The irreducibilities of Appell's F_4

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THE IRREDUCIBILITIES OF APPELL's F_4

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1. Introduction.

In this paper we give a necessary and sufficient condition for the irreducibility of the system of Appell's differential equations of type F_4.

Appell's system of hypergeometric differential equations of type F_4 is defined by

$$(F_4)\quad \begin{cases} X(1-X)z_{XX}-Y^2z_{YY}-2XYz_{XY}+cz_X-(a+b+1)(Xz_X+Yz_Y)-abz=0 \\ Y(1-Y)z_{YY}-X^2z_{XX}-2XYz_{XY}+c'z_Y-(a+b+1)(Xz_X+Yz_Y)-abz=0 \end{cases}$$

which has four dimensional solution space $\mathcal{F}_4$.

We say that (F_4) (or $\mathcal{F}_4$) is irreducible if there is no proper subspace of $\mathcal{F}_4$ which is invariant under operations of the monodromy group. Then irreducibility conditions are given by the following theorem.

THEOREM 1. (F_4) is irreducible if and only if none of $a, b, a-c, a-c', a-c-c', b-c, b-c', b-c-c'$ is an integer.

In [4], we have proved THEOREM 1 under the assumption

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that $c, c' \notin \mathbb{Z}$. This restriction arises from the fact that the fundamental solutions ω_i $1 \le i \le 4$ adopted there are invalid when c or c' is an integer. But the "only if" part holds even if c or c' is an integer by continuity. We prove the "if" part of THEOREM 1 by chosing suitable fundamental solutions g_i $1 \le i \le 4$.

2. Proof of THEOREM 1

In this section we prove the "if part" of THEOREM 1. So we assume that none of $a, b, a-c, b-c, a-c', b-c', a-c-c', b-c-c'$ is an integer.

In [4], we have proved that (F_4) is irreducible when $c, c' \notin \mathbb{Z}$. It is also true that if $c, a-b \notin \mathbb{Z}$ or if $c', a-b \notin \mathbb{Z}$ then (F_4) is irreducible by exchanging L_Y and L_∞ or L_X and L_∞ respectively.

So it suffices to prove for the case when two of $c, c', a-b$ are integers. We may assume for example that c and c' are integer.

Since (F_4) has the symmetries

$(a, b, c, c') \longrightarrow (1+a-c, 1+b-c, 2-c, c')$,

$(a, b, c, c') \longrightarrow (1+a-c', 1+b-c', c, 2-c')$,

$(a, b, c, c') \longrightarrow (2+a-c-c', 2+b-c-c', 2-c, 2-c')$,

we may assume that $c, c' \ge 1$. So we will prove the "if" part of THEOREM 1 under the condition that $c, c' \in \mathbb{N}$.

2.1 A system of Fundamental solutions for $c, c' \in N$

We will find a basis of $\mathcal{F}_4$ in a neighborhood of $c=1+m$,

$c'=1+m'$, $m, m' \geq 0$.

For $c, c' \notin \mathbf{Z}$, put

$f_1' = \Gamma(a)\Gamma(b)F_4(a,b,c,c';X,Y)$,

$f_2' = \Gamma(1+a-c)\Gamma(1+b-c)X^{1-c}F_4(1+a-c,1+b-c,2-c,c';X,Y)$,

$f_3' = \Gamma(1+a-c')\Gamma(1+b-c')Y^{1-c'}F_4(1+a-c',1+b-c',c,2-c';X,Y)$,

$f_4' = \Gamma(2+a-c-c')\Gamma(2+b-c-c')$

$$X^{1-c}Y^{1-c'}F_4(2+a-c-c',2+b-c-c',2-c,2-c';X,Y).$$

Then

$$f_1'/\Gamma(c)\Gamma(c') = \sum \frac{\Gamma(a+k+k')\Gamma(b+k+k')}{\Gamma(c+k)\Gamma(c'+k')\Gamma(1+k)\Gamma(1+k')} X^k Y^{k'},$$

$$f_2'/\Gamma(2-c)\Gamma(c') = \sum \frac{\Gamma(1+a-c+k+k')\Gamma(1+b-c+k+k')}{\Gamma(2-c+k)\Gamma(c'+k')\Gamma(1+k)\Gamma(1+k')} X^{1-c+k} Y^{k'},$$

If $c \longrightarrow 1+m$ $(m=0,1,2,...)$ then

$f_2'/\Gamma(2-c)\Gamma(c') \longrightarrow f_1'/\Gamma(c)\Gamma(c')$. Similarly

$f_4'/\Gamma(2-c)\Gamma(2-c') \longrightarrow f_3'/\Gamma(c)\Gamma(2-c')$.

If $c' \longrightarrow 1+m'$ $(m'=0,1,2,...)$ then

$f_3'/\Gamma(c)\Gamma(2-c') \longrightarrow f_1'/\Gamma(c)\Gamma(c')$,

$f_4'/\Gamma(2-c)\Gamma(2-c') \longrightarrow f_2'/\Gamma(2-c)\Gamma(c')$.

Hence

$\Gamma(1-c)(f_2'/\Gamma(2-c)\Gamma(c') - f_1'/\Gamma(c)\Gamma(c'))$ and

$\Gamma(1-c)(f_4'/\Gamma(2-c)\Gamma(2-c') - f_3'/\Gamma(c)\Gamma(2-c'))$ are holomorphic at $c=1+m$. Equivalently

$\Gamma(c-1)f_2' + \Gamma(1-c)f_1'$ and $\Gamma(c-1)f_4' + \Gamma(1-c)f_3'$ are holomorphic at $c=1+m$.

By the same reason,

$\Gamma(c'-1)f_3' + \Gamma(1-c')f_1'$ and $\Gamma(c'-1)f_4' + \Gamma(1-c')f_2'$ are holomorphic at $c'=1+m'$.

Thus we have the following four solutions of (F_4) in a neighborhood of $c=1+m$, $c'=1+m'$:

$g_1=f_1'$,

$g_2= \Gamma(c-1)f_2' + \Gamma(1-c)f_1'$,

$g_3= \Gamma(c'-1)f_3' + \Gamma(1-c')f_1'$,

$g_4= \Gamma(c-1)\Gamma(c'-1)f_4' + \Gamma(1-c)\Gamma(c'-1)f_3'$
$\quad + \Gamma(c-1)\Gamma(1-c')f_2' + \Gamma(1-c)\Gamma(1-c')f_1'$.

Here g_4 is holomorphic at $c=1+m$, $c'=1+m'$ by Hartogs' theorem.

We will show that at $c=1+m, c'=1+m'$, g_i are of the following forms:

$g_1= \Gamma(a)\Gamma(b)F_4(a,b,1+m,1+m';X,Y)$,

$g_2 = (-1)^{1+m} \log X\, g_1/\Gamma(1+m) + X^{-m} g_{21}$,

$g_3 = (-1)^{1+m'} \log Y\, g_1/\Gamma(1+m') + Y^{-m'} g_{31}$,

$g_4 = (-1)^{m+m'} \log X \log Y\, g_1/\Gamma(1+m)\Gamma(1+m')$

$$+ Y^{-m'} \log X\, g_{41} + X^{-m} \log Y\, g_{42} + X^{-m} Y^{-m'} g_{43},$$

where g_{ij} are holomorphic at X=0, Y=0. This proves that g_1, g_2, g_3, g_4 form a system of fundamental solutions of (F_4).

Put $c=1+m-\delta$, $c'=1+m'-\delta'$.

$$g_2/\Gamma(c') = -\pi/\sin \pi c \{$$

$$\sum \frac{\Gamma(a+1-c+k+k')\Gamma(b+1-c+k+k')}{\Gamma(2-c+k)\Gamma(1+k)\Gamma(c'+k')\Gamma(1+k')} X^{1-c+k} Y^{k'}$$

$$- \sum \frac{\Gamma(a+k+k')\Gamma(b+k+k')}{\Gamma(c+k)\Gamma(1+k)\Gamma(c'+k')\Gamma(1+k')} X^k Y^{k'} \}$$

$$= (-1)^{1+m} \pi/\sin \pi\delta \{$$

$$X^{-m} \sum_{k<m} \frac{\Gamma(a-m+\delta+k+k')\Gamma(b-m+\delta+k+k')}{\Gamma(1-m+k+\delta)\Gamma(1+k)\Gamma(c'+k')\Gamma(1+k')} X^{k+\delta} Y^{k'}$$

$$+ \sum (\frac{\Gamma(a+k+k'+\delta)\Gamma(b+k+k'+\delta)}{\Gamma(1+k+\delta)\Gamma(1+m+k)\Gamma(c'+k')\Gamma(1+k')} X^{\delta}$$

$$- \frac{\Gamma(a+k+k')\Gamma(b+k+k')}{\Gamma(1+k)\Gamma(1+m+k-\delta)\Gamma(c'+k')\Gamma(1+k')}) X^k Y^{k'} \}$$

If $\delta \longrightarrow 0$ then

$$g_2/\Gamma(c') \longrightarrow (-1)^{1+m} \{ \log X\, f_1' / \Gamma(c)\Gamma(c')$$

$$- X^{-m} \sum_{k<m} (-1)^{m-k} \frac{\Gamma(m-k)\Gamma(a-m+k+k')\Gamma(b-m+k+k')}{\Gamma(1+k)\Gamma(c'+k')\Gamma(1+k')} X^k Y^{k'}$$

$$+ \frac{\partial}{\partial\delta} \sum \frac{\Gamma(a+k+k'+\delta)\Gamma(b+k+k'+\delta)}{\Gamma(1+k+\delta)\Gamma(1+m+k+\delta)\Gamma(c'+k')\Gamma(1+k')} X^k Y^{k'} \Big|_{\delta=0} \}$$

Similarly if $\delta \longrightarrow 0$ then

$$(\Gamma(c-1) f_4' + \Gamma(1-c) f_3') / \Gamma(2-c') \longrightarrow (-1)^{1+m} \{ \log X\, f_3' / \Gamma(c)\Gamma(2-c')$$

$$- X^{-m} \sum_{k<m} (-1)^{m-k} \frac{\Gamma(m-k)\Gamma(1+a-c'-m+k+k')\Gamma(1+b-c'-m+k+k')}{\Gamma(1+k)\Gamma(2-c'+k')\Gamma(1+k')}$$

$$X^k Y^{1-c'+k'}$$

$$+\frac{\partial}{\partial\delta}\sum\frac{\Gamma(1+a-c'+k+k'+\delta)\Gamma(1+b-c'+k+k'+\delta)}{\Gamma(1+k+\delta)\Gamma(1+m+k+\delta)\Gamma(2-c'+k')\Gamma(1+k')}X^k Y^{1-c'+k'}\Big|_{\delta=0}$$

If $\delta' \longrightarrow 0$ then

$$g_3/\Gamma(c) \longrightarrow (-1)^{1+m'}\{\log Y\ f_1'/\Gamma(c)\Gamma(c')$$

$$-Y^{-m'}\sum_{k'<m'}(-1)^{m'-k'}\frac{\Gamma(m'-k')\Gamma(a-m'+k+k')\Gamma(b-m'+k+k')}{\Gamma(1+k)\Gamma(c+k)\Gamma(1+k')}X^k Y^{k'}$$

$$+\frac{\partial}{\partial\delta'}\sum\frac{\Gamma(a+k+k'+\delta')\Gamma(b+k+k'+\delta')}{\Gamma(1+k)\Gamma(c+k)\Gamma(1+m'+k'+\delta')\Gamma(1+k'+\delta')}X^k Y^{k'}\Big|_{\delta'=0}\}$$

$$(\Gamma(c'-1)f_4'+\Gamma(1-c')f_2')/\Gamma(2-c)\longrightarrow(-1)^{1+m'}\{\log Y\ f_2'/\Gamma(2-c)\Gamma(c')$$

$$-Y^{-m'}\sum_{k'<m'}(-1)^{m'-k'}$$

$$\frac{\Gamma(m'-k')\Gamma(1+a-c-m'+k+k')\Gamma(1+b-c-m'+k+k')}{\Gamma(1+k)\Gamma(2-c+k)\Gamma(1+k')}X^k Y^{1-c'+k'}$$

$$+\frac{\partial}{\partial\delta'}\sum\frac{\Gamma(1+a-c'+k+k'+\delta)\Gamma(1+b-c'+k+k'+\delta)}{\Gamma(1+k)\Gamma(2-c+k)\Gamma(1+m'+k'+\delta')\Gamma(1+k'+\delta')}X^k Y^{k'}\Big|_{\delta=0}\}.$$

Hence if $\delta \longrightarrow 0$ and $\delta' \longrightarrow 0$, then

$$g_4 \longrightarrow (-1)^{m+m'}\{\log X\ \log Y\ g_1/\Gamma(1+m)\Gamma(1+m')$$

$$+\log X\ ($$

$$-Y^{-m'}\sum_{k'<m'}(-1)^{m'-k'}\frac{\Gamma(m'-k')\Gamma(a-m'+k+k')\Gamma(b-m'+k+k')}{\Gamma(1+k)\Gamma(1+m+k)\Gamma(1+k')}X^k Y^{k'}$$

$$+\frac{\partial}{\partial\delta'}\sum\frac{\Gamma(a+k+k'+\delta')\Gamma(b+k+k'+\delta')}{\Gamma(1+k)\Gamma(1+m+k)\Gamma(1+m'+k'+\delta')\Gamma(1+k'+\delta')}X^k Y^{k'}\Big|_{\delta'=0})$$

$$+\log Y\ ($$

$$-X^{-m}\sum_{k<m}(-1)^{m-k}\frac{\Gamma(m-k)\Gamma(a-m+k+k')\Gamma(b-m+k+k')}{\Gamma(1+k)\Gamma(1+m'+k')\Gamma(1+k')}X^k Y^{k'}$$

$$+\frac{\partial}{\partial\delta}\sum\frac{\Gamma(a+k+k'+\delta)\Gamma(b+k+k'+\delta)}{\Gamma(1+k+\delta)\Gamma(1+m+k+\delta)\Gamma(1+m'+k')\Gamma(1+k')}X^k Y^{k'}\Big|_{\delta'=0})$$

$$+X^{-m}Y^{-m'}\sum_{k<m,\,k'<m'}$$

$$(-1)^{m+m'} \frac{\Gamma(m-k)\Gamma(m'-k')\Gamma(a-m-m'+k+k')\Gamma(b-m-m'+k+k')}{\Gamma(1+k)\Gamma(1+k')} X^k Y^{k'}$$

$$- X^{-m} \frac{\partial}{\partial\delta'} \Sigma_{k'<m'} (-1)^{m'-k'}$$

$$\frac{\Gamma(m'-k')\Gamma(a-m'+k+k'+\delta')\Gamma(b-m'+k+k'+\delta')}{\Gamma(1+k)\Gamma(1+k'+\delta')\Gamma(1+m'+k'+\delta')} X^k Y^{k'} \Big|_{\delta'=0}$$

$$- Y^{-m'} \frac{\partial}{\partial\delta} \Sigma_{k<m} (-1)^{m-k}$$

$$\frac{\Gamma(m-k)\Gamma(a-m+k+k'+\delta)\Gamma(b-m+k+k'+\delta)}{\Gamma(1+k+\delta)\Gamma(1+m+k+\delta)\Gamma(1+k')} X^k Y^{k'} \Big|_{\delta=0}$$

$$+ \frac{\partial^2}{\partial\delta\partial\delta'} \Sigma \frac{\Gamma(a+k+k'+\delta+\delta')\Gamma(b+k+k'+\delta+\delta')}{\Gamma(1+k+\delta)\Gamma(1+m+k+\delta)\Gamma(1+m'+k'+\delta')\Gamma(1+k'+\delta')} X^k Y^{k'} \Big|_{\substack{\delta=0\\ \delta'=0}}$$

Hence g_i are of the desired forms.

2.2 Proof of THEOREM 1

From now on we assume that c and c' are in a small neiborhoods of $1+m$ and $1+m'$ respectively.

In [4], we have considered a fundamental solutions $\omega_1, \omega_2, \omega_3, \omega_4$. Here

$g_1 = \omega_1/\Gamma(1-c)\Gamma(1-c')$, $g_2 = \omega_2/\Gamma(1-c')$, $g_3 = \omega_3/\Gamma(1-c)$,

$g_4 = \omega_2+\omega_3-\omega_4$ for $c\neq 1+m, c'\neq 1+m'$.

We recall here the fundamental group

$\pi_1=\pi_1(C^2-L_X \cup L_Y \cup C, P_0)$ where $L_X=\{X=0\}$, $L_Y=\{Y=0\}$,

$C=\{(X-Y)^2-2(X+Y)+1=0\}$, $P_0=(1/100, 1/100)$. Let γ_1 be the horizontal $(Y\equiv 1/100)$ loop surrounding $L_X=0$ positively and γ_2 be the vertical loop surrounding $L_Y=0$. Let γ_3 be the diagonal

(X≡Y) loop surrounding C positively.

The analytic continuation $\gamma_i g$ of $g={}^t(g_1, g_2, g_3, g_4)$ along γ_i are then derived from $\gamma_i \omega$ (see [4]) as follows:

$$\gamma_1 g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ (1-e(-c))\Gamma(1-c) & e(-c) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & (1-e(-c))\Gamma(1-c) & e(-c) \end{pmatrix} g,$$

$$\gamma_2 g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ (1-e(-c'))\Gamma(1-c') & 0 & e(-c') & 0 \\ 0 & (1-e(-c'))\Gamma(1-c') & 0 & e(-c') \end{pmatrix} g,$$

$$\gamma_3 g = \begin{pmatrix} -e(\varepsilon) & e_2 & e_3 & e_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} g,$$

where $e(a)=\exp(2\pi\sqrt{-1}\ a)$, $\varepsilon=c+c'-a-b-1$,

$e_2=e(\varepsilon/2)\ \pi\ \Gamma(c-1)\ \sin\pi(a+b-2c-c')$,

$e_3=e(\varepsilon/2)\ \pi\ \Gamma(c'-1)\ \sin\pi(a+b-c-2c')$,

$e_4=e(\varepsilon/2)\ \pi^2\ \Gamma(c-1)\Gamma(c'-1)\ \sin\pi(a-c-c')\ \sin\pi(b-c-c')$.

Assume there exists a non trivial proper subspace W invariant under γ_1, γ_2, γ_3. W contains an eigen vector $\alpha g_1+\beta g_3$ of γ_1.

If $\beta=0$, then $g_1 \in W$. If $\beta\neq 0$ then $\gamma_2(\alpha g_1+\beta g_3)-(\alpha g_1+\beta g_3)\in W$, which implies $g_1\in W$. Hence we have

$$g_1 \in W.$$

$\gamma_3(g_1)=-e(\varepsilon)g_1+e_2g_2+e_3g_3+e_4g_4\in W$, $e_4\neq 0$.

$\gamma_1\gamma_3(g_1)-\gamma_3(g_3)=\beta_1 g_1+\beta_3 g_3\in W$, $\beta_3\neq 0$. Since $g_1\in W$, we have

$$g_3 \in W.$$

Similarly we have

$$g_2 \in W.$$

This contradicts to the fact that W is a proper subspace. And this proves theorem 1.4.

3. Reducible Cases.

As stated in [4], we know that

if one of $a, b, 1+a-c, 1+b-c, 1+a-c', 1+b-c', 2+a-c-c', 2+b-c-c'$ is a non positive (resp. positive) integer then $\mathcal{F}_4$ has a one dimensional (resp. three dimensional) invariant subspace of $\mathcal{F}_4$ under the operations of the monodromy group.

The three dimensional invariant spaces can be expressed by Appell's F_1 (see [4]).

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