Generalized states and measuring processes for continuous observables

Ryukyu Math. J., 5(1992), 53-63

GENERALIZED STATES AND MEASURING PROCESSES FOR CONTINUOUS OBSERVABLES

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Abstract. An extension of the concept of physical state is introduced. It is shown that if we accept all the "generalized states" as initial states of an apparatus, then we can circumvent the Ozawa's theorem which rules out the existence of repeatable measuring processes for continuous observables. A generalization of the Wigner-Araki-Yanase theorem is argued.

1. Introduction

In its usual formulation, the theory of measurement of an observable (i.e., a self-adjoint operator) A of an object system (with a Hilbert space $\mathcal{H}_1$) starts with fixing a measuring apparatus (with a Hilbert space $\mathcal{H}_2$), its initial state σ (i.e., a density operator on $\mathcal{H}_2$), an observable $\tilde{A}$ of the apparatus, and a unitary operator U on $\mathcal{H}_1 \otimes \mathcal{H}_2$.

DEFINITION 1.1: The four-tuple $M = \langle \mathcal{H}_2, \tilde{A}, \sigma, U \rangle$ is a *measuring process* of A if it satisfies the relation

$$\mathrm{Tr}[U(\rho \otimes \sigma)U^{\dagger}(I \otimes P^{\tilde{A}}(\Delta))] = \mathrm{Tr}[\rho P^{A}(\Delta)] \tag{1.1}$$

for all density operators ρ on $\mathcal{H}_1$ and all Borel subsets $\Delta \subset \mathbf{R}$, where I is the identity operator on $\mathcal{H}_1$ and $\Delta \to P^{A}(\Delta)$ ($P^{\tilde{A}}(\Delta)$) is the

Received October 31, 1992

spectral measure associated with the self-adjoint operator A (resp. $\tilde{A}$).

It can be proved that there is a measuring process of A for any observable A. (Although the above statement can be proved in more general formulation (see Ref. 1, p.84), we restrict ourselves to the case of standard quantum theory.) Considering von Neumann's repeatability hypothesis[2], we say that a measuring process $M = \langle \mathcal{H}_2, \tilde{A}, \sigma, U \rangle$ of A is *weakly repeatable* if

$$\mathrm{Tr}[U(\rho \otimes \sigma)U^\dagger(P^A(\Delta) \otimes P^{\tilde{A}}(\Delta'))] = \mathrm{Tr}[\rho P^A(\Delta \cap \Delta')] \qquad (1.2)$$

for all density operators ρ on $\mathcal{H}_1$ and all Borel sets Δ, $\Delta' \subset \mathbf{R}$.

When we deal with observables with discrete spectra, the concept of weakly repeatable measuring process works well, however, Ozawa[1] has shown the following theorem.

THEOREM 1.1: There does not exist any weakly repeatable measuring process of an observable with continuous spectrum .

On the other hand, most of the observables one deals with ordinarily in either non-relativistic quantum mechanics or in quantum field theory are those with continuous spectra, therefore, the above theorem forces us to modify or generalize the concept of repeatable measuring process so that it can be applied for any observable. In fact, Davies and Lewis[3] introduced the *approximate repeatability condition.*

In this article, we attempt to circumvent the above Ozawa's theorem without appealing to any approximation. We shall show that it can be done if we accept an extension of the concept of physical state.

2. Generalized states

Consider a physical system which is based on a separable Hilbert space $\mathcal{H}$. We denote by $T(\mathcal{H})$ the Banach space (under the trace norm) of all trace class operators on $\mathcal{H}$, and by $B(\mathcal{H})$ the Banach space (under the operator norm) of all bounded operators on $\mathcal{H}$. It is well-known that the linear functional on $B(\mathcal{H})$

$$f_v : B(\mathcal{H}) \ni L \mapsto \mathrm{Tr}[vL] \tag{2.1}$$

is bounded for each $v \in T(\mathcal{H})$, and that the correspondence

$$v \mapsto f_v \tag{2.2}$$

gives an isomorphism between $T(\mathcal{H})$ and a subspace of the dual space $B(\mathcal{H})^*$ of $B(\mathcal{H})$.

In the conventional formulation of quantum theory, physical states of the system are characterized by density operators, i.e., positive trace class operators of unit trace. It is well known that each positive trace class operator of unit trace is identified, via (2.2), with a positive and normal linear functional on $B(\mathcal{H})$ of unit norm , and vice versa[4]. (A linear functional on $B(\mathcal{H})$ is said to be *normal* if it is continuous under the ultraweak topology of $B(\mathcal{H})$.)

In this article, we have to consider non-normal linear functionals. Let $\mathcal{N}$ be a subspace of $B(\mathcal{H})$ such that :

(i)$\mathcal{N} \ni I$.

(ii)If $L \in \mathcal{N}$, then $L^\dagger \in \mathcal{N}$.

We define a *generalized state* on $\mathcal{N}$ to be a linear functional $\rho : \mathcal{N} \to \mathbf{C}$ which is continuous under the operator norm and satisfies :

(i)$\langle \rho, L^\dagger \rangle = \langle \rho, L \rangle^*$ for all $L \in \mathcal{N}$.

(ii)If L is a nonnegative element in $\mathcal{N}$ then $\langle \rho, L \rangle \geq 0$. (2.3)

(iii)$\langle \rho, I \rangle = 1$.

(A linear functional which satisfies the above conditions is designated "a state on $\mathcal{N}$" in mathematical literature, however, we use the term "*generalized state*" in order to avoid confusion.)

For each generalized state ρ on $\mathcal{N}$, and for each self-adjoint operator A on $\mathcal{H}$ such that $P^A(\Delta) \in \mathcal{N}$ for all Borel subsets $\Delta \subset \mathbf{R}$, we define $\Pr_A^\rho(\Delta)$ according to the formula

$$\Pr_A^\rho(\Delta) \equiv \langle \rho, P^A(\Delta) \rangle. \tag{2.4}$$

If ρ is a physical state in the conventional meaning (i.e., a density operator on $\mathcal{H}$) then $\Pr_A^\rho(\Delta) = \mathrm{Tr}[\rho P^A(\Delta)]$, therefore, $\Pr_A^\rho(\Delta)$ is nothing but the probability that the outcome of an experiment to observe A on an ensemble in the state ρ lies in the Borel set Δ. A simple example of generalized state is as follows.

EXAMPLE: We set $\mathcal{H} = L^2(\mathbf{R})$. For two arbitrary real numbers λ and q, we define density operators $\rho_\epsilon = |\phi_\epsilon\rangle\langle\phi_\epsilon|$ ($\epsilon > 0$), where

$$\phi_\epsilon(x) \equiv (\pi\hbar\epsilon^2)^{-1/4} \exp[\{i\lambda x - \frac{(x-q)^2}{2\epsilon}\}/(\epsilon\hbar)].$$

Then we have $\|\phi_\epsilon\| = 1$ and

$$|\langle \rho_\epsilon, L \rangle| = |\mathrm{Tr}[\rho_\epsilon L]| \leq \|L\| \tag{2.5}$$

for all $L \in B(\mathcal{H})$, where $\|L\|$ is the operator norm of L. Let $\mathcal{N}$ be the subspace of $B(\mathcal{H})$ constituted by all elements L for which the limit

$$\langle \rho, L \rangle \equiv \lim_{\epsilon \to 0} \langle \rho_\epsilon, L \rangle \tag{2.6}$$

exists. The inequality (2.5) says that the linear functional ρ is continuous on $\mathcal{N}$, and it can be easily checked that ρ satisfies the condition (2.3), that is, ρ is a generalized state on $\mathcal{N}$. Furthermore, if a is a real number then we have

$$\Pr_{\hat{x}}^\rho((-\infty, a]) = \langle \rho, P^{\hat{x}}((-\infty, a]) \rangle = \begin{cases} 1 & \text{if } a > q, \\ 1/2 & \text{if } a = q, \\ 0 & \text{if } a < q, \end{cases} \tag{2.7}$$

$$\langle \rho, e^{it(\hat{x}-q)} \rangle = 1 \quad \text{for all } \ t \in \mathbf{R}, \tag{2.8}$$

$$\Pr_{\hat{p}}^\rho((-\infty, a]) = \langle \rho, P^{\hat{p}}((-\infty, a]) \rangle = (\pi\hbar)^{-1/2} \int_{-\infty}^{-\lambda} e^{-p^2/\hbar} dp. \tag{2.9}$$

The interesting feature of the above formulae (2.7)-(2.9) is that they clearly show that the position distribution is entirely concentrated at $\{q\}$ and the momentum distribution at $\{\pm\infty\}$ in the generalized state ρ. The formula (2.7) or (2.9) also demonstrates the fact that the linear functionals ρ $(-\infty < \lambda,\ q < \infty)$ are not normal i.e., they are not continuous under the ultra-weak topology of $B(\mathcal{H})$. Non-normal states play an essential role when we generalize the conventional von Neumann-Lüders collapse postulate to observables with continuous spectra[5] (see also Ref. 6). As we shall see in the next section, the concept of generalized state would be important also in the theory of measuring process of continuous observable.

3. Generalized measuring process

DEFINITION 3.1: Let A be a self-adjoint operator in a separable Hilbert space $\mathcal{H}_1$. A *generalized measuring process* M of A is a five-tuple $M = \langle \mathcal{H}_2, \tilde{A}, \mathcal{N}, \sigma, U \rangle$ consisting of a Hilbert space $\mathcal{H}_2$, a self-adjoint operator $\tilde{A}$ on $\mathcal{H}_2$, a subspace $\mathcal{N}$ of $B(\mathcal{H}_2)$, a generalized state σ on $\mathcal{N}$, and a unitary operator U on $\mathcal{H}_1 \otimes \mathcal{H}_2$ for which the values

$$\langle \rho \otimes \sigma, U^\dagger (P^A(\Delta) \otimes P^{\tilde{A}}(\Delta'))U \rangle$$

exist and

$$\langle \rho \otimes \sigma, U^\dagger (I \otimes P^{\tilde{A}}(\Delta))U \rangle = \mathrm{Tr}[\rho P^A(\Delta)] \tag{3.1}$$

for all density operators ρ on $\mathcal{H}_1$ and all Borel sets $\Delta, \Delta' \subset \mathbf{R}$.

If σ is an ordinary density operator on $\mathcal{H}_2$, then $\rho \otimes \sigma$ is a density operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$ and the formula (3.1) reduces to (1.1).

The generalization of repeatability condition (1.2) to the generalized measuring process $M = \langle \mathcal{H}_2, \tilde{A}, \mathcal{N}, \sigma, U \rangle$ is now straightforward.

DEFINITION 3.2: A generalized measuring process $M = \langle \mathcal{H}_2, \tilde{A}, \mathcal{N}, \sigma, U \rangle$ of A is *weakly repeatable* if

$$\langle \rho \otimes \sigma, U^\dagger (P^A(\Delta) \otimes P^{\tilde{A}}(\Delta'))U \rangle = \mathrm{Tr}[\rho P^A(\Delta \cap \Delta')] \tag{3.2}$$

for all density operators ρ on $\mathcal{H}_1$ and all Borel subsets $\Delta, \Delta' \subset \mathbf{R}$.

The next example demonstrates the fact that there is a weakly repeatable generalized measuring process of A even if A has a continuous spectrum. Thus, the acceptance of generalized states as initial states of an apparatus enables us to circumvent the Ozawa's theorem mentioned in section 1.

EXAMPLE: Let $\mathcal{H}_1 = \mathcal{H}_2 = L^2(\mathbf{R})$. The object system which is based on $\mathcal{H}_1$, as well as the observer system which is based on $\mathcal{H}_2$, is characterized by a single variable x and y respectively, running continuously from $-\infty$ to $+\infty$. That is, let both be one-dimensional particles. Let $A = \hat{x}$, $\tilde{A} = \hat{y}$, and $U = \exp(-iH/\hbar)$, where the Hamiltonian H is given by $H = \hat{x}\hat{p}_y$.

Then we have $U^\dagger(A \otimes I)U = A \otimes I$, and hence

$$U^\dagger(P^A(\Delta) \otimes I)U = P^A(\Delta) \otimes I, \tag{3.3}$$

for each Borel subset Δ of $\mathbf{R}$. As is calculated in Ref. 2, $(U\Phi)(x,y) = \Phi(x, y-x)$ for all $\Phi \in \mathcal{H}_1 \otimes \mathcal{H}_2$. Therefore we have

$$U^\dagger(I \otimes P^{\tilde{A}}(\Delta))U \;\Phi(x,y) = \chi_\Delta(y+x)\Phi(x,y), \tag{3.4}$$

where χ_Δ is the characteristic function of the set Δ.

Let $\sigma_\epsilon \equiv |\psi_\epsilon\rangle\langle\psi_\epsilon|$ $(\epsilon > 0)$, where

$$\psi_\epsilon(y) \equiv (\pi\hbar\epsilon^2)^{-1/4}\exp\{-y^2/2\hbar\epsilon^2\}. \tag{3.5}$$

Then, as in the example in the preceding section, we can get a generalized state σ on a subspace $\mathcal{N}$ of $B(\mathcal{H}_2)$ satisfying

$$\langle\sigma, L\rangle = \lim_{\epsilon\to 0}\langle\sigma_\epsilon, L\rangle \tag{3.6}$$

for all $L \in \mathcal{N}$.

Now if $\rho = |\phi\rangle\langle\phi|$ for a wave function $\phi \in \mathcal{H}_1$, then the formulae (3.3)-(3.6) imply that

$$
\begin{aligned}
&\langle \rho \otimes \sigma, U^\dagger(P^A(\Delta) \otimes P^{\tilde{A}}(\Delta'))U\rangle \\
&\quad = \lim_{\epsilon \to 0} \int dx \int dy \chi_\Delta(x)|\phi(x)|^2 \chi_{\Delta'}(y+x)|\psi_\epsilon(y)|^2 \\
&\quad = (\pi\hbar)^{-1/2} \lim_{\epsilon \to 0} \int dx \int dy \chi_\Delta(x)|\phi(x)|^2 \chi_{\Delta'}(\epsilon y + x) \exp(-y^2/\hbar) \\
&\quad = \int dx \chi_{\Delta \cap \Delta'}(x)|\phi(x)|^2 \\
&\quad = \mathrm{Tr}[\rho P^A(\Delta \cap \Delta')].
\end{aligned}
$$

The above equation means that $M = \langle \mathcal{H}_2, \tilde{A}, \mathcal{N}, \sigma, U\rangle$ is a weakly repeatable generalized measuring process of A.

4. Wigner-Araki-Yanase's theorem

The quantum theory of measurement shows up some limitations on the measurability of physical quantities. A type of limitation arises from the existence of conservation laws, like the conservation of linear momentum or angular momentum. This type of limitation was discovered by Wigner[7] and it has long been discussed by several authors[8-11]. Recently, Ozawa[10] claimed that this type of limitation is not generally valid if the conserved quantity is not bounded.

In this section we restrict ourselves to the case of bounded conserved quantity, and show that Wigner-Araki-Yanase's theorem is valid for the generalized measuring processes.

Let $M = \langle \mathcal{H}_2, \tilde{A}, \mathcal{N}, \sigma, U\rangle$ be a generalized measuring process of a self-adjoint operator A on $\mathcal{H}_1$. Let

$$\mathcal{B}_1 \equiv \{L \in B(\mathcal{H}_1) : U^\dagger(L \otimes P^{\tilde{A}}(\Delta))U \in B(\mathcal{H}_1) \otimes \mathcal{N} \text{ for all } \Delta\},$$

$$\mathcal{B}_2 \equiv \{L \in B(\mathcal{H}_2) : U^\dagger(P^A(\Delta) \otimes L)U \in B(\mathcal{H}_1) \otimes \mathcal{N} \text{ for all } \Delta\}.$$

The linear functional

$$T(\mathcal{H}_1) \ni v \longmapsto \langle v \otimes \sigma, K\rangle \tag{4.1}$$

is continuous (under the trace norm) for each $K \in B(\mathcal{H}_1) \otimes \mathcal{N}$. Therefore the fact that the duality between the Banach space $T(\mathcal{H}_1)$ and its dual $B(\mathcal{H}_1)$ can be expressed by the bilinear form $\mathrm{Tr}[vL]$ $(v \in T(\mathcal{H}_1), L \in B(\mathcal{H}_1))$ guarantees the existence of a linear mapping $\mathcal{E}^M(\Delta) : \mathcal{B}_1 \to B(\mathcal{H}_1)$ such that

$$\langle \rho \otimes \sigma, U^\dagger(L \otimes P^{\tilde{A}}(\Delta))U \rangle = \mathrm{Tr}[\rho \mathcal{E}^M(\Delta)L] \tag{4.2}$$

for all density operators ρ on $\mathcal{H}_1$ and all $L \in \mathcal{B}_1$.

The conditions (2.3) for the generalized state σ immediately imply the following properties.

$$\{\mathcal{E}^M(\Delta)L\}^\dagger = \mathcal{E}^M(\Delta)(L^\dagger) \quad \text{for all} \quad L \in \mathcal{B}_1. \tag{4.3}$$

$$\text{If} \quad L \geq 0 \quad \text{then} \quad \mathcal{E}^M(\Delta)L \geq 0. \tag{4.4}$$

$$\mathcal{E}^M(\mathbf{R})I = I. \tag{4.5}$$

Moreover, the condition (3.1) for the generalized measuring process indicates that

$$\mathcal{E}^M(\Delta)I = P^A(\Delta) \tag{4.6}$$

for all Borel sets $\Delta \subset \mathbf{R}$.

THEOREM 4.1: For all $L \in \mathcal{B}_1$ and all Borel sets $\Delta, \Delta' \subset \mathbf{R}$, we have

$$\mathcal{E}^M(\Delta)L = P^A(\Delta)\mathcal{E}^M(\mathbf{R})L, \tag{4.7}$$

$$[\mathcal{E}^M(\Delta)L, P^A(\Delta')] = 0. \tag{4.8}$$

Proof: Let L be a self-adjoint operator in $\mathcal{B}_1$. Since $-\|L\|I \leq L \leq \|L\|I$, it follows from (4.4) and (4.6) that

$$-\|L\|P^A(\Delta) \leq \mathcal{E}^M(\Delta)L \leq \|L\|P^A(\Delta) \tag{4.9}$$

for all Borel sets $\Delta \subset \mathbf{R}$. Hence we have

$$\mathcal{E}^M(\Delta)L = P^A(\Delta)\mathcal{E}^M(\Delta)L = [\mathcal{E}^M(\Delta)L]P^A(\Delta). \tag{4.10}$$

On the other hand, we have

$$P^A(\Delta)\mathcal{E}^M(\mathbf{R})L = P^A(\Delta)\mathcal{E}^M(\Delta)L + P^A(\Delta)\mathcal{E}^M(\Delta^c)L, \qquad (4.11)$$

where Δ^c is the complement of Δ. Since $P^A(\Delta)P^A(\Delta^c) = 0$, it follows from (4.9) that $P^A(\Delta)\mathcal{E}^M(\Delta^c)L = 0$. Thus we obtain (4.7). Moreover (4.10) implies

$$[\mathcal{E}^M(\Delta \cap \Delta')L]P^A(\Delta') = P^A(\Delta')\mathcal{E}^M(\Delta \cap \Delta')L,$$

$$[\mathcal{E}^M(\Delta \cap \Delta'^c)L]P^A(\Delta') = P^A(\Delta')\mathcal{E}^M(\Delta \cap \Delta'^c)L = 0.$$

From the above equations we get (4.8).

THEOREM 4.2: Let $M = \langle \mathcal{H}_2, \tilde{A}, \mathcal{N}, \sigma, U \rangle$ be a generalized measuring process of a self-adjoint operator A on $\mathcal{H}_1$. Let $L = L_1 \otimes I + I \otimes L_2$ $(L_1 \in \mathcal{B}_1, L_2 \in \mathcal{B}_2)$ be a bounded self-adjoint operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that $[L, U] = 0$, i.e., a constant of motion of the system $\mathcal{H}_1 \otimes \mathcal{H}_2$ with respect to U. Assume that

$$U^\dagger(A \otimes I)U = A \otimes I. \qquad (4.12)$$

Then we have $[L_1, P^A(\Delta)] = 0$ for all Borel sets $\Delta \subset \mathbf{R}$.

Proof: The condition $[L, U] = 0$ implies that

$$L_1 \otimes I = -I \otimes L_2 + U^\dagger(L_1 \otimes I)U + U^\dagger(I \otimes L_2)U, \qquad (4.13)$$

and, therfore, we have

$$\langle \rho \otimes \sigma, L_1 \otimes I \rangle = -\langle \rho \otimes \sigma, I \otimes L_2 \rangle + \langle \rho \otimes \sigma, U^\dagger(L_1 \otimes I)U \rangle + \langle \rho \otimes \sigma, U^\dagger(I \otimes L_2)U \rangle$$

for all density operators ρ on $\mathcal{H}_1$. Hence, we have

$$\mathrm{Tr}[\rho L_1] = -\langle \sigma, L_2 \rangle + \mathrm{Tr}[\rho \mathcal{E}^M(\mathbf{R})L_1] + \mathrm{Tr}[\rho E_\sigma(U^\dagger(I \otimes L_2)U)], \qquad (4.14)$$

where $E_\sigma : B(\mathcal{H}_1) \otimes \mathcal{N} \to B(\mathcal{H}_1)$ is a linear mapping such that

$$\mathrm{Tr}[vE_\sigma(K)] = \langle v \otimes \sigma, K\rangle \qquad (K \in B(\mathcal{H}_1) \otimes \mathcal{N})$$

for all trace class operators v on $\mathcal{H}_1$ (see (4.1)). Equation (4.14) means

$$L_1 = -\langle \sigma, L_2\rangle I + \mathcal{E}^M(\mathbf{R})L_1 + E_\sigma(U^\dagger(I \otimes L_2)U). \tag{4.15}$$

Since we have already shown in Theorem 4.1 that $[\mathcal{E}^M(\mathbf{R})L_1, P^A(\Delta)] = 0$, all we must do now is to prove the following:

$$[E_\sigma(U^\dagger(I \otimes L_2)U), P^A(\Delta)] = 0 \tag{4.16}$$

for all Borel sets $\Delta \subset \mathbf{R}$. Using the assumption (4.12), we have

$$\begin{aligned}
&\mathrm{Tr}[\rho E_\sigma(U^\dagger(I \otimes L_2)U)P^A(\Delta)] \\
&\qquad = \langle (P^A(\Delta)\rho) \otimes \sigma, U^\dagger(I \otimes L_2)U\rangle \\
&\qquad = \langle \rho \otimes \sigma, U^\dagger(I \otimes L_2)U(P^A(\Delta) \otimes I)\rangle \\
&\qquad = \langle \rho \otimes \sigma, U^\dagger(P^A(\Delta) \otimes L_2)U\rangle \\
&\qquad = \langle \rho \otimes \sigma, (P^A(\Delta) \otimes I)U^\dagger(I \otimes L_2)U\rangle \\
&\qquad = \langle (\rho P^A(\Delta)) \otimes \sigma, U^\dagger(I \otimes L_2)U\rangle \\
&\qquad = \mathrm{Tr}[\rho P^A(\Delta)E_\sigma(U^\dagger(I \otimes L_2)U)].
\end{aligned}$$

In the above, ρ is an arbitrary density operator on $\mathcal{H}_1$, thus we get (4.16).

Acknowledgements

The author would like to express his sincere gratitude to Drs. Kiyotaka Kakazu and Shoju Kudaka for their useful comments and encouragement.

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