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Abstract. We formulate some fluctuations of space-time which do not generate any fluctuations in the curvature, but which nevertheless do give rise to a spontaneous breakdown of the quantum superposition principle for macroscopic objects.

Various arguments about the quantum uncertainty of space-time geometry have appeared over a long period of time in connection with quantization of gravity or a reexamination of the concepts of general relativity. On the other hand, it has been recently shown that such uncertainty may sometimes actually cause the quantum wave-packet reduction for macroscopic objects. Károlyházy [1, 2] introduced a model of the fluctuation of space-time in which he showed that the fluctuation gives rise to a spontaneous breakdown of the quantum superposition principle for macroscopic systems.

In 1993 however, Diósi and Lukács [3] pointed out that the Károlyházy's spacetime model is not very plausible since, for example, the fluctuation of metric in his model generates some extremely high curvatures comparable to or higher than those in neutron stars. Since then, it has generally been judged that this kind of approach to the problem of wave-packet reduction is unrealistic [4].

Our objective in this letter is to construct a desirable model. That is to say, we will formulate some fluctuations of the metric which do

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not generate any fluctuations in the curvature, but which nevertheless do give rise to a spontaneous breakdown of the quantum superposition principle for macroscopic systems in space-time. The essential approach in the following discussions is that the existence of the quantum uncertainty of space-time geometry implies the existence of some uncertainty of coordinate transformation functions between two coordinate systems. Such uncertainty gives rise to some fluctuation of the metric which will be shown to be desirable.

Firstly we briefly summarize the main features of the quantum uncertainty of space-time geometry in order to clarify our basic standpoint.

When we attempt to describe a physical state of space-time, we have to construct a net of events as a coordinate system and have to express the metric tensor through measuring the distances between those events. Here an event means a coincidence, that is to say, a collision of particles. Moreover physical instruments such as clocks are necessary to measure the distances between such events. Neither particles nor clocks can be independent of quantum physics, and the various uncertainty relations impose some limitations on the accuracy of the description of space-time.

Many investigations of such uncertainty of the space-time geometry have been made over a period of some fifty years, resulting in a variety of expressions for it. In this letter, we use a result originally indicated by Wigner [5, 6] and recently analyzed by Diósi and Lukács [7]. In Wigner's gedanken experiment, a net of events (i.e. a coordinate system) is composed of time-like geodesics of some real bodies (clocks). Since those bodies are subject to various quantum uncertainty relations, their geodesics are concluded to fluctuate to some extent. Therefore when we measure a world line segment on this fluctuating coordinate system, we cannot avoid some uncertainty in the length s of the segment. Wigner indicated that

$$\Delta s \approx \sqrt{\hbar \langle s \rangle / Mc} \tag{1}$$

(see also [8]), where $\langle s \rangle$ is the average value of s and M is the mass of the clock.

Hereafter we restrict ourselves to a space-time which is Minkowskian in the classical sense and to coordinate systems which are inertial at the macroscopic level.

We can make the uncertainty Δs as small as we wish by increasing the mass M. However, large masses bend space-time more, and a space-time equipped with very heavy clocks will be non-Minkowskian. This excludes the use of a net composed of too heavy clocks as a coordinate system in our flat space-time.

In order to estimate a realistic upper bound to the mass M, let $\phi_{\mu\nu}$ be the difference between the Minkowski metric $\eta_{\mu\nu}$ and the metric generated by the distribution of those bodies. Further let α be the spatial distance between two adjacent geodesics (i.e. two adjacent coordinate axes). Then we get from Einstein's gravitational equation

$$\phi_{00}/\alpha^2 \approx \kappa M c^2/\alpha^3. \tag{2}$$

Therefore the mass M should satisfy the condition

$$\kappa M c^2/\alpha \ll 1. \tag{3}$$

Combining conditions (1) and (3), we get [7, 8]

$$\Delta s \approx l_g \sqrt{\langle s \rangle/\alpha}, \tag{4}$$

where $l_g = \sqrt{G\hbar/c^3}$ is the Planck length.

In this letter this uncertainty (4) is the starting point of our discussions. Since we cannot measure a world line segment to higher precision than the uncertainty Δs in (4), it would be natural to think that the metric of our space-time (even if it is flat in the classical sense) cannot be definitely determined but fluctuates in a real sense.

In fact Károlyházy [1, 2] elaborated a model of a hazy space-time in which the metric tensor fluctuates statistically and a certain uncertainty similar to (4) is realized. Unfortunately it was pointed out by

Diósi and Lukács that the fluctuation of the Káloryházy's metric generates too large fluctuation of the curvature. It seems to the author however that the Káloryházy's metric includes some needless fluctuation. Our objective in this letter is to formulate some fluctuations of the metric which satisfy the uncertainty (4), but which nevertheless do not generate any fluctuation of the curvature.

Now as stated above, in Wigner's gedanken experiment, the uncertainty (1) originated in the fact that the coordinate axes generally fluctuate when we attempt to set a coordinate system in our space-time. A more exact expression would be as follows : If we consider two coordinate systems, the coordinate axes of one of them must generally be seen to fluctuate when viewed from another system. Therefore the transformation between two coordinate systems (say, (x^μ) and (x'^μ)) must generally have some uncertainty around an average Lorentz transformation. In order to express this uncertainty of the coordinate transformation, we introduce some statistically fluctuating functions ξ^μ and assume that

$$x'^\mu = a^\mu{}_\nu x^\nu + \xi^\mu(x). \tag{5}$$

In the following we will formulate some desirable fluctuations of the metric using these fluctuating functions ξ^μ.

First, once we assume the uncertainty (5) of the coordinate transformation, we have to adopt the existence of some fluctuation in the metric as a simple logical consequence. The reasons are as follows: Let $g_{\mu\nu}(x)$ and $g'_{\mu\nu}(x')$ be the metric coefficients on (x^μ) and (x'^μ) respectively. Then we have the relation between them :

$$g_{\mu\nu}(x) = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} g'_{\alpha\beta}(x') = (a^\alpha{}_\mu + \xi^\alpha{}_{,\mu})(a^\beta{}_\nu + \xi^\beta{}_{,\nu}) g'_{\alpha\beta}(x'). \tag{6}$$

Here the functions $\xi^\mu(x)$ are fluctuating, hence even if we fix the coefficients $g'_{\alpha\beta}(x')$, the relation (6) claims that the value of $g_{\mu\nu}(x)$ must fluctuate. The inverse relation to (6) alike requires some fluctuation of the value of $g'_{\mu\nu}(x')$. On the other hand, since our space-time is assumed to be Minkowskian at the classical level, it is natural to expect that the averages of $g_{\mu\nu}$ and $g'_{\mu\nu}$ are both $\eta_{\mu\nu}$. Therefore we should conclude that the coefficients $g_{\mu\nu}$ have at least the fluctuation

$$g_{\mu\nu}(x) = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} \eta_{\alpha\beta} = (a^{\alpha}{}_{\mu} + \xi^{\alpha}{}_{,\mu})(a^{\beta}{}_{\nu} + \xi^{\beta}{}_{,\nu})\eta_{\alpha\beta}. \tag{7}$$

(The same can be said of $g'_{\mu\nu}$.) Hereafter we restrict ourselves to this minimum uncertainty of the metric coefficients.

The formula (7) means that the metric $g_{\mu\nu}$ on the x-coordinate is isometric with the flat metric $\eta_{\alpha\beta}$ on the x'-coordinate. That is to say, we consider many metrics on the x-coordinate each of which is isometric with the flat metric $\eta_{\alpha\beta}$ via the transformation (5).

Secondly, let us note that our minimum uncertainty of the metric does not generate any fluctuation of the curvature. This is trivial if we recall that each metric $g_{\mu\nu}$ on the x-coordinate is isometric with the flat metric $\eta_{\alpha\beta}$ and that the curvature corresponding to $\eta_{\alpha\beta}$ is 0.

Before continuing, we should note the following in order to avoid misunderstanding: The formula (6) means that

$$\begin{aligned} R_{\mu\nu\sigma\rho}(x) = & \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} \frac{\partial x'^{\gamma}}{\partial x^{\sigma}} \frac{\partial x'^{\delta}}{\partial x^{\rho}} R'_{\alpha\beta\gamma\delta}(x') \\ & \left(\frac{\partial x'^{\alpha}}{\partial x^{\mu}} = a^{\alpha}{}_{\mu} + \xi^{\alpha}{}_{,\mu} \right), \end{aligned} \tag{8}$$

where $R_{\mu\nu\sigma\rho}$ and $R'_{\alpha\beta\gamma\delta}$ are curvature tensors of $g_{\mu\nu}$ and $g'_{\alpha\beta}$ respectively. Hence it is trivial that the curvature $R_{\mu\nu\sigma\rho}$ is always 0 under our assumption $g'_{\alpha\beta} = \eta_{\alpha\beta}$ introduced in the formula (7). On the other hand, if we relax our condition and argue about more general space-time which is not Minkowskian, then $R'_{\alpha\beta\gamma\delta}$ is not necessarily 0 and therefore the curvature $R_{\mu\nu\sigma\rho}$ in (8) fluctuates to some extent since ξ^{μ} fluctuate. Even then, however, the deviation of $R_{\mu\nu\sigma\rho}$ from its average must be small as far as the condition $|\xi_{\mu,\nu}| \ll 1$ is satisfied.

Thus the metric fluctuation induced from the uncertainty of coordinate transformation does not generate such high curvatures that Diósi and Lúkács mentioned in Ref. [3].

Finally, we wish to show that there exist some models of fluctuation of $\xi^\mu(x)$ from which we can derive the uncertainty (4). For simplicity we consider two coordinate systems (x^μ) and (x'^μ) such that they are seen to be at rest at the classical level when viewed from each other. Then the formulas (5) and (7) reduce to

$$x'^\mu = x^\mu + \xi^\mu(x) \tag{9}$$

and

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \xi_{\mu,\nu}(x) + \xi_{\nu,\mu}(x) \equiv \eta_{\mu\nu} + h_{\mu\nu}(x), \tag{10}$$

where $\xi_{\mu,\nu}$ are assumed to be so small that the higher terms can be neglected, and raising and lowering of suffixes are made with $\eta_{\mu\nu}$ here and hereafter.

Before advancing to the main subject, let us show that the formula

$$\xi^\mu(b^\nu{}_\rho x^\rho) = b^\mu{}_\rho \xi^\rho(x^\nu) \tag{11}$$

holds for each Lorentz matrix $(b^\mu{}_\nu)$. Let (X^μ) and (X'^μ) be coordinate systems denoted by

$$X^\mu = b^\mu{}_\nu x^\nu \quad \text{and} \quad X'^\mu = b^\mu{}_\nu x'^\nu, \tag{12}$$

respectively. It would be natural to think that the physical relation between (X^μ) and (X'^μ) is the same as the relation between (x^μ) and (x'^μ). Therefore the uncertainty in the coordinate transformation between (X^μ) and (X'^μ) should be represented by the same functions ξ^μ as in (9):

$$X'^\mu = X^\mu + \xi^\mu(X). \tag{13}$$

Combining (13), (12) and (9), we have the formula (11).

Now we consider a world line segment which starts from the origin of the system (x^μ) and ends at a point $x^\mu = p^\mu$. (For simplicity it is assumed to be time-like.) Let s be the length of this segment $x^\mu(t) = tp^\mu$ $(0 \le t \le 1)$ measured by the fluctuating metric $g_{\mu\nu}$ in (10) and let $L = \sqrt{-p^\mu p_\mu}$, then we have

$$s = \int_0^1 \sqrt{-g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}dt = L - \frac{1}{L}p^\mu\{\xi_\mu(p) - \xi_\mu(0)\}, \tag{14}$$

where again we have assumed that $\xi_{\mu,\nu}$ are so small that their higher order terms can be neglected. This means that $\langle s\rangle = L$ and

$$(\Delta s)^2 = \langle\ [p^\mu\{\xi_\mu(p) - \xi_\mu(0)\}]^2\ \rangle/\langle s\rangle^2, \tag{15}$$

and therefore if the formula

$$\langle\ [p^\mu\{\xi_\mu(p) - \xi_\mu(0)\}]^2\ \rangle \approx l_g^2\langle s\rangle^3/\alpha \tag{16}$$

is satisfied, then the uncertainty (4) is derived.

Combining the formulas (11) and (16), we have

$$\langle\ \left[\xi^0(p_\tau) - \xi^0(0)\right]^2\ \rangle \approx l_g^2 c\tau/\alpha \tag{17}$$

for all real numbers τ, where we set

$$p_\tau = (c\tau, 0, 0, 0). \tag{18}$$

Here, making the assumption that $\partial_\nu\partial^\nu\xi^\mu = 0$, we denote by

$$\xi^\mu(x) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\left[c^\mu(\mathbf{k})e^{i(\mathbf{kx}-kx^0)} + c^\mu(\mathbf{k})^* e^{-i(\mathbf{kx}-kx^0)}\right] \qquad (k \equiv |\mathbf{k}|) \tag{19}$$

the Fourier expansion of $\xi^\mu(x)$ in a large box of volume V in three dimensional space. (The summation in (19) should cover only the range $k \le 1/\alpha$. See [8].) Each complex coefficient $c^\mu(\mathbf{k})$ is supposed to vary around the average value 0, and each set of specific values for every $c^\mu(\mathbf{k})$ determines a $\xi^\mu(x)$. Then we have

$$\langle\left[\xi^0(p_\tau) - \xi^0(0)\right]^2\rangle = \frac{2}{\pi^2(c\tau)^3}\int dk k^2 G(k/(c\tau))(1 - \cos k), \tag{20}$$

where we have assumed that

$$\langle c^0(\mathbf{k})c^0(\mathbf{k}')\rangle = 0, \qquad \langle c^0(\mathbf{k})c^0(\mathbf{k}')^*\rangle = 0 \quad (\mathbf{k} \neq \mathbf{k}'), \tag{21}$$

and that $G(k) \equiv \langle|c^0(\mathbf{k})|^2\rangle$ depends only on $|\mathbf{k}| = k$. Hence if we take $\xi^0(x)$ such that

$$G(k) \approx \frac{l_g{}^2}{\alpha}\frac{1}{k^4}, \tag{22}$$

then the condition (17) is satisfied.

For instance, if each $c^0(\mathbf{k})$ distributes independently of $\mathbf{k}$ under the Gaussian probability density

$$(\alpha k^4/\pi l_g{}^2)e^{-\alpha k^4|c^0(\mathbf{k})|^2/l_g{}^2}, \tag{23}$$

then the conditions (21) and (22) are both satisfied.

Thus we have succeeded in constructing a model of metric fluctuation which does not generate any fluctuation of the curvature but which nevertheless realizes the uncertainty (4).

Lastly we should consider the order of magnitude of α. At present we do not know how densely the coordinate axes can be set up in our space-time. However an estimate of the order of α may be made by assuming our clock has the same order of precision as an atomic clock, and we have

$$\alpha \approx 10^{-24} \text{ to } 10^{-19}\text{cm} \tag{24}$$

(see [8] for the details). The formulas (19), (21) and (22) lead us to

$$\langle\ \xi_{0,0}(x)^2\ \rangle \approx (l_g/\alpha)^2. \tag{25}$$

On the other hand, if we take the formula (11) into consideration, it would be natural to think that the functions $\xi_{\mu,\nu}(x)$ have the same order of magnitude as $\xi_{0,0}(x)$. Therefore the range for α of (24) is consistent with the condition

$$|\xi_{\mu,\nu}(x)| \ll 1 \tag{26}$$

which we have assumed in this letter. Moreover such a metric fluctuation causes the localization of wave functions of macroscopic objects as we have showed in Ref. [8].

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