} \coprod D^{(2)} / \sim$). We think of g as $g : S^1 \to S^1$. Then g is given by the following:

Lemma 2.6. *g is homotopic to the antipodal map.*

Proof. Note that $CS^1 -$ {vertex} is parametrized by α, β, and the freedom S^1 in the case (iii) is of course parametrized by x_2. Moreover note that x_2 and x_2' corresponds to each other in the case (i). By using these facts, it is easy to see that g is homotopic to the antipodal map. □

If we consider the situation around $p_2^{(1)}$ and $p_2^{(2)}$ in the same manner, then finally we have the following:

Theorem 2.7. *Let X be $T^2 - \{e_1^2, e_2^2, e_3^2\}$, where $\{e_1^2, e_2^2, e_3^2\}$ are small open disks. Then M' is homeomorphic to $X \coprod X / \cong$, where $\cong$ is meant to identify the boundaries of two X's via one identity map* *and two maps which are homotopic to the antipodal map.*

Note that antipodal map preserves orientation. Hence it is easy to see that M is orientable.

This completes the proof of Theorem 1.

References

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