A note on iterations of mappings on an interval

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A note on iterations of mappings on an interval

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Abstract. We consider iterations of functions on an interval and propose a conjecture. In conjunction with the conjecture, some properties of the Feigenbaum constants are discussed.

We will consider functions $f(x)$ which satisfy
(1) $f(x)$ is a continuously differentiable function of $[-1,1]$ into itself,
(2) $f(x)$ has a unique maximum at $x=0$, and $f(0)-f(x)\sim|x|^{\rho}$ for x sufficiently small,
(3) $f(-1)=-1$; $f(x)$ is strictly increasing on $[-1,0]$ and $f(x)=f(-x)$,
where $\rho>1$ is a fixed real number.The space of all such functions will be denoted by $\mathcal{F}$.

Let $f(c,x)$ ($c\in[0,1]$) be a one-parameter family of elements of $\mathcal{F}$. Assuming that $f(c,0)$ is a strictly increasing differentiable function of c and that $-1=f(0,0)<f(1,0)=1$, we define a new family $Rf(c,x)$ in the following manner.

There is a unique number p such that

$$f(p,0)=0. \tag{1}$$

For each $c>p$, define

$$a(c)\equiv\min\{\ x>0\mid f(c,x)=x\ \}. \tag{2}$$

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Moreover, define

$$q \equiv \min\{\ c > p \mid f^2(c,0) = -a(c)\ \}, \tag{3}$$

where the functions $f^n(c,x)$ $(n = 1,2,\cdots)$ are defined in the usual way:

$$f^1(c,x) = f(c,x), \qquad f^{n+1}(c,x) = f^n(c, f(c,x)). \tag{4}$$

Here we need a function $b(\alpha,\beta,c)$ of α,β,c which satisfies
(1) (α,β) varies in the region $0 < \alpha < \beta < 1$, and c in the region $0 \le c \le 1$.
(2) For each pair (α,β), $b(\alpha,\beta,c)$ is a strictly increasing function of c.
(3)

$$b(\alpha,\beta,0) = \alpha, \qquad b(\alpha,\beta,1) = \beta.$$

Now, under these preparations, we can define $Rf(c,x)$:

$$(Rf)(c,x) = -\frac{1}{a(b(c))} f^2(b(c), -a(b(c))x) \qquad (-1 \le x \le 1,\ \ 0 \le c \le 1), \tag{5}$$

where $b(c) \equiv b(p,q,c)$.

It is easily verified that the above procedure can be applied to $f^2(c,x)$ though f^2 is not generally a family of elements of $\mathcal{F}$ and that we can define $Rf^2(c,x)$. Assuming the natural condition

$$Rf^2 = R^2 f, \tag{6}$$

we will deduce a condition on the function $b(\alpha,\beta,c)$.

Replacing f with f^2 in the equations (1), (2) and (3), we can define three quantities p, q and $a(c)$ for the family $f^2(c,x)$. They are denoted by p', q' and $a'(c)$, respectively. The three quantities for $Rf(c,x)$ are similarly denoted by $\hat{p}, \hat{q}$ and $\hat{a}(c)$.Then from the equation

$$0 = Rf(\hat{p},0) = -\frac{1}{a(b(\hat{p}))} f^2(b(\hat{p}),0) \tag{7}$$

we have

$$p' = b(p, q, \hat{p}), \tag{8}$$

and from the equation

$$\begin{aligned} \hat{a}(c) &= Rf(c, \hat{a}(c)) \\ &= -\frac{1}{a(b(c))} f^2(b(c), -a(b(c))\hat{a}(c)) \end{aligned} \tag{9}$$

we have

$$a'(c) = -a(c)\hat{a}(b^{-1}(c)). \tag{10}$$

Furthermore the definition (5) of $Rf(c, x)$ claims

$$\begin{aligned} f^4(b(c), 0) &= f^2(b(c), f^2(b(c), 0)) \\ &= f^2(b(c), -a(b(c))Rf(c, 0)) \\ &= -a(b(c))(Rf)^2(c, 0), \end{aligned} \tag{11}$$

that is to say

$$(Rf)^2(b^{-1}(q'), 0) = -\hat{a}(b^{-1}(q')). \tag{12}$$

This equation means that

$$q' = b(p, q, \hat{q}). \tag{13}$$

For simplicity we set

$$b'(c) \equiv b(p', q', c), \qquad \hat{b}(c) \equiv b(\hat{p}, \hat{q}, c), \tag{14}$$

then

$$Rf^2(c, x) = -\frac{1}{a'(b'(c))} f^4(b'(c), -a'(b'(c))x), \tag{15}$$

$$R^2 f(c, x) = -\frac{1}{\hat{a}(\hat{b}(c))} (Rf)^2(\hat{b}(c), -\hat{a}(\hat{b}(c))x). \tag{16}$$

A straightforward calculation shows that

$$
\begin{aligned}
R^2 f(c,x) &= -\frac{1}{\hat{a}(\hat{b}(c))} Rf(\hat{b}(c), Rf(\hat{b}(c), -\hat{a}(\hat{b}(c))x)) \\
&= \frac{1}{\hat{a}(\hat{b}(c))} \frac{1}{a(b(\hat{b}(c)))} f^4(b(\hat{b}(c)), a(b(\hat{b}(c)))\hat{a}(\hat{b}(c))x).
\end{aligned} \tag{17}
$$

Therefore a necessary and sufficient condition for $Rf^2 = R^2 f$ is

$$b(\hat{b}(c)) = b'(c), \tag{18}$$

that is to say

$$b(p,q,b(\hat{p},\hat{q},c)) = b(b(p,q,\hat{p}), b(p,q,\hat{q}), c). \tag{19}$$

In the following, we make use of

$$b(p,q,c) = (q-p)c + p \tag{20}$$

as a solution of (19).

We conjecture that there is a limiting function $\lim_{n\to\infty} R^n f$ for each $f \in \mathcal{F}$. If an element $f \in \mathcal{F}$ equals $\lim R^n g$ for a $g \in \mathcal{F}$, then f should be a fixed point of R:

$$Rf = f. \tag{21}$$

In the following, we will investigate some properties of such a family $f(c,x)$.

As c is increased from a sufficiently small value, a fixed point $(\neq -1)$ is stable until a value c_1 is reached when it becomes unstable. As c increases above c_1, a two-point cycle is stable, until it becomes unstable at c_2 when a stable four-point cycle occurs. As c is increased, this phenomenon recurs, with a 2^n-point cycle stable for

$$c_n \leq c < c_{n+1}. \tag{22}$$

In paticular we have

$$\frac{df}{dx}(c_1, a(c_1)) = -1. \tag{23}$$

On the other hand, the assumption $Rf = f$ claims that

$$\frac{df}{dx}(c, x) = \frac{df^2}{dx}(b(c), -a(b(c))x), \tag{24}$$

thus we have

$$c_2 = b(c_1), \tag{25}$$

and generally we have

$$c_n = b^{n-1}(c_1) \qquad (n = 1, 2, \cdots). \tag{26}$$

Accordingly, the equality

$$\frac{c_{n+2} - c_{n+1}}{c_{n+1} - c_n} = q - p \tag{27}$$

holds, and $q-p$ turns out to be the reciprocal of one of the Feigenbaum constants;

$$q - p = \delta^{-1}. \tag{28}$$

Let $\lim c_n$ be denoted by c_∞, then it is easily seen that $b(p, q, c_\infty) = c_\infty$ and that

$$b^n(c) = c_\infty - (c_\infty - c)/\delta^n \qquad (0 \leq c \leq 1). \tag{29}$$

Thus the equality

$$f(c_\infty, x) = -\frac{1}{a(c_\infty)} f^2(c_\infty, -a(c_\infty)x) \tag{30}$$

is deduced from the assumption $Rf = f$. This equation means that

$$a(c_\infty) = \alpha^{-1}, \tag{31}$$

where α is another Feigenbaum's constant.

Finally, the equation $R^n f = R^{n-1} f$ means

$$-\frac{1}{a(b^n(c))} f^{2^n}(b^n(c), -a(b^n(c))x) = f^{2^{n-1}}(b^{n-1}(c), x) \qquad (32)$$
$$(0 \le c \le 1, |x| \le a(b(c)) \cdots a(b^{n-1}(c))\ ),$$

therefore, for n sufficiently large, we have

$$\begin{aligned} -\alpha f^{2^n}&(c_\infty - (c_\infty - c)/\delta^n, -x/\alpha) \\ &\approx f^{2^{n-1}}(c_\infty - (c_\infty - c)/\delta^{n-1}, x). \end{aligned} \qquad (33)$$

The final asymptotic relation (33) suggests the existence of some fractal structure in the set of periodic points at $c \ge c_\infty$ [1-4].

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