The Homfly invariant of closed tangles

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THE HOMFLY INVARIANT OF CLOSED TANGLES *

Masashi Kosuda

Abstract

In the papers [3, 4], the author constructed a complete set of irreducible representations of the Hecke category. These representations also define invariants of oriented tangles. Since links are special examples of the oriented tangles, their invariant, which eventually coincides with the HOMFLY invariant of links, can be calculated by the same method. In this article, we calculate the invariant of the Hopf link, the Whitehead link and the Borromean link by this new method.

1 Preliminaries

We recall some definitions about tangles. For details we refer the reader to the papers [1, 3, 6, 8]. Those who are familiar with these papers can skip this section.

Let r and s be non-negative integers. An oriented (r, s)-*tangle* T is a finite set of disjoint oriented arcs and circles properly embedded (up to isotopy) in $\mathbf{R}^2 \times [0, 1]$ such that

$$\partial T = \{(i, 0, 0) | i = 1, 2, \ldots, r\} \cup \{(j, 0, 1) | j = 1, 2, \ldots, s\},$$

and such that T is perpendicular to $\mathbf{R}^2 \times \{0\}$ and $\mathbf{R}^2 \times \{1\}$ at every boundary point of ∂T. (See Figure 1.) With each (r, s)-tangle T, we associate two sequences, $\partial_- T = (\epsilon_1(T), \epsilon_2(T), \ldots, \epsilon_r(T))$ and $\partial^+ T = (\epsilon^1(T), \epsilon^2(T), \ldots, \epsilon^s(T))$, consisting of ± 1. Here $\epsilon_i(T) = +1$ if

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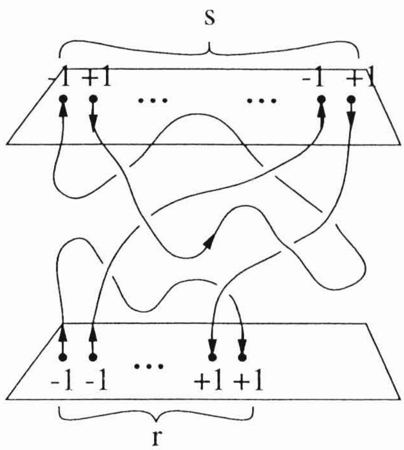

Figure 1: An oriented tangle

the tangent vector of T at $(i, 0, 0)$ is outward with respect to $\mathbf{R}^2 \times [0, 1]$ and $\epsilon_i(T) = -1$ otherwise. Similarly $\epsilon^j(T) = -1$ if the tangent vector of T at $(j, 0, 1)$ is outward and $\epsilon^j(T) = +1$ otherwise. If $r = 0$ (resp. $s = 0$), then $\partial_- T$ (resp. $\partial^+ T$) is the empty set $\emptyset$. We can easily find that there exists no (r, s)-tangle T such that $\sum_1^r \epsilon_i(T) \neq \sum_1^s \epsilon^j(T)$.

We define the category $\mathcal{OTA}$ of the oriented tangles. The *objects* of $\mathcal{OTA}$ are defined as the sequences $\{(\epsilon_1, \ldots \epsilon_r) | r = 0, 1, \ldots\}$ with $\epsilon_i = \pm 1$ including the empty sequence and denoted by $Ob(\mathcal{OTA})$. We denote the number of ones in ϵ by $\mathrm{Pos}(\epsilon)$, and the number of minus ones in ϵ by $\mathrm{Neg}(\epsilon)$. A *morphism* from $\epsilon = (\epsilon_1, \ldots, \epsilon_r)$ to $\epsilon' = (\epsilon'_1, \ldots, \epsilon'_s)$ is a $\mathbf{C}$-linear combination of oriented (r, s)-tangles in which each tangle T satisfies $\partial_- T = \epsilon$ and $\partial^+ T = \epsilon'$. The set of morphisms from ϵ to ϵ' is denoted by $Mor_{\mathcal{OTA}}(\epsilon, \epsilon')$. We define the composition product $T_1 T_2$ of tangles T_1 and T_2 by placing T_1 on T_2, gluing the corresponding boundaries and shrinking half along the vertical axis. The composition $T_1 T_2$ is defined only when $\partial_- T_1 = \partial^+ T_2$. The composition product will be extended $\mathbf{C}$-linearly.

Turaev's paper [8] assures that every oriented tangle T can be presented by a composition product of special tangles as in Figure 2. In other words, these special tangles are generators of $\mathcal{OTA}$.

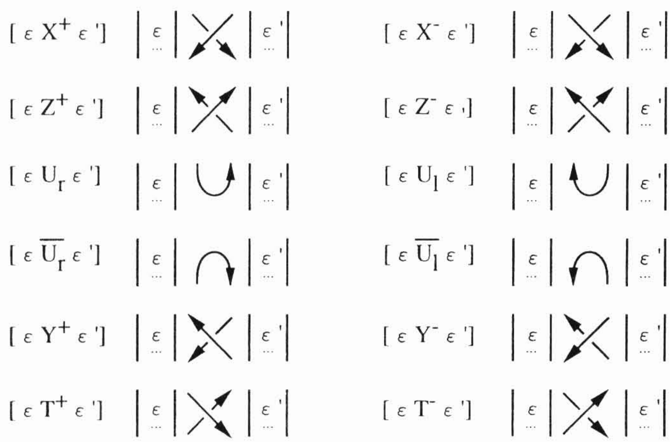

Figure 2: Special tangles

2 Invariants of the special tangles

In the papers [3, 4], the author defined a complete set of irreducible representations $\{\mathbf{P}^\gamma = (\mathbf{P}^\gamma, \mathcal{L}^\gamma)\}$ of the Hecke category. These representations are also considered as the ones of oriented tangles. This means that each representation of the Hecke category defines an invariant of tangles. Since $\mathcal{OTA}$ is generated by the special tangles, if we get the invariants of them, we can obtain the invariant of an arbitrary tangle. In the following, we give the invariant of the special tangles for each irreducible representation $\mathbf{P}^\gamma = (\mathbf{P}^\gamma, \mathcal{L}^\gamma)$.

The representation spaces $\{\mathcal{L}^\gamma\}$ are defined over $\mathbf{C}$. The objects of $\mathcal{L}^\gamma$ are $\mathbf{C}$-vector spaces $\{\mathbf{C}\Omega(\epsilon)^\gamma | \epsilon \in Ob(\mathcal{H})\}$ and the morphisms of $\mathcal{L}^\gamma$ are linear maps between every two objects of $\mathcal{L}^\gamma$. In the following we define the set $\Omega(\epsilon)^\gamma$ of tableaux. Before defining a tableau, we introduce some terminology.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition of the *length* $l(\lambda) = n$. Let $\gamma = [\alpha, \beta]$ be a pair of partitions. We call this *staircase.* The size of staircase is defined by $|\gamma| = |\alpha| - |\beta|$. If we consider a staircase as two sets of coordinates in matrix style, then we can define the inclusion $\gamma \subset \gamma'$ by $\alpha \subset \alpha'$ and $\beta' \subset \beta$. Staircases $\{\gamma = [\alpha, \beta]\}$ are partially ordered with respect to this inclusion.

A *tableau* is a sequence of staircases which is defined as follows:

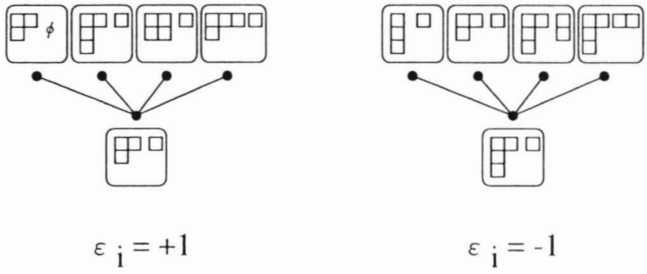

Figure 3: The branching rule

DEFINITION 2.1. (Stembridge [7]) Let $\gamma^{(0)}$ be the staircase defined by the pair of the null partitions $[\emptyset, \emptyset]$. A *tableau* ξ *of length* r *and shape* γ is a sequence $(\gamma^{(1)}, \ldots, \gamma^{(r)} = \gamma)$ of staircases in which either $\gamma^{(i)} \supset \gamma^{(i-1)}, |\gamma^{(i)}| - |\gamma^{(i-1)}| = 1$ or $\gamma^{(i)} \subset \gamma^{(i-1)}$, $|\gamma^{(i)}| - |\gamma^{(i-1)}| = -1$ for $1 \leq i \leq r$. The tableau ξ is said to be of *type* $\epsilon = (\epsilon_1, \ldots, \epsilon_r)$, where $\epsilon_i = |\gamma^{(i)}| - |\gamma^{(i-1)}|$.

For a fixed staircase γ, $\Omega(\epsilon)^\gamma$ is a set of all the tableaux whose shapes are γ and whose types are ϵ. The objects of $\mathcal{L}^\gamma$ are the $\mathbf{C}$-vector spaces $\{\mathbf{C}\Omega(\epsilon)^\gamma | \epsilon \in Ob(\mathcal{OTA})\}$. If $\Omega(\epsilon)^\gamma = \emptyset$, then $\mathbf{C}\Omega(\epsilon)^\gamma = 0$. We denote the natural basis of $\mathbf{C}\Omega(\epsilon)^\gamma$ defined by the tableaux $\{\xi | \xi \in \Omega(\epsilon)^\gamma\}$ by $\{v_\xi\}$.

Figure 3 shows how $\gamma^{(i)}$ is generated from $\gamma^{(i-1)}$ according to the signature ϵ_i in making a tableau. We call this generation rule the *branching rule.* All the tableaux of type ϵ are conveniently described using the graph Γ_ϵ. Figure 4 is an example of Γ_ϵ, where $\epsilon = (+1, -1, +1, -1, +1)$. In the picture, $\Omega(\epsilon)^{[1,0]}$ is a set of five paths from the bottom vertex $[0, 0]$ to the top vertex $[1, 0]$.

As it defined in [2, 5], each vertex of Γ_ϵ has its *weight.* These weights are defined by the indices $\{\gamma\}$ which are assigned to the vertices. Let

$$\Lambda_{k,l} = \coprod_{m=0}^{\min(k,l)} \{[\alpha, \beta]; \alpha, \beta \text{ partitions}, |\alpha| = k - m, |\beta| = l - m\}$$

be a set of pairs of partitions and let Λ be the set of all the pairs of

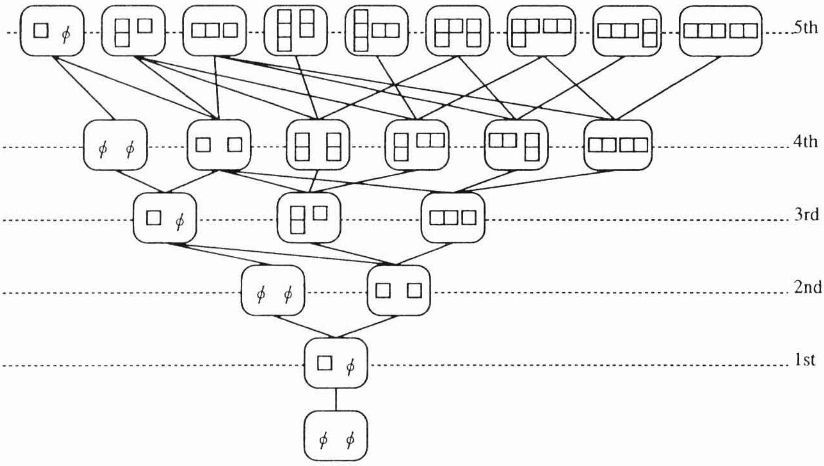

Figure 4: $\Gamma_{(+1,-1,+1,-1,+1)}$

partitions:

$$\Lambda = \bigcup_{r=0}^{\infty} \Big(\bigcup_{\substack{k \geq 0, l \geq 0 \\ k+l=r}} \Lambda_{k,l} \Big) = \coprod_{p=-\infty}^{\infty} \Big(\bigcup_{\substack{k \geq 0, l \geq 0 \\ k-l=p}} \Lambda_{k,l} \Big).$$

As is well known, any partition λ is expressed by a Young diagram. We denote the coordinates of boxes in a Young diagram in matrix style. For example, if a box is in the i-th row and in the j-th column of a Young diagram λ, it is denoted by $(i,j) \in \lambda$. Each box $(i,j) \in \lambda$ has its *hook length* $h_\lambda(i,j)$. Let $\gamma = [\lambda, \mu]$ be a staircase such that $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$. Then the weight $s[\gamma]$ of γ is defined by

$$s[\gamma] = \frac{\prod_{(i,j)\in\mu} \left([a; j-i-l(\lambda)] \prod_{k=1}^{l(\lambda)} \frac{[a; j-i+\lambda_k-k+1]}{[a; j-i+\lambda_k-k]} \right) \prod_{(i,j)\in\lambda} [a; j-i]}{\prod_{(i,j)\in\lambda} [h_\lambda(i,j)] \prod_{(i,j)\in\mu} [h_\mu(i,j)]},$$

where

$$[a; m] = \frac{a^{-1}q^m - aq^{-m}}{q - q^{-1}} \text{ and } [m] = [1; m] = \frac{q^m - q^{-m}}{q - q^{-1}}.$$

The following are examples of the weights $\{s[\gamma]\}$.

$$s[0,0] = 1,$$

$$\begin{aligned}
s[1,0] = s[0,1] &= [a;0], \\
s[11,0] = s[0,11] &= \frac{[a;0][a;-1]}{[2]}, \\
s[2,0] = s[0,2] &= \frac{[a;0][a;1]}{[2]}, \\
s[1,1] &= [a;-1][a;1], \\
s[2,1] &= \frac{[a;-1][a;2][a;0]}{[2]}, \\
s[11,1] &= \frac{[a;-2][a;1][a;0]}{[2]}.
\end{aligned}$$

An object $\epsilon = (\epsilon_1, \dots, \epsilon_k)$ of $\mathcal{OTA}$ is mapped by $\mathbf{P}^\gamma$ to an object $\mathbf{P}^\gamma(\epsilon) = \mathbf{C}\Omega(\epsilon)^\gamma$ of $\mathcal{L}^\gamma$. If either $\mathbf{C}\Omega(\epsilon)^\gamma$ or $\mathbf{C}\Omega(\epsilon')^\gamma$ is the 0 space, then $Mor_{\mathcal{L}^\gamma}(\epsilon, \epsilon') = \{0\}$. Hence if either $\Omega(\epsilon)^\gamma = \emptyset$ or $\Omega(\epsilon')^\gamma = \emptyset$, then $\mathbf{P}^\gamma(T) = 0$ for any tangle T such that $\partial_- T = \epsilon$ and $\partial^+ T = \epsilon'$.

2.1 $\mathbf{P}^\gamma([\epsilon X^+ \epsilon'])$ and $\mathbf{P}^\gamma([\epsilon X^- \epsilon'])$

Let $x = (\epsilon, +1, +1, \epsilon')$ be an object on which $[\epsilon X^+ \epsilon']$ is defined. Suppose that $\mathrm{Pos}(x) = k$, $\mathrm{Neg}(x) = l$ and $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_{i-1})$, $\epsilon' = (\epsilon_{i+2}, \epsilon_{i+3}, \dots, \epsilon_{k+l})$.

If $\gamma \notin \Lambda_{k,l}$, then define $\mathbf{P}^\gamma([\epsilon X^+ \epsilon']) = 0$.

Otherwise, each of the generators of the form $\{[\epsilon X^+ \epsilon']\}$ is mapped to a morphism from the object $\mathbf{C}\Omega(x)^\gamma$ to itself. Let

$$\xi = (\gamma^{(1)}, \dots, \gamma^{(i-1)}, \gamma^{(i)}, \gamma^{(i+1)}, \dots, \gamma^{(k+l)} = \gamma)$$

be a tableau of shape γ and of type x. Then according to the branching rule as in Figure 3, the staircase $\gamma^{(i+1)}$ is obtained from $\gamma^{(i-1)}$ one of the following ways:

case 1.1 By adding two boxes to the same row of $\alpha^{(i-1)}$.

case 1.2 By removing two boxes from the same row of $\beta^{(i-1)}$.

case 2.1 By adding two boxes to the same column of $\alpha^{(i-1)}$.

case 2.2 By removing two boxes from the same column of $\beta^{(i-1)}$.

case 3.1 By adding two boxes in different rows and columns of $\alpha^{(i-1)}$.

case 3.2 By removing two boxes from different rows and columns of $\beta^{(i-1)}$.

case 4.1 By adding one box to $\alpha^{(i-1)}$ first then removing one box from $\beta^{(i-1)}$.

case 4.2 By removing one box from $\beta^{(i-1)}$ first then adding one box to $\alpha^{(i-1)}$.

In case 3.1, 3.2, 4.1 or 4.2, there exists exactly one tableau

$$\xi' = (\gamma^{(1)}, \ldots, \gamma^{(i-1)}, (\gamma^{(i)})', \gamma^{(i+1)}, \ldots, \gamma^{(k+l)}),$$

which differs from ξ in its i-th coordinate only.

In case 3.1, if the box $(r_i, c_i) \in \alpha^{(i+1)}$ is added first and then $(r_{i+1}, c_{i+1}) \in \alpha^{(i+1)}$ is added, then we define the axis distance $d(\xi, i)$ by

$$d(\xi, i) = (c_{i+1} - r_{i+1}) - (c_i - r_i).$$

In case 3.2, if the box $(r_i, c_i) \in \beta^{(i-1)}$ is removed first and then $(r_{i+1}, c_{i+1}) \in \beta^{(i-1)}$ is removed, then we define the axis distance $d(\xi, i)$ by

$$d(\xi, i) = (c_i - r_i) - (c_{i+1} - r_{i+1}).$$

In case 4.1 or in case 4.2, we define the axis distance $d(\xi, i)$ as follows: Suppose that $(r_\alpha, c_\alpha) \in \alpha^{(i+1)} \setminus \alpha^{(i-1)}$ and $(r_\beta, c_\beta) \in \beta^{(i-1)} \setminus \beta^{(i+1)}$. Then $d(\xi, i)$ is defined by

$$d = d(\xi, i) = (c_\alpha - r_\alpha) + (c_\beta - r_\beta).$$

Using this axis distance d, we define $\mathbf{P}^\gamma([\epsilon X^+ \epsilon'])$ and $\mathbf{P}^\gamma([\epsilon X^- \epsilon'])$as follows:

$$\begin{aligned}
\mathbf{P}^\gamma([\epsilon X^+ \epsilon'])v_\xi &= a \cdot a_{\xi(i)} v_\xi + a \cdot b_{\xi(i)} v_{\xi'} \\
&= \begin{cases}
a \cdot q v_\xi & \text{case 1.1, 1.2,} \\
-a \cdot q^{-1} v_\xi & \text{case 2.1, 2.2,} \\
a \cdot \frac{q^d}{[d]} v_\xi + a \cdot \frac{[d-1]}{[d]} v_{\xi'} & \text{case 3.1, 3.2,} \\
a \cdot \frac{a q^{-d}}{[a^{-1}; -d]} v_\xi + a \cdot \frac{[a^{-1}; -d-1]}{[a^{-1}; -d]} v_{\xi'} & \text{case 4.1,} \\
a \cdot \frac{a^{-1} q^d}{[a; d]} v_\xi + a \cdot \frac{[a; d-1]}{[a; d]} v_{\xi'} & \text{case 4.2,}
\end{cases}
\end{aligned}$$

and

$$\mathbf{P}^\gamma([\epsilon X^- \epsilon']) = a^{-2} \mathbf{P}^\gamma([\epsilon X^+ \epsilon']) - a^{-1}(q - q^{-1}).$$

For the definition of $[a;m]$ and $[m]$, see the one of the weight $s[\gamma]$.

We note that

$$\begin{aligned}\mathbf{P}^{\gamma}([\epsilon X^{-}\epsilon'])v_{\xi} &= a^{-1}\cdot a'_{\xi(i)}v_{\xi} - a^{-1}\cdot b_{\xi(i)}v_{\xi'} \\ &= a^{-1}\{a_{\xi(i)} - (q-q^{-1})\}v_{\xi} - a^{-1}\cdot b_{\xi(i)}v_{\xi'}.\end{aligned}$$

REMARK 2.2. The above $a_{\xi(i)}$, $b_{\xi(i)}$ and $a'_{\xi(i)}$ are defined by the triple $\gamma^{(i-1)}$, $\gamma^{(i)}$ and $\gamma^{(i+1)}$. We sometimes use $a_{(\gamma^{(i-1)},\gamma^{(i)},\gamma^{(i+1)})}$ (resp. $b_{(\gamma^{(i-1)},\gamma^{(i)},\gamma^{(i+1)})}$, $a'_{(\gamma^{(i-1)},\gamma^{(i)},\gamma^{(i+1)})}$), instead of $a_{\xi(i)}$ (resp. $b_{\xi(i)}$, $a'_{\xi(i)}$).

2.2 $\mathbf{P}^{\gamma}([\epsilon Z^{+}\epsilon'])$ and $\mathbf{P}^{\gamma}([\epsilon Z^{-}\epsilon'])$

Let $x = (\epsilon, -1, -1, \epsilon')$ be an object on which $[\epsilon Z^{+}\epsilon']$ is defined. Suppose that $\mathrm{Pos}(x) = k$, $\mathrm{Neg}(x) = l$ and $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{i-1})$, $\epsilon' = (\epsilon_{i+2}, \epsilon_{i+3}, \ldots, \epsilon_{k+l})$.

If $\gamma \notin \Lambda_{k,l}$, then define $\mathbf{P}^{\gamma}([\epsilon Z^{+}\epsilon']) = 0$.

Otherwise, each of the generators of the form $\{[\epsilon Z^{+}\epsilon']\}$ is mapped to a morphism from the object $\mathbf{C}\Omega(x)^{\gamma}$ to itself. Let

$$\xi = (\gamma^{(1)}, \ldots, \gamma^{(i-1)}, \gamma^{(i)}, \gamma^{(i+1)}, \ldots, \gamma^{(k+l)} = \gamma)$$

be a tableau of shape γ and of type x. Then according to the branching rule as in Figure 3, the staircase $\gamma^{(i+1)}$ is obtained from $\gamma^{(i-1)}$ one of the following ways:

case 1.1 By removing two boxes from the same row of $\alpha^{(i-1)}$.

case 1.2 By adding two boxes to the same row of $\beta^{(i-1)}$.

case 2.1 By removing two boxes from the same column of $\alpha^{(i-1)}$.

case 2.2 By adding two boxes to the same column of $\beta^{(i-1)}$.

case 3.1 By removing two boxes from different rows and columns of $\alpha^{(i-1)}$.

case 3.2 By adding two boxes from different rows and columns of $\beta^{(i-1)}$.

case 4.1 By removing one box from $\alpha^{(i-1)}$ first then adding one box to $\beta^{(i-1)}$.

case 4.2 By adding one box to $\beta^{(i-1)}$ first then removing one box from $\alpha^{(i-1)}$.

In case 3.1, 3.2, 4.1 or 4.2, there exists exactly one tableau

$$\xi' = (\gamma^{(1)}, \dots, \gamma^{(i-1)}, (\gamma^{(i)})', \gamma^{(i+1)}, \dots, \gamma^{(k+l)})$$

which differs from ξ in its i-th coordinate only.

In case 3.1, if the box $(r_i, c_i) \in \alpha^{(i-1)}$ is removed first and then $(r_{i+1}, c_{i+1}) \in \alpha^{(i-1)}$ is removed, then we define the axis distance $d(\xi, i)$ by

$$d(\xi, i) = (c_{i+1} - r_{i+1}) - (c_i - r_i).$$

In case 3.2, if the box $(r_i, c_i) \in \beta^{(i+1)}$ is added first and then $(r_{i+1}, c_{i+1}) \in \beta^{(i+1)}$ is added, then we define the axis distance $d(\xi, i)$ by

$$d(\xi, i) = (c_i - r_i) - (c_{i+1} - r_{i+1}).$$

In case 4.1 or in case 4.2, we define the axis distance $d(\xi, i)$ as follows: Suppose that $(r_\alpha, c_\alpha) \in \alpha^{(i-1)} \setminus \alpha^{(i+1)}$ and $(r_\beta, c_\beta) \in \beta^{(i+1)} \setminus \beta^{(i-1)}$. Then $d(\xi, i)$ is defined by

$$d = d(\xi, i) = (c_\alpha - r_\alpha) + (c_\beta - r_\beta).$$

Using this axis distance d, we define $\mathbf{P}^\gamma([\epsilon Z^+ \epsilon'])$ and $\mathbf{P}^\gamma([\epsilon Z^- \epsilon'])$ as follows:

$$\mathbf{P}^\gamma([\epsilon Z^+ \epsilon'])v_\xi = \begin{cases} a \cdot q v_\xi & \text{case 1.1, 1.2,} \\ -a \cdot q^{-1} v_\xi & \text{case 2.1, 2.2,} \\ a \cdot \frac{q^d}{[d]} v_\xi + a \cdot \frac{[d+1]}{[d]} v_{\xi'} & \text{case 3.1, 3.2,} \\ a \cdot \frac{a^{-1}q^d}{[a;d]} v_\xi + a \cdot \frac{[a;d+1]}{[a;d]} v_{\xi'} & \text{case 4.1,} \\ a \cdot \frac{aq^{-d}}{[a^{-1};-d]} v_\xi + a \cdot \frac{[a^{-1};-d+1]}{[a^{-1};-d]} v_{\xi'} & \text{case 4.2,} \end{cases}$$

and

$$\mathbf{P}^\gamma([\epsilon Z^- \epsilon']) = a^{-2}\mathbf{P}^\gamma([\epsilon Z^+ \epsilon']) - a^{-1}(q - q^{-1}).$$

2.3 $\mathbf{P}^{\gamma}([\epsilon U_r \epsilon'])$ and $\mathbf{P}^{\gamma}([\epsilon U_l \epsilon'])$

Let $x = (\epsilon, \epsilon')$, $x_r = (\epsilon, +1, -1, \epsilon')$ $x_l = (\epsilon, -1, +1, \epsilon')$ be objects such that $[\epsilon U_r \epsilon'] : x \to x_r$ and $[\epsilon U_l \epsilon'] : x \to x_l$ are defined. Suppose that $\mathrm{Pos}(x) = k, \mathrm{Neg}(x) = l$ and that

$$\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_i)$$

and

$$\epsilon' = (\epsilon_{i+1}, \epsilon_{i+2}, \ldots, \epsilon_{k+l}).$$

If $\gamma \notin \Lambda_{k,l}$, then define $\mathbf{P}^{\gamma}([\epsilon U_r \epsilon']) = 0$ and $\mathbf{P}^{\gamma}([\epsilon U_l \epsilon']) = 0$.

A generator $[\epsilon U_r \epsilon']$ (resp. $[\epsilon U_l \epsilon']$) is mapped by $\mathbf{P}^{\gamma}$ to a morphism from the object $\mathbf{C}\Omega(x)^{\gamma}$ to the object $\mathbf{C}\Omega(x_r)^{\gamma}$ (resp. $\mathbf{C}\Omega(x_l)^{\gamma}$). For each tableau

$$\xi = (\gamma^{(1)}, \ldots, \gamma^{(i-1)}, \mu, \gamma^{(i+1)}, \ldots, \gamma^{(k+l)} = \gamma)$$

of shape γ and of type x, we define the tableau $\xi(j)$ (resp. $\xi'(j')$) of shape γ and of type x_r (resp. x_l) as follows:

$$\begin{aligned} \xi(j) &= (\gamma^{(1)}, \ldots, \gamma^{(i-1)}, \mu, \lambda(j), \mu, \gamma^{(i+1)}, \ldots, \gamma^{(k+l)} = \gamma), \\ (\text{resp. } \xi'(j') &= (\gamma^{(1)}, \ldots, \gamma^{(i-1)}, \mu, \nu(j'), \mu, \gamma^{(i+1)}, \ldots, \gamma^{(k+l)} = \gamma),) \end{aligned}$$

where $\{\lambda(j)\}$ $(j = 1, 2, \ldots, p(\mu))$ (resp. $\{\nu(j')\}$ $(j' = 1, 2, \ldots, p'(\mu))$ are all the staircases such that $\lambda(j) \supset \mu$ and $|\lambda(j)| - |\mu| = 1$ (resp. $\nu(j') \subset \mu$ and $|\nu(j')| - |\mu| = -1$). See the branching rule pictured in Figure 3. Under these notation $\mathbf{P}^{\gamma}([\epsilon U_r \epsilon'])$ (resp. $\mathbf{P}^{\gamma}([\epsilon U_l \epsilon'])$) is defined as follows:

$$\begin{aligned} \mathbf{P}^{\gamma}([\epsilon U_r \epsilon'])v_{\xi} &= \sum_{j}^{p(\mu)} v_{\xi(j)}. \\ \left(\text{resp. } \mathbf{P}^{\gamma}([\epsilon U_l \epsilon'])v_{\xi} \right. &= \left. \sum_{j'}^{p'(\mu)} \frac{s[\nu(j')]}{s[\mu]} v_{\xi'(j')} \right). \end{aligned}$$

2.4 $\mathbf{P}^{\gamma}([\epsilon \bar{U}_r \epsilon'])$ and $\mathbf{P}^{\gamma}([\epsilon \bar{U}_l \epsilon'])$

Let $x_r = (\epsilon, -1, +1, \epsilon')$, $x_l = (\epsilon, +1, -1, \epsilon')$, $\hat{x} = (\epsilon, \epsilon')$ be objects such that $[\epsilon \bar{U}_r \epsilon'] : x_r \to \hat{x}$ and $[\epsilon \bar{U}_l \epsilon'] : x_l \to \hat{x}$ are defined. Suppose that $\mathrm{Pos}(x_r) = \mathrm{Pos}(x_l) = k, \mathrm{Neg}(x_r) = \mathrm{Neg}(x_l) = l$ and that

$$\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{i-1})$$

and

$$\epsilon' = (\epsilon_{i+2}, \epsilon_{i+3}, \ldots, \epsilon_{k+l}).$$

If $\gamma \notin \Lambda_{k-1,l-1}$, then define $\mathbf{P}^\gamma([\epsilon \bar{U}_r \epsilon']) = 0$ and $\mathbf{P}^\gamma([\epsilon \bar{U}_l \epsilon']) = 0$.

A generator $[\epsilon \bar{U}_r \epsilon']$ (resp. $[\epsilon \bar{U}_l \epsilon']$) is mapped by $\mathbf{P}^\gamma$ to a morphism from the object $\mathbf{C}\Omega(x_r)^\gamma$ (resp. $\mathbf{C}\Omega(x_l)^\gamma$) to the object $\mathbf{C}\Omega(\hat{x})^\gamma$. For each tableau

$$\xi = (\gamma^{(1)}, \ldots, \gamma^{(i-1)} = \nu, \gamma^{(i)} = \mu, \gamma^{(i+1)}, \ldots, \gamma^{(k+l)} = \gamma)$$

of shape γ and of type x_r (resp. x_l), we define

$$\mathbf{P}^\gamma([\epsilon \bar{U}_r \epsilon'])v_\xi = \begin{cases} 0, & \text{if } \gamma^{(i-1)} \neq \gamma^{(i+1)}, \\ v_{\hat{\xi}}, & \text{if } \gamma^{(i-1)} = \gamma^{(i+1)}, \end{cases}$$

$$\left(\text{resp. } \mathbf{P}^\gamma([\epsilon \bar{U}_l \epsilon'])v_\xi = \begin{cases} 0, & \text{if } \gamma^{(i-1)} \neq \gamma^{(i+1)}, \\ (s[\mu]/s[\nu])v_{\hat{\xi}}, & \text{if } \gamma^{(i-1)} = \gamma^{(i+1)}, \end{cases} \right)$$

where $\hat{\xi}$ is a tableau of shape γ and of type $\hat{x}$ which is obtained from the tableau ξ by removing the i-th coordinate $\gamma^{(i)} = \mu$ and the $(i+1)$-st coordinate $\gamma^{(i+1)}$.

2.5 $[\epsilon Y^+ \epsilon']$, $[\epsilon Y^- \epsilon']$ and $[\epsilon T^+ \epsilon']$, $[\epsilon T^- \epsilon']$

Suppose that $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{i-1})$ and $\epsilon' = (\epsilon_{i+2}, \epsilon_{i+3}, \ldots, \epsilon_{k+l})$. For a tableau

$$\xi = (\gamma^{(1)}, \ldots, \gamma^{(i-2)}, \mu_0, \lambda_0, \mu_1, \gamma^{(i+2)}, \ldots, \gamma^{(k+l)}),$$

of shape γ and of type $(\epsilon, +1, -1, \epsilon')$, we define tableaux

$$\{\xi(j') = (\gamma^{(1)}, \ldots, \gamma^{(i-2)}, \mu_0, \nu(j'), \mu_0, \lambda_0, \mu_1, \gamma^{(i+2)}, \ldots, \gamma^{(k+l)}) \mid j' = 1, 2, \ldots, p'(\mu_0)\}$$

of shape γ and of type $(\epsilon, -1, +1, +1, -1, \epsilon')$ by duplicating the $(i-1)$-st coordinate μ_0 of ξ and then inserting $\nu(j')$ between them, where $\{\nu(j')\}$ are all the staircases such that $\nu(j') \subset \mu_0$ and $|\nu(j')| - |\mu_0| = -1$. If $\mu_0 \neq \mu_1$, then there exists an index j'_0 such that $\mu_0 \supset \nu(j'_0)$ and $\mu_1 \supset \nu(j'_0)$. We put $\nu_0 = \nu(j'_0)$. For $\xi(j')$ we define $\xi'(j')$ by replacing the $(i+1)$-st coordinate μ_0 of $\xi(j')$ with $\mu_0(j')$ such that $\nu(j') \subset \mu_0(j') \subset \lambda_0$ and $\mu_0(j') \neq \mu_0$ (if it exists).

Similarly, for a tableau

$$\eta = (\gamma^{(1)}, \ldots, \gamma^{(i-2)}, \mu_0, \nu_0, \mu_1, \gamma^{(i+2)}, \ldots, \gamma^{(k+l)})$$

of shape γ and of type $(\epsilon, -1, +1, \epsilon')$, we define tableaux

$$\{\eta(j) = (\gamma^{(1)}, \ldots, \gamma^{(i-2)}, \mu_0, \nu_0, \mu_1, \lambda(j), \mu_1, \gamma^{(i+2)}, \ldots, \gamma^{(k+l)}) \mid j = 1, 2, \ldots, p(\mu_1)\}$$

of shape γ and of type $(\epsilon, -1, +1, +1, -1, \epsilon')$ by duplicating the $(i+1)$-st coordinate μ_1 of η and then inserting $\lambda(j)$ between them, where $\{\lambda(j)\}$ are all the staircases such that $\lambda(j) \supset \mu_1$ and $|\lambda(j)| - |\mu_1| = 1$. For $\eta(j)$ we define $\eta'(j)$ by replacing the $(i+1)$-st coordinate μ_1 of $\eta(j)$ with $\mu_1(j)$ such that $\nu_0 \subset \mu_1(j) \subset \lambda(j)$ and $\mu_1(j) \neq \mu_1$ (if it exists). If $\mu_0 \neq \mu_1$, then there exists an index j_0 such that $\lambda(j_0) \supset \mu_0$ and $\lambda(j_0) \supset \mu_1$. We put $\lambda_0 = \lambda(j_0)$.

We note that if $\mu_0 \neq \mu_1$, then for the tableau ξ, the tableau η is uniquely determined by replacing the i-th coordinate $\lambda_0 = \mu_0 \cup \mu_1$ with $\nu_0 = \mu_0 \cap \mu_1$ and vice versa. In case $\mu_0 = \mu_1$, we set $\xi_j, (j = 1, 2, \ldots, p(\mu_1))$ and $\eta_{j'}(j' = 1, 2, \ldots, p'(\mu_0))$ as follows:

$$\begin{aligned} \xi_j &= (\gamma^{(1)}, \ldots, \gamma^{(i-2)}, \mu_0, \lambda(j), \mu_1, \gamma^{(i+2)}, \ldots, \gamma^{(k+l)}), \\ \eta_{j'} &= (\gamma^{(1)}, \ldots, \gamma^{(i-2)}, \mu_0, \nu(j'), \mu_1, \gamma^{(i+2)}, \ldots, \gamma^{(k+l)}). \end{aligned}$$

Then we obtain

$$\mathbf{P}^\gamma([\epsilon Y^+ \epsilon'])v_\xi = \begin{cases} a \sum_{j'} a_{(\nu(j'), \mu_0, \lambda_0)} \frac{s[\nu(j')]s[\lambda_0]}{s[\mu_0]^2} v_{\eta_{j'}} & (\text{if } \mu_0 = \mu_1), \\ a \cdot b_{(\nu_0, \mu_0, \lambda_0)} \frac{s[\nu_0]s[\lambda_0]}{s[\mu_0]s[\mu_1]} v_\eta & (\text{otherwise}), \end{cases}$$

$$\mathbf{P}^\gamma([\epsilon Y^- \epsilon'])v_\xi = \begin{cases} a^{-1} \sum_{j'} a'_{(\nu(j'), \mu_0, \lambda_0)} \frac{s[\nu(j')]s[\lambda_0]}{s[\mu_0]^2} v_{\eta_{j'}} & (\text{if } \mu_0 = \mu_1), \\ a^{-1} \cdot b_{(\nu_0, \mu_0, \lambda_0)} \frac{s[\nu_0]s[\lambda_0]}{s[\mu_0]s[\mu_1]} v_\eta & (\text{otherwise}), \end{cases}$$

$$\mathbf{P}^\gamma([\epsilon T^+ \epsilon'])v_\eta = \begin{cases} a \sum_j a_{(\nu_0, \mu_1, \lambda(j))} v_{\xi_j} & (\text{if } \mu_0 = \mu_1), \\ a \cdot b_{(\nu_0, \mu_1, \lambda_0)} v_\xi & (\text{otherwise}). \end{cases}$$

$$\mathbf{P}^\gamma([\epsilon T^- \epsilon'])v_\eta = \begin{cases} a^{-1} \sum_j a'_{(\nu_0, \mu_1, \lambda(j))} v_{\xi_j} & (\text{if } \mu_0 = \mu_1), \\ a^{-1} \cdot b_{(\nu_0, \mu_1, \lambda_0))} v_\xi & (\text{otherwise}). \end{cases}$$

3 Examples

According to the definition in the previous section, we calculate the invariant of some famous links. Since oriented links can be considered as $(0, 0)$-tangles, and since $\Lambda_{0,0} = \{(\emptyset, \emptyset)\}$, the only non zero

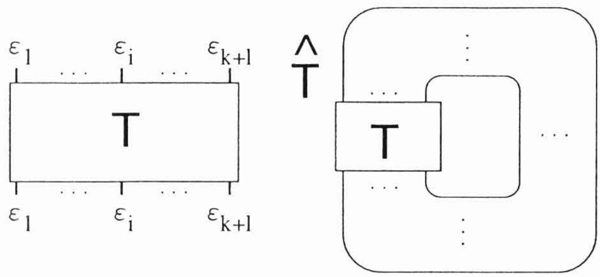

Figure 5: Closed Tangle

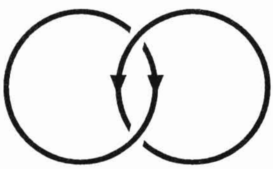

Figure 6: Hopf Link

representation of links is one dimensional. This means the invariant of a link L is given by a scalar $P(L)$.

Moreover, suppose that a link L is given by a closed tangle $\hat{T}$ as in Figure 5. In this case, it is easily checked that the invariant is given by the formula

$$P(\hat{T}) = \frac{1}{[a;0]} \sum_{\gamma \in \Lambda_{k,l}} s[\gamma] tr(\mathbf{P}^{\gamma}(T)),$$

where $k = \mathrm{Pos}(\partial_{-}T)$ and $l = \mathrm{Neg}(\partial_{-}T)$.

Using this formula, we calculate the invariant of the Hopf link, the Whitehead link and the Borromean link.

3.1 The Hopf link

The easiest non trivial example is the Hopf link, L_{Ho}, which is pictured in Figure 6.

This link is obtained from the tangle $[X^+][X^+]$ by closing the corresponding boundaries. Since the type of $[X^+][X^+]$ is $(+1, +1)$, there

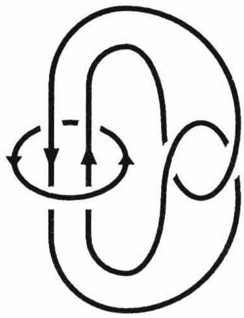

Figure 7: Whitehead Link

are only two non zero representations, $\gamma = [2,0]$ and $\gamma' = [11,0]$, and these are both one dimensional. The special tangle $[X^+]$ acts on these spaces by scalar multiples aq and $-aq^{-1}$ respectively. Hence we obtain

$$\begin{aligned} P(L_{Ho}) &= \frac{s[\gamma]tr(\mathbf{P}^{\gamma}([X^+][X^+])) + s[\gamma']tr(\mathbf{P}^{\gamma'}([X^+][X^+]))}{[a;0]} \\ &= \frac{1}{[a;0]}\left\{\frac{[a;0][a;1]}{[2]}(aq)^2 + \frac{[a;0][a;-1]}{[2]}(-aq^{-1})^2\right\} \\ &= \frac{1}{[2]}([a;1]a^2q^2 + [a;-1]a^2q^{-2}) \\ &= a(q-q^{-1}) + a^2\frac{a^{-1}-a}{q-q^{-1}}. \end{aligned}$$

3.2 The Whitehead link

The next example is the Whitehead link, L_{Wh}, which is pictured in Figure 7. Put

$$[Wh1] = [(+1)\overline{U_l}(-1)]([X^-(-1,-1)][(+1,+1)Z^+])^2[(+1)U_r(-1)]$$

and

$$[Wh2] = [(+1)\overline{U_l}(-1)][X^+(-1,-1)][(+1,+1)Z^+][(+1)U_r(-1)].$$

The Whitehead link is obtained from the tangle

$$[Wh] = [Wh1][Wh2]$$

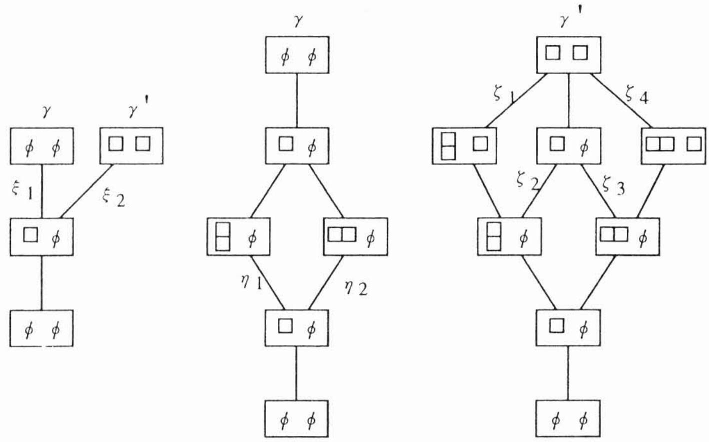

Figure 8: Representation spaces to calculate the Whitehead link

by closing the corresponding boundaries.

Since the type of the tangle $[Wh]$ is $(+1, -1)$, there are only two possible representations, $\gamma = [0, 0]$ and $\gamma' = [1, 1]$ and these are both one dimensional spaces. We denote the bases by v_{ξ_1} and v_{ξ_2} respectively. However, unlike the case of the Hopf link, to calculate the invariant, we need two more spaces, a two dimensional space and a four dimensional space. (See Figure 8.) We denote the standard bases of these spaces by $\{v_{\eta_1}, v_{\eta_2}\}$ and $\{v_{\zeta_1}, v_{\zeta_2}, v_{\zeta_3}, v_{\zeta_4}\}$ respectively. From the definition of previous section, we have the following:

$$\begin{aligned}
\mathbf{P}^{\gamma}([(+1)U_r(-1)])v_{\xi_1} &= v_{\eta_1} + v_{\eta_2}, \\
\mathbf{P}^{\gamma'}([(+1)U_r(-1)])v_{\xi_2} &= v_{\zeta_2} + v_{\zeta_3}, \\
\mathbf{P}^{\gamma}([(+1,+1)Z^+])v_{\eta_1} &= -aq^{-1}v_{\eta_1}, \\
\mathbf{P}^{\gamma}([(+1,+1)Z^+])v_{\eta_2} &= aqv_{\eta_2}, \\
\mathbf{P}^{\gamma'}([(+1,+1)Z^+])v_{\zeta_1} &= -\frac{a^2q}{[a;-1]}v_{\zeta_1} + \frac{a[a;-2]}{[a;-1]}v_{\zeta_2}, \\
\mathbf{P}^{\gamma'}([(+1,+1)Z^+])v_{\zeta_2} &= \frac{a[a;0]}{[a;-1]}v_{\zeta_1} + \frac{q^{-1}}{[a;-1]}v_{\zeta_2}, \\
\mathbf{P}^{\gamma'}([(+1,+1)Z^+])v_{\zeta_3} &= \frac{q}{[a;1]}v_{\zeta_3} + \frac{a[a;2]}{[a;1]}v_{\zeta_4},
\end{aligned}$$

$$
\begin{aligned}
\mathbf{P}^{\gamma'}([(+1,+1)Z^+])v_{\zeta_4} &= \frac{a[a;0]}{[a;1]}v_{\zeta_3} - \frac{a^2q^{-1}}{[a;1]}v_{\zeta_4},\\
\mathbf{P}^{\gamma}([X^+(-1,-1)])v_{\eta_1} &= -aq^{-1}v_{\eta_1},\\
\mathbf{P}^{\gamma}([X^+(-1,-1)])v_{\eta_2} &= aqv_{\eta_2},\\
\mathbf{P}^{\gamma'}([X^+(-1,-1)])v_{\zeta_1} &= -aq^{-1}v_{\zeta_1},\\
\mathbf{P}^{\gamma'}([X^+(-1,-1)])v_{\zeta_2} &= -aq^{-1}v_{\zeta_2},\\
\mathbf{P}^{\gamma'}([X^+(-1,-1)])v_{\zeta_3} &= aqv_{\zeta_3},\\
\mathbf{P}^{\gamma'}([X^+(-1,-1)])v_{\zeta_4} &= aqv_{\zeta_4},\\
\mathbf{P}^{\gamma}([X^-(-1,-1)])v_{\eta_1} &= -a^{-1}qv_{\eta_1},\\
\mathbf{P}^{\gamma}([X^-(-1,-1)])v_{\eta_2} &= a^{-1}q^{-1}v_{\eta_2},\\
\mathbf{P}^{\gamma'}([X^-(-1,-1)])v_{\zeta_1} &= -a^{-1}qv_{\zeta_1},\\
\mathbf{P}^{\gamma'}([X^-(-1,-1)])v_{\zeta_2} &= -a^{-1}qv_{\zeta_2}\\
\mathbf{P}^{\gamma'}([X^-(-1,-1)])v_{\zeta_3} &= a^{-1}q^{-1}v_{\zeta_3},\\
\mathbf{P}^{\gamma'}([X^-(-1,-1)])v_{\zeta_4} &= a^{-1}q^{-1}v_{\zeta_4},\\
\mathbf{P}^{\gamma}([(+1)\overline{U_l}(-1)])v_{\eta_1} &= \frac{s[11,0]}{s[1,0]}v_{\xi_1} = \frac{[a;-1]}{[2]}v_{\xi_1},\\
\mathbf{P}^{\gamma}([(+1)\overline{U_l}(-1)])v_{\eta_2} &= \frac{s[2,0]}{s[1,0]}v_{\xi_1} = \frac{[a;1]}{[2]}v_{\xi_1},\\
\mathbf{P}^{\gamma'}([(+1)\overline{U_l}(-1)])v_{\zeta_1} &= 0,\\
\mathbf{P}^{\gamma'}([(+1)\overline{U_l}(-1)])v_{\zeta_2} &= \frac{s[11,0]}{s[1,0]}v_{\xi_2} = \frac{[a;-1]}{[2]}v_{\xi_2},\\
\mathbf{P}^{\gamma'}([(+1)\overline{U_l}(-1)])v_{\zeta_3} &= \frac{s[2,0]}{s[1,0]}v_{\xi_2} = \frac{[a;1]}{[2]}v_{\xi_2},\\
\mathbf{P}^{\gamma'}([(+1)\overline{U_l}(-1)])v_{\zeta_4} &= 0.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\mathbf{P}^{\gamma}([Wh2])v_{\xi_1} &= a^2\left(q^{-2}\frac{[a;-1]}{[2]} + q^2\frac{[a;1]}{[2]}\right)v_{\xi_1}\\
&= \frac{a^2}{[2]}(q^{-2}[a;-1] + q^2[a;1])v_{\xi_1},\\
\mathbf{P}^{\gamma}([Wh1])v_{\xi_1} &= \frac{1}{[2]}([a;-1] + [a;1])v_{\xi_1}\\
&= [a;0]v_{\xi_1},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{P}^{\gamma'}([Wh2])v_{\xi_2} &= \left(-\frac{aq^{-2}}{[a;-1]}\frac{[a;-1]}{[2]} + \frac{aq^2}{[a;1]}\frac{[a;1]}{[2]}\right)v_{\xi_2} \\
&= \frac{1}{[2]}(-aq^{-2}+aq^2)v_{\xi_2} \\
&= a(q-q^{-1})v_{\xi_2}, \\
\mathbf{P}^{\gamma'}([Wh1])v_{\xi_2} &= \frac{[a;-1]}{[2]}\left(\frac{a^{-2}}{[a;-1]^2} + \frac{q^2[a;0][a;-2]}{[a;-1]^2}\right)v_{\xi_2} \\
&\quad + \frac{[a;1]}{[2]}\left(\frac{a^{-2}}{[a;1]^2} + \frac{q^{-2}[a;0][a;2]}{[a;1]^2}\right)v_{\xi_2} \\
&= \left(\frac{a^{-2}+[a;0][a;-2]q^2}{[2][a;-1]} + \frac{a^{-2}+[a;0][a;2]q^{-2}}{[2][a;1]}\right)v_{\xi_2} \\
&= [a;0](q^2-1+q^2)v_{\xi_2}.
\end{aligned}
$$

Hence

$$
\mathbf{P}^{\gamma}([Wh])v_{\xi_1} = \mathbf{P}^{\gamma}([Wh1][Wh2])v_{\xi_1} = \frac{a^2[a;0]}{[2]}([a;-1]q^{-2}+[a;1]q^2)v_{\xi_1}
$$

and

$$
\mathbf{P}^{\gamma'}([Wh])v_{\xi_2} = \mathbf{P}^{\gamma'}([Wh1][Wh2])v_{\xi_2} = [a;0](q^2-1+q^{-2})a(q-q^{-1})v_{\xi_2}.
$$

Finally, we obtain the invariant of the Whitehead link as follows:

$$
\begin{aligned}
P(L_{Wh}) &= \frac{1}{[a;0]}\left\{s[\gamma]tr(\mathbf{P}^{\gamma}([Wh])) + s[\gamma']tr(\mathbf{P}^{\gamma'}([Wh]))\right\} \\
&= \frac{1}{[a;0]}\left\{1\cdot\frac{a^2[a;0]}{[2]}([a;-1]q^{-2}+[a;1]q^2)\right\} \\
&\quad + \frac{1}{[a;0]}\left\{[a;-1][a;1]\cdot[a;0](q^2-1+q^{-2})a(q-q^{-1})\right\} \\
&= \frac{a^2([a;-1]q^{-2}+[a;1]q^2)}{[2]} \\
&\quad +[a;-1][a;1](q^2-1+q^{-2})a(q-q^{-1}) \\
&= \frac{a(-a^2q^2+q^4-q^2+1)}{q(q^2-1)} \\
&\quad + \frac{(a^2q^2-1)(a^2-q^2)(q^4-q^2+1)}{aq^3(q^2-1)}
\end{aligned}
$$

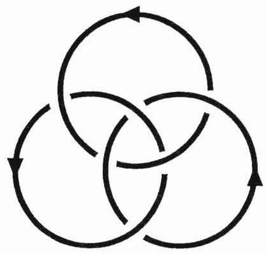

Figure 9: Borromean Link

$$= \frac{(q^6 - 2\,q^4 + q^2)a^4}{aq^3(q^2-1)} + \frac{(-q^8 + 2\,q^6 - 3\,q^4 + 2\,q^2 - 1)a^2 + q^6 - q^4 + q^2}{aq^3(q^2-1)}.$$

3.3 Borromean link

Finally, we calculate the invariant of the Borromean link L_{Bor}, which is pictured in Figure 9. This link is obtained from the tangle

$$[Bor] = [Y^-(+1)][(+1)Y^+][X^+(-1)][(+1)T^+][T^-(+1)][(-1)X^-]$$

by closing the corresponding boundaries.

Since the type of the above tangle is $(-1, +1, +1)$, there are three non zero representations, $\gamma = [1, 0]$, $\gamma' = [2, 1]$ and $\gamma'' = [11, 1]$. The natural bases of these spaces are defined in Figure 10. According to the formula in Section 2, the special tangles which are involved in $[Bor]$ are mapped as follows:

$$\begin{aligned}
\mathbf{P}^\gamma([(-1)X^-])v_{\xi_1} &= \frac{1}{[a;0]}v_{\xi_1} + a^{-1}\frac{[a;-1]}{[a;0]}\xi_2,\\
\mathbf{P}^\gamma([(-1)X^-])v_{\xi_2} &= a^{-1}\frac{[a;1]}{[a;0]}v_{\xi_1} - \frac{a^{-2}}{[a;0]}v_{\xi_2},\\
\mathbf{P}^{\gamma'}([(-1)X^-])v_{\xi_3} &= a^{-1}q^{-1}v_{\xi_3},\\
\mathbf{P}^{\gamma''}([(-1)X^-])v_{\xi_4} &= -a^{-1}qv_{\xi_4},\\
\mathbf{P}^\gamma([T^-(+1)])v_{\xi_1} &= \frac{1}{[a;0]}v_{\eta_1},
\end{aligned}$$

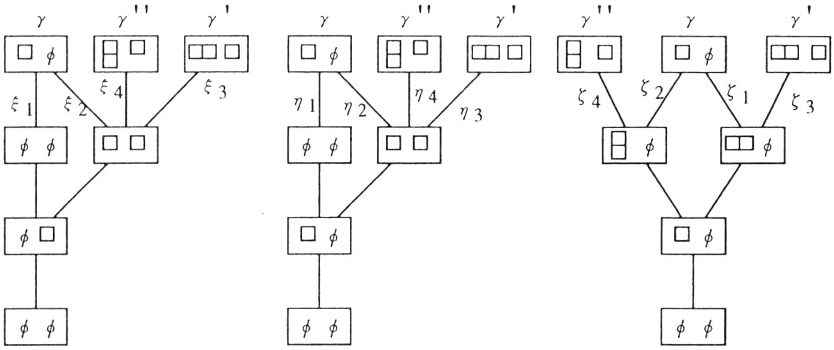

Figure 10: Representation spaces to calculate Borromean Link

$$\mathbf{P}^{\gamma}([T^{-}(+1)])v_{\xi_2} = a^{-1}\frac{[a;1]}{[a;0]}v_{\eta_2},$$

$$\mathbf{P}^{\gamma'}([T^{-}(+1)])v_{\xi_3} = a^{-1}\frac{[a;1]}{[a;0]}v_{\eta_3},$$

$$\mathbf{P}^{\gamma''}([T^{-}(+1)])v_{\xi_4} = a^{-1}\frac{[a;1]}{[a;0]}v_{\eta_4},$$

$$\mathbf{P}^{\gamma}([(+1)T^{+}])v_{\eta_1} = aqv_{\zeta_1} - aq^{-1}v_{\zeta_2},$$

$$\mathbf{P}^{\gamma}([(+1)T^{+}])v_{\eta_2} = \frac{q}{[a;1]}v_{\zeta_1} + \frac{q^{-1}}{[a;-1]}v_{\zeta_2},$$

$$\mathbf{P}^{\gamma'}([(+1)T^{+}])v_{\eta_3} = a\frac{[a;2]}{[a;1]}v_{\zeta_3},$$

$$\mathbf{P}^{\gamma''}([(+1)T^{+}])v_{\eta_4} = a\frac{[a;0]}{[a;-1]}v_{\zeta_4},$$

$$\mathbf{P}^{\gamma}([X^{+}(-1)])v_{\zeta_1} = aqv_{\zeta_1},$$

$$\mathbf{P}^{\gamma}([X^{+}(-1)])v_{\zeta_2} = -aq^{-1}v_{\zeta_2},$$

$$\mathbf{P}^{\gamma'}([X^{+}(-1)])v_{\zeta_3} = aqv_{\zeta_3},$$

$$\mathbf{P}^{\gamma''}([X^{+}(-1)])v_{\zeta_4} = -aq^{-1}v_{\zeta_4}$$

$$\mathbf{P}^{\gamma}([(+1)Y^{+}])v_{\zeta_1}$$

$$= aq\frac{s[0,0]s[2,0]}{s[1,0]^2}v_{\eta_1} + a\frac{a^{-1}q}{[a;1]}\frac{s[1,1]s[2,0]}{s[1,0]^2}v_{\eta_2}$$

$$= \frac{aq[a;1]}{[2][a;0]} v_{\eta_1} + \frac{q[a;-1][a;1]}{[2][a;0]} v_{\eta_2},$$

$$\mathbf{P}^{\gamma}([(+1)Y^+])v_{\zeta_2}$$

$$= a(-q^{-1}) \frac{s[0,0]s[11,0]}{s[1,0]^2} v_{\eta_1} + a \frac{a^{-1}q^{-1}}{[a;-1]} \frac{s[1,1]s[11,0]}{s[1,0]^2} v_{\eta_2}$$

$$= \frac{-aq^{-1}[a;-1]}{[2][a;0]} v_{\eta_1} + \frac{q^{-1}[a;-1][a;1]}{[2][a;0]} v_{\eta_2},$$

$$\mathbf{P}^{\gamma'}([(+1)Y^+])v_{\zeta_3} = a \frac{[a;0]}{[a;1]} \frac{s[1,1]s[2,0]}{s[1,0]s[2,1]} v_{\eta_3} = \frac{a[a;1]}{[a;2]} v_{\eta_3},$$

$$\mathbf{P}^{\gamma''}([(+1)Y^+])v_{\zeta_4} = a \frac{[a;-2]}{[a;-1]} \frac{s[1,1]s[11,0]}{s[1,0]s[11,1]} = \frac{a[a;-1]}{[a;0]} v_{\eta_4},$$

$$\mathbf{P}^{\gamma}([Y^-(+1)])v_{\eta_1} = a^{-1} \frac{a}{[a;0]} \frac{s[0,1]s[1,0]}{s[0,0]^2} v_{\xi_1} = [a;0]v_{\xi_1},$$

$$\mathbf{P}^{\gamma}([Y^-(+1)])v_{\eta_2} = a^{-1} \frac{[a;-1]}{[a;0]} \frac{s[0,1]s[1,0]}{s[0,0]s[1,1]} v_{\xi_2} = \frac{a^{-1}[a;0]}{[a;1]} v_{\xi_2},$$

$$\mathbf{P}^{\gamma'}([Y^-(+1)])v_{\eta_3} = a^{-1} \frac{[a;-1]}{[a;0]} \frac{s[0,1]s[1,0]}{s[0,0]s[1,1]} v_{\xi_3} = \frac{a^{-1}[a;0]}{[a;1]} v_{\xi_3},$$

$$\mathbf{P}^{\gamma''}([Y^-(+1)])v_{\eta_4} = a^{-1} \frac{[a;-1]}{[a;0]} \frac{s[0,1]s[1,0]}{s[0,0]s[1,1]} v_{\xi_4} = \frac{a^{-1}[a;0]}{[a;1]} v_{\xi_4}.$$

Hence we have

$$\mathbf{P}^{\gamma}([Bor])v_{\xi_1}$$

$$= \frac{a^3(q^3[a;1] - q^{-3}[a;-1]) + [a;1][a;-1](q^3+q^{-3})}{[a;0]^2[2]} v_{\xi_1}$$

$$+ \frac{[a;-1]\{a^{-2}(q^3[a;-1] - q^{-3}[a;1]) + a(q^3+q^{-3})\}}{[a;0]^2[2]} v_{\xi_2},$$

$$\mathbf{P}^{\gamma}([Bor])v_{\xi_2}$$

$$= \frac{[a;1]\{a^2(q^3[a;1] - q^{-3}[a;-1]) - a^{-1}(q^3+q^{-3})\}}{[a;0]^2[2]} v_{\xi_1}$$

$$+ \frac{a^{-3}(q^{-3}[a;1] - q^3[a;-1]) + [a;1][a;-1](q^3+q^{-3})}{[a;0]^2[2]} v_{\xi_2}$$

and

$$\mathbf{P}^{\gamma'}([Bor])v_{\xi_3} = v_{\xi_3}, \quad \mathbf{P}^{\gamma''}([Bor])v_{\xi_4} = v_{\xi_4}.$$

Using the formula for a closed tangle, we finally obtain

$$\begin{aligned}
P(L_{Bor}) &= \frac{s[\gamma]}{[a;0]} \cdot tr(\mathbf{P}^{\gamma}([Bor])) \\
&\quad + \frac{s[\gamma']}{[a;0]} \cdot tr(\mathbf{P}^{\gamma'}([Bor])) + \frac{s[\gamma'']}{[a;0]} \cdot tr(\mathbf{P}^{\gamma''}([Bor])) \\
&= 1 \cdot tr(\mathbf{P}^{\gamma}([Bor])) + \frac{[a;-1][a;2]}{[2]} \cdot 1 + \frac{[a;-2][a;1]}{[2]} \cdot 1 \\
&= tr(\mathbf{P}^{\gamma}([Bor])) + [a;1][a;-1] - 1 \\
&= \cdots \\
&= \frac{(-q^{10} + 4\,q^{8} - 5\,q^{6} + 4\,q^{4} - q^{2})(a^{4}+1)}{a^{2}(q^{2}-1)^{2}q^{4}} \\
&\quad + \frac{(q^{12} - 4\,q^{10} + 7\,q^{8} - 10\,q^{6} + 7\,q^{4} - 4\,q^{2} + 1)a^{2}}{a^{2}(q^{2}-1)^{2}q^{4}}.
\end{aligned}$$

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Department of Mathematical Sciences
College of Science
University of the Ryukyus
Nishihara-cho, Okinawa 903-0213
JAPAN