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Ryukyu Math. J., 5(1992), 1-6

A REMARK ON THEOREM OF JARNIK

TAKANORI HINOKUMA AND HIROO SHIGA

1. Introduction

In this note we consider the Hausdorff dimension of a set arising from Diophantine approximation. Set

$$E_\alpha = \{x \in (0,1) : |x - \frac{p}{q}| < \frac{1}{q^\alpha} \text{ for infinitely many positive integers } p, q\}.$$

Then the Hausdorff dimension of E_α, denoted by $dim_H E_\alpha$, is one when $\alpha \le 2$ by Khinchine's theorem[3]. If $\alpha > 2$, $dim_H E_\alpha = \frac{2}{\alpha}$ by Jarnik's theorem[1]. The Purpose of the present note is to show the following result:

Proposition 1: *Set*

$$S_\alpha = \{x \in (0,1) : |x - \frac{p}{q}| < \frac{|\sin q|}{q^\alpha} \textit{ for infinitely many positive integers } p, q\}.$$

Then

$$dim_H S_\alpha = \begin{cases} 1 & \textit{if } \alpha \le 2 \\ \frac{2}{\alpha} & \textit{if } \alpha > 2. \end{cases}$$

In the case $\alpha \le 2$, the above result is contained in Khinchine's theorem since $\int_1^\infty \frac{|\sin x|}{x^{\alpha-1}} dx = \infty$.

The outline of the proof is based on [1]. But the essential part of our estimation is changed anew to avoid discussing the distribution of the set of prime numbers p satisfying $|\sin p| > c$ for a positive constant c.

Received October 31, 1992.

2. Preliminaries

We first show the inequality $dim_H S_\alpha \leq \frac{2}{\alpha}$. For each positive integer q, we set

$$S_q = \{x \in [0,1] : |x - \frac{p}{q}| < \frac{|\sin q|}{q^\alpha}\}.$$

Then S_q consists of $q-1$ intervals of length $\frac{2|\sin q|}{q^\alpha}$ and two end intervals of length $\frac{|\sin q|}{q^\alpha}$. It is clear that $S_\alpha \subset \bigcup_{q=n}^{\infty} G_q$ for each positive integer n. Hence if we choose n so that $\frac{|\sin n|}{2n^\alpha} \leq \frac{1}{2n^\alpha} \leq \delta$, then

$$\mathcal{H}^s_\delta(S_\alpha) \leq \sum_{q=n}^{\infty}(q+1)\left(\frac{2|\sin q|}{q^\alpha}\right)^s \leq \sum_{q=n}^{\infty}(q+1)\left(\frac{2}{q^\alpha}\right)^s.$$

When $s > \frac{2}{\alpha}$, the above series is convergent. So $\mathcal{H}^s(S_\alpha) = 0$ and we have $dim_H S_\alpha \leq \frac{2}{\alpha}$.

To prove converse inequality we prepare some lemmata.

Definition 2.1: *Let $\{x_n\}$, $n = 1, 2, \cdots$ be a sequence of real numbers. The sequence $\{x_n\}$ is said to be well distrbuted mod 1 if for any pair of numbers with $0 \leq a < b \leq 1$ we can choose a number N_0 which does not depend on k such that*

$$\left|\frac{\#\{x_i \in [a,b) : i = k+1, \cdots, k+N\}}{N} - (b-a)\right| < \epsilon$$

for $N \geq N_0$, where $\#$ denotes the cardinality.

Lemma 2.2: *Let c be a constant with $0 < c < 1$. Then for any $\epsilon > 0$ there exists N_0 so that if $N \geq N_0$*

$$\left|\frac{\#\{N < q \leq 2N : |\sin q| > c\}}{N} - l_c\right| < \epsilon$$

for some positive number l_c depending only on c.

Proof: Let J_c be the subinterval of $[0,\pi]$ satisfying $|\sin x| > c$. Then we have

$$\begin{aligned}&\#\{N < q \leq 2N : |\sin q| > c\}\\ &= \#\{N < q \leq 2N : q \in J_c \bmod \pi\}\\ &= \#\{N < q \leq 2N : \frac{q}{\pi} \in \tilde{J}_c \bmod 1\},\end{aligned}$$

where $\tilde{J}_c$ denotes the subinterval of $[0,1]$ such that $|\sin \pi x| > c$. Since $\{\frac{q}{\pi}\}$ $q = 1, 2, \cdots$ is well distributed by Weyl's Criterion ([2], Example 5.2), for any $\epsilon > 0$, there is a number N_0 such that if $N > N_0$

$$\left| \frac{\#\{N < q \leq 2N : \frac{q}{\pi} \in \tilde{J}_c \bmod 1\}}{n} - l_c \right| < \epsilon$$

where l_c is the length of $\tilde{J}_c$. ∎

Let $[0,1] = E_0 \supset E_1 \supset E_2 \supset \cdots$ be a decreasing sequence of sets satisfying the following conditions:

(1) each E_k is a disjoint union of finite number of closed intervals,

(2) each interval of E_k contains at least two intervals of E_{k+1},

(3) the maximal length of intervals in E_k tends to zero as $k \to \infty$. Then the set $F = \bigcap_{k=1}^{\infty} E_k$ is totally disconnected subset of $[0,1]$. The following lemma is contained in [1].

Lemma 2.3: *Suppose in the above construction each interval of E_{k-1} contains at least m_k intervals of E_k ($k = 1, 2, \cdots$) which are separated by the distance at least ϵ_k with $0 < \epsilon_{k+1} < \epsilon_k$ for each k. Then*

$$dim_H F \geq \underline{\lim}_{k\to\infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k \epsilon_k)}.$$

3. Proof of Proposition 1

Throughout this section we fix a some constant real number c with $0 < c < 1$. We translate S_α to the right by one and set

$$G_q = \{x \in (1+q^{-\alpha}, 2-q^{-\alpha}) : |x - \frac{p}{q}| < \frac{c}{q^\alpha} \text{ with reduced fraction } \frac{p}{q}\}$$

Let n be a positive integer and q_i $(i = 1, 2)$ be positive integers satisfying $n < q_1 < q_2 \le 2n$. If $\frac{r_1}{q_1} \ne \frac{r_2}{q_2}$ then

$$\left|\frac{r_1}{q_1} - \frac{r_2}{q_2}\right| = \frac{1}{q_1 q_2}|r_1 q_2 - r_2 q_1| \ge \frac{1}{q_1 q_2} \ge \frac{1}{4n^2}$$

Hence the distance between distinct centers of any pairs of intervals in G_{q_1} and G_{q_2} is at least $\frac{1}{4n^2}$. Since these intervals have lengths at most $2cn^{-\alpha}$, the distance between any point of G_{q_1} and any point of G_{q_2} is at least

$$\frac{1}{4}n^{-2} - 2cn^{-\alpha} \ge \frac{1}{8}n^{-2}$$

provided for sufficiently large $n \ge n_0$ for some large n_0. Set

$$C_n = \{n < q \le 2n : |\sin q| > c\}.$$

By Lemma 2.2 the cardinality of the set C_n is at least $\frac{n}{2}l_c$ if $n \ge n_0'$ for some large $n_0' \ge n_0$. For such n the set

$$H_n = \bigcup_{q \in C_n} G_q$$

consists of intervals of length at least $c(2n)^{-\alpha}$ which are separated by distance at least $\frac{1}{8}n^{-2}$. Let $I = [a, b] \subset [1, 2]$ be an interval. Applying the prime number theorem there are at least

$$\frac{1}{2}\left(\frac{bq}{\log bq} - \frac{aq}{\log aq}\right)$$

primes in the range $[aq, bq]$ for sufficiently large q. Moreover we have

$$\frac{1}{2}\left(\frac{bq}{\log bq} - \frac{aq}{\log aq}\right)$$
$$= \frac{1}{2}\frac{(b-a)q\log q + q(b\log a - a\log b)}{(\log q + \log b)(\log q + \log a)}$$
$$> \frac{1}{4}\frac{(b-a)q}{\log q} = \frac{1}{4}\frac{q|I|}{\log q}$$

for sufficiently large q. Thus there are at least

$$\frac{n|I|}{4\log n}$$

reduced fractions whose denominators are q in the range $[a, b]$ if $n < q \leq 2n$ and $n \geq n_1$ for some large $n_1 > n_0'$. Summarizing the above estimates, at least

$$\frac{n^2 l_c}{8\log n}|I|$$

intervals of H_n are contained in $I = [a, b]$ provided that $n \geq n_1$.

Let n_1 be as above and $n_k = n_{k-1}^k$ for $k = 2, 3, \cdots$. Let $E_0 = [\delta, 1]$ and for $k = 1, 2, \cdots$ let E_k consist of those intervals of H_{n_k} which are completely contained in E_{k-1}. The intervals of E_k are of lengths at least $c(2n_k)^{-\alpha}$ and are separated by distance at least $\epsilon_k = \frac{1}{8}n_k^{-2}$. From the above argument each interval of E_{k-1} contains at least m_k intervals of E_k, where

$$m_k = \frac{n_k^2 l_c}{8\log n_k}c(2n_{k-1})^{-\alpha} = \frac{K n_k^2 n_{k-1}^{-\alpha}}{\log n_k}$$

if $k \geq 2$, where $K = \frac{1}{8}2^{-\alpha}cl_c$. Note that we take $m_1 = 1$ and $m_k \geq 2$ for sufficiently large k. By Lemma 2.3

$$
\begin{aligned}
&dim_H(\bigcap_{k=1}^{\infty} E_k) \\
&\geq \varliminf_{k\to\infty} \frac{\log\left[K^{k-2}n_1^{-\alpha}(n_2\cdots n_{k-2})^{2-\alpha}n_{k-1}^2(\log n_2\cdots\log n_{k-1})^{-1}\right]}{-\log\left[Kn_{k-1}^{-\alpha}(8\log n_k)^{-1}\right]} \\
&= \varliminf_{k\to\infty} \frac{\log\left[K^{k-2}n_1^{-\alpha}(n_2\cdots n_{k-2})^{2-\alpha}(\log n_2\cdots\log n_{k-1})^{-1}\right] + 2\log n_{k-1}}{-\log(K/8) + \log k(\log n_{k-1}) + \alpha\log n_{k-1}} \\
&= \frac{2}{\alpha}
\end{aligned}
$$

since $\log n_k = n_1 k!$. If $x \in E_k \subset H_{n_k}$, then $|x - \frac{p}{q}| < \frac{c}{q^\alpha}$ and $c \leq |\sin q|$ where $n_k < q \leq 2n_k$ and $\frac{p}{q}$ is a reduced fraction. Thus $\bigcap_{k=1}^{\infty} E_k \subset S_\alpha$. Hence $dim_H S_\alpha \geq \frac{2}{\alpha}$.

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Department of Mathematics
College of Science
University of the Ryukyus
Nishihara, Okinawa 903-01
JAPAN