A note on endomorphism rings of H-separable extensions

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A NOTE ON ENDOMORPHISM RINGS OF H-SEPARABLE EXTENSIONS

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1. Introduction

Let $R \supset S$ be a ring extension. R is said to be an $H-$separable extension over S if as $R-R-$modules $R \otimes_S R \langle\oplus (R \oplus \cdots \oplus R)$. There is an equivalent definition of $H-$separable extension in terms of the natural $R-R-$homomorphism $\varphi : R \otimes_S R \longrightarrow Hom(\Delta_C, R_C)$ where $\varphi(a \otimes b)(x) = axb, C$ is the center of R and Δ is the commutor ring of S in R, i.e. $\Delta = R^S = \{x \in R : xa = ax$ for all $a \in S\}$. R is $H-$separable extension over S if and only if Δ is C-finitely generated projective and φ is an isomorphism. Let Q be a ring and P a $Q-Q$-bimodule. P is said to be $Q-$centrally projective if as $Q-Q-$bimodules $P \langle\oplus (Q \oplus \cdots \oplus Q)$. Furthermore, if P is ring extension over Q then $P \cong Q \otimes_C \Delta^\circ$ where Δ° is the commutor ring of Q in P and C is the center of P. For details see [**1**].

In this paper, we give some one to one correspondence between some class of intermediate rings of H-separable extension and some class of intermediate rings of centrally projective extension, by taking the endomorphism ring.

2. Main Results

Let $P \supset Q$ be a ring extension, Δ° the commutor ring of Q in P and C the center of Q. Furthermore, let $\mathcal{B}_r^\circ$ be the set of subrings

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B of P such that $Q \subset B$, ${}_Q B_B \langle\oplus\, {}_Q P_B$ and the multiplication map $P \otimes_Q B \to P$ splits as a $P - B-$homomorphism, and $\mathcal{D}^\circ_\tau$ the set of $C-$subalgebras D of Δ° such that $D_D \langle\oplus\, \Delta^\circ_D$ and the multiplication map $\Delta^\circ \otimes_C D \to \Delta^\circ$ splits as a $\Delta^\circ - D-$homomorphism.

Lemma. *Let P be Q-centrally projective, that is*

$$P \langle\oplus\, (Q \oplus \cdots \oplus Q)$$

as $Q-Q$-bimodules. Then we have one to one correspondence between $\mathcal{B}^\circ_\tau$ and $\mathcal{D}^\circ_\tau$, by taking $B \cap \Delta^\circ$ for $B \in \mathcal{B}^\circ_\tau$ and QD for $D \in \mathcal{D}^\circ_\tau$.

Proof: Let $B \in \mathcal{B}^\circ_\tau$ and p the $Q - B-$projection of P to B. Since, for any $q \in Q$ and $\delta \in \Delta^\circ$,

$$qp(\delta) = p(q\delta) = p(\delta q) = p(\delta)q,$$

we have $D \subset \Delta^\circ$ and $D_D \langle\oplus\, \Delta^\circ_D$, where $D = p(\Delta^\circ)$. By [**2**, Proposition 5.5], $P \cong Q \otimes_C \Delta^\circ$. Then we have

$$B = p(P) = p(Q\Delta^\circ) = Qp(\Delta^\circ) = QD.$$

Since $D \langle\oplus\, \Delta^\circ$, we have the commutative diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & Q \otimes_C D & \longrightarrow & Q \otimes_C \Delta^\circ \\
 & & \downarrow & & \downarrow{\scriptstyle \alpha} \\
0 & \longrightarrow & QD & \longrightarrow & P \\
 & & \| & & \\
 & & B & &
\end{array}$$

where two rows are exact and α is an isomorphism. Therefore $B \cong Q \otimes_C D$. On the other hand, it is easy to check that $D = B \cap \Delta^\circ$. Now, for any $Q-Q$-bimodule M, we denote by M^Q the subset $\{m \in M; qm = mq \text{ for all } q \in Q\}$ of M. Then we have the following isomorphisms

$$\begin{aligned}
& Hom({}_{Q-\Delta^\circ}Q \otimes_C \Delta^\circ_{Q-D}, {}_{Q-\Delta^\circ} Q \otimes_C (\Delta^\circ \otimes_C D)_{Q-D}) \\
& \cong Hom({}_{\Delta^\circ}\Delta^\circ_D, {}_{\Delta^\circ} Hom({}_Q Q_Q, {}_Q Q \otimes_C (\Delta^\circ \otimes_C D)_Q)_D) \\
& \cong Hom({}_{\Delta^\circ}\Delta^\circ_D, {}_{\Delta^\circ} (Q \otimes_C (\Delta^\circ \otimes_C D))^Q_D) \\
& \cong Hom({}_{\Delta^\circ}\Delta^\circ_D, {}_{\Delta^\circ} Q^Q \otimes_C (\Delta^\circ \otimes_C D)_D) \\
& = Hom({}_{\Delta^\circ}\Delta^\circ_D, {}_{\Delta^\circ} C \otimes_C (\Delta^\circ \otimes_C D)_D) \\
& \cong Hom({}_{\Delta^\circ}\Delta^\circ_D, {}_{\Delta^\circ} \Delta^\circ \otimes_C D_D).
\end{aligned}$$

The inverse map of the composition of these isomorphism is given by $f \longmapsto [q \otimes \delta \mapsto q \otimes f(\delta)]$ where $f \in Hom({}_{\Delta^\circ}\Delta^\circ_D, {}_{\Delta^\circ} \Delta^\circ \otimes_C D_D), q \in Q$ and $\delta \in \Delta^\circ$. Furthermore, since

$$P \otimes_Q B \cong (Q \otimes_C \Delta^\circ) \otimes_Q (Q \otimes_C D) \cong Q \otimes_C (\Delta^\circ \otimes_C D)$$

and the multiplication map $P \otimes_Q B \longrightarrow P$ splits as a $P-B$-homomorphism, there exist a splitting map of the multiplication map $\Delta^\circ \otimes_C D \longrightarrow \Delta^\circ$ as a $\Delta^\circ - D$-homomorphism. Therefore $D \in \mathcal{D}^\circ_\tau$.

Next let $D \in \mathcal{D}^\circ_\tau$ and $B = QD$. Since $D \langle\oplus \Delta^\circ$, we have the commutative diagram

$$
\begin{array}{ccccc}
0 & \longrightarrow & Q \otimes_C D & \longrightarrow & Q \otimes_C \Delta^\circ \\
 & & \downarrow & & \downarrow \alpha \\
0 & \longrightarrow & B & \longrightarrow & P
\end{array}
$$

where two rows are exact and α is an isomorphism. Therefore $B \cong Q \otimes_C D$. Then we have

$$
{}_Q B_B \cong_Q (Q \otimes_C D)_{Q-D} \langle\oplus\, {}_Q(Q \otimes_C \Delta^\circ)_{Q-D} \cong_Q P_B
$$

and

$$
\begin{aligned}
{}_P P \otimes_Q B_B &\cong_{Q-\Delta^\circ} (Q \otimes_C \Delta^\circ) \otimes_Q (Q \otimes_C D)_{Q-D} \\
&\cong_{Q-\Delta^\circ} Q \otimes_C (\Delta^\circ \otimes_C D)_{Q-D} \oplus\rangle\ {}_{Q-\Delta^\circ} Q \otimes_C \Delta^\circ_{Q-D} \\
&\cong_P P_B .
\end{aligned}
$$

Therefore $B \in \mathcal{B}^\circ_\tau$. If $D' = B \cap \Delta^\circ$ then $D' \in \mathcal{D}^\circ_\tau$, for the result of the first half. Since $D \langle\oplus \Delta^\circ$ and $D \subset D'$, $D \langle\oplus D'$. So, tensoring Q over C to the short exact sequence

$$
0 \longrightarrow D \longrightarrow D' \longrightarrow D'/D \longrightarrow 0,
$$

we have the short exact sequence

$$
0 \longrightarrow Q \otimes_C D \longrightarrow Q \otimes_C D' \longrightarrow Q \otimes_C D'/D \longrightarrow 0.
$$

But $Q \otimes_C D \cong B \cong Q \otimes_C D'$. Then $Q \otimes_C D'/D = 0$. On the other hand, D'/D is $C-$finitely generated projective,for so are D' and D and $D \langle \oplus D'$. Hence D'/D is $C-$flat and

$$\begin{array}{ccccc} 0 & \longrightarrow & C \otimes_C D'/D & \longrightarrow & Q \otimes_C D'/D \\ & & \| & & \| \\ & & D'/D & & 0 \end{array}$$

Therefore $D'/D = 0$ and $D' = D$. This completes the proof. ∎

Next theorem is the purpose of this paper. Let $R \supset S$ be a ring extension, Δ the commutor ring of S in R and C the center of R. Furthermore, let $\mathcal{B}_l$ be the set of subrings B of R such that $S \subset B$, ${}_BB_S \langle \oplus {}_BR_S$ and the multiplication map $B \otimes_S R \to R$ splits as a $B - R-$homomorphism, and $\mathcal{D}_l$ the set of $C-$subalgebras D of Δ such that ${}_DD \langle \oplus {}_D\Delta$ and the multiplication map $D \otimes_C \Delta \to \Delta$ splits as a $D - \Delta-$homomorphism.

THEOREM. *Let $R \supset S$ be an H-separable extension and M a right R-module such that faithfully balanced. And let $P = Hom(M_S, M_S)$, $Q = Hom(M_R, M_R)$. Then we have one to one correspondence between $\mathcal{B}_l$ and $\mathcal{B}_r^\circ$, by taking the endomorphism ring.*

PROOF: First we have

$$P^Q = Hom(M_S, M_S)^Q = Hom({}_QM,{}_QM)^S \cong R^S = \Delta.$$

Considering the ring structure, we have that $P^Q \cong \Delta^\circ$ (it means that the opposite ring of Δ). By the same way, we have that $Q^Q = C$, that is the center of Q. By [**3**, proposition 2.1], P is Q–centrally projective. By [**2**, Proposition 5.6], $P \cong Q \otimes_C \Delta^\circ$.

Let $B \in \mathcal{B}_l$ and $D = R^B$. By [**4**, (1.3)], $D \in \mathcal{D}_l$ and $D^\circ \in \mathcal{D}_r^\circ$. Then we have

$$\begin{aligned}\tilde{B}; &= Hom(M_B, M_B) = Hom(M_S, M_S)^B = P^B \\ &\cong (Q \otimes_C \Delta^\circ)^B = Q \otimes_C \Delta^{\circ B} = Q \otimes_C D^\circ .\end{aligned}$$

By Lemma, $\tilde{B} \in \mathcal{B}_r^\circ$. By [**4**, (1,3)],

$$\begin{aligned}&Hom({}_{\tilde{B}}M, {}_{\tilde{B}}M) = Hom({}_{Q-D^\circ}M, {}_{Q-D^\circ}M) \\ &Hom({}_QM, {}_QM)^{D^\circ} \cong R^{D^\circ} = R^D \\ &= B.\end{aligned}$$

Next let $B \in \mathcal{B}_r^\circ$ and $D = \Delta^\circ \cap B$. By Lemma, $D \in \mathcal{D}_r^\circ$ and $D^\circ \in \mathcal{D}_l$. Then, by [**4**, (1.3)], we have

$$\begin{aligned}\tilde{B}; &= Hom({}_BM, {}_BM) = Hom({}_QM, {}_QM)^B \\ &= Hom({}_QM, {}_QM)^{Q-D} = Hom({}_QM, {}_QM)^D \\ &\cong R^D = R^{D^\circ} \in \mathcal{B}_l\end{aligned}$$

By [**4**, (1.3)], $R^{\tilde{B}} = D^\circ$, and by the above result,

$$Hom(M_{\tilde{B}}, M_{\tilde{B}}) \cong Q \otimes_C D^{\circ\circ}$$
$$= Q \otimes_C D \cong B.$$

This completes the proof. ∎

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