On the dodecahedral group

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ON THE DODECAHEDRAL GROUP [1]

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ABSTRACT. Using the fact that five regular hexahedrons are inscribed in a regular dodecahedron, we prove that the rotation group of regular dodecahedron is isomorphic with the alternating group of degree five.

Klein proved [1] the theorem that the rotation group of a regular dodecahedron is isomorphic with the alternating group A_5 of degree five by using transformations between the orthogonal coordinate systems. But it seems to the authors that his proof is somewhat difficult to follow. In this article we wish to present another proof of the theorem, which is so useful when we study Galois theory [2].

To begin with, we verify that the number of symmetries of a regular dodecahedron is sixty. (We imply a rotation by the term symmetry.) We number the faces of the dodecahedron from 1 to 12. Since any symmetry maps the first face onto one of the twelve faces, the whole set of symmetries may be classified into twelve classes, each classes having five symmetries. Thus there exist a total of 60 ($= 12 \times 5$) symmetries.

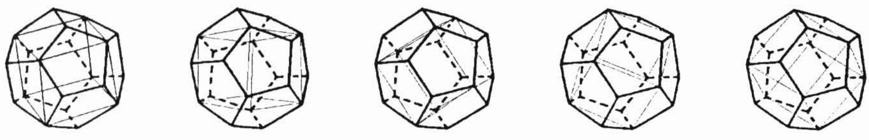

Figure 1

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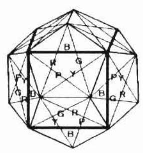
We can see from Figure 1 that exactly five regular hexahedrons may be inscribed in a regular dodecahedron, if their vertices coincide. And any symmetry gives rise to a permutation of these five hexahedrons. We thus get a homomorphism between the group of symmetries of the dodecahedron and the permutation group S_5 of degree five. We will show that the homomorphism is an injection onto the alternating group A_5.

Since we need to distinguish these five hexahedrons from one another in the following arguments, we assign different colors to each, namely red, yellow, green, blue, and purple.

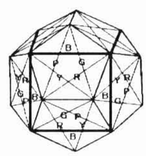
Lemma 1. *The only symmetry which leaves all five hexahedrons invariant is the identity transformation.*

Proof. All five colored edges appear on each face of the dodecahedron, as shown in Figure 2 and 3.



R	G
P	Y

Figure 2



P	G
Y	R

Figure 3

We consider any symmetry which maps the blue hexahedron onto itself. One symmetry does indeed transform the positions shown in Figure 2 to those in Figure 3. Of course such symmetry leaves the blue hexahedron invariant, we thus represent the relative positions of the four colored edges on the central faces with the exception of blue as in the right half of Figure 2 and 3.

Figure 4 is the list of the relative positions of the four colored edges on all of the twelve faces corresponding to the arrangement shown in Figure 2.

R	G	P	Y	Y	P	G	R
P	Y	R	G	G	R	Y	P
R	Y	P	G	Y	R	G	P
G	P	Y	R	P	G	R	Y
R	P	P	R	Y	G	G	Y
Y	G	G	Y	R	P	P	R

Figure 4

Any symmetry which maps the blue hexahedron onto itself transforms the positions of the four colors shown in Figure 2 to one of the arrangements Figure 4. It is easy to check that the twelve arrangements listed in Figure 4 are all different, and this means that the only symmetry which maps all five hexahedrons onto themselves is the identity transformation.

In this way we have shown that our homomorphism is injective.

Lemma 2

1. *The only symmetry which leaves all five hexhedrons invariant is the identity transformation.*

2. *There is no symmetry which leaves just three hexahedrons invariant.*

3. *There are twenty symmetries which leave just two hexahedrons invariant. (These are all even permutations.)*

4. *There are fifteen symmetries which leave just one hexahedron invariant. (These are all even permutations.)*

5. *There are twenty-four symmetries which leave no hexahedron invariant. (These are all even permutations.)*

Proof.

1. See Lemma 1.

2. This can be easily checked from Figure 4.

3. Figure 4 shows that there are only two symmetries which fix the blue and red hexahedrons and not any others. There are hence 20 $(= 2 \times_5 C_2)$ symmetries that fix just two hexahedrons. Since such a symmetry moves three hexahedrons, it is a permutation of type $(1\ 2\ 3)$, which is even.

4. Figure 4 shows that there are only three symmetries that fix the blue hexahedron but not any others. Therefore, there are 15 $(= 3 \times_5 C_1)$ symmetries that fix just one hexahedron. Besides, it can be seen from Figure 4 that those symmetries are of type $(R\ P)(G\ Y)$, which is even.

5. We have already shown that 36 ($= 1 + 20 + 15$) symmetries are even. Then we have to consider other 24 ($= 60 - 36$) symmetries.

   - Step 1. We consider a symmetry that does not fix any hexahedron. It maps the red hexahedron onto another. Suppose it is the yellow.
   - Step 2. Here we have two cases; the first case in which the yellow is mapped onto the red, and the second case in which it is mapped onto another. Suppose the first case.
   - Step 3. We can then suppose that the green one is mapped onto the blue. And then the blue has to be mapped onto the purple, the purple onto the green.

   The permutation thus obtained is

$$\sigma \equiv \begin{pmatrix} R & Y & G & B & P \\ Y & R & B & P & G \end{pmatrix}.$$

   We have to consider about the second case in Step2.

   - Step2′. Suppose the yellow is mapped onto the green.
   - Step3′. If the green hexahedron is mapped onto the red, then the blue is automatically mapped onto the purple, and the purple onto the blue. The permutation thus obtained is

$$\begin{pmatrix} R & Y & G & B & P \\ Y & G & R & P & B \end{pmatrix} = \begin{pmatrix} B & P & R & Y & G \\ P & B & Y & G & R \end{pmatrix},$$

     which has the same type as the permutation σ. We can therefore suppose that the green is mapped onto the blue. Then the blue is automaticaly mapped onto the purple, and the purple onto the red. The permutation thus obtained is

$$\tau \equiv \begin{pmatrix} R & Y & G & B & P \\ Y & G & B & P & R \end{pmatrix}.$$

   It is therefore sufficient to consider these two permutations σ and τ. First, it is easily shown that $\sigma^3 = (R\ Y)$, which is contradictory to the result of 2. That is to say, σ cannot be a member of rotation group of the dodecahedron. Second, it is quite easy to check that τ is an even permutation.

The above lemma shows that each symmetry corresponds to an even permutation, and this means that the image of our homomorphism coincides with the alternating group A_5. We have hence proved the next theorem:

Theorem *The rotation group of a regular dodecahedron is isomorphic with the alternating group A_5 of degree five.*

References

[1] F.Klein, *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*, Nachdr. der Ausg. Teubner, Leipzig, 1884.

[2] E.Artin, *Galoissche theorie*, B.G. Teubner Verlagsgesellschaft, Leipzig, 1959.

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