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A NOTE ON ENDOMORPHISM RINGS OF H-SEPARABLE EXTENSIONS

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1. Introduction

Let $R \supset S$ be a ring extension. R is said to be an H-separable extension over S if as R - R-modules $R \otimes_S R \ (\oplus (R \oplus \cdots \oplus R))$. There is an equivalent definition of H-separable extension in terms of the natural R - R-homomorphism $\varphi : R \otimes_S R \longrightarrow Hom(\Delta_C, R_C)$ where $\varphi(a \otimes b)(\mathbf{x}) = a\mathbf{x}b, C$ is the center of R and Δ is the commutor ring of S in R, i.e. $\Delta = R^S = \{\mathbf{x} \in R : \mathbf{x}a = a\mathbf{x} \text{ for all } a \in S\}$. Ris H-separable extension over S if and only if Δ is C-finitely generated projective and φ is an isomorphism. Let Q be a ring and P a Q - Q-bimodule. P is said to be Q-centrally projective if as Q - Q-bimodules $P \ (\oplus (Q \oplus \cdots \oplus Q))$. Furthermore, if P is ring extension over Q then $P \cong Q \otimes_C \Delta^\circ$ where Δ° is the commutor ring of Q in P and C is the center of P. For details see [1].

In this paper, we give some one to one correspondence between some class of intermediate rings of H-separable extension and some class of intermediate rings of centrally projective extension, by taking the endomorphism ring.

2. Main Results

Let $P \supset Q$ be a ring extension, Δ° the commutor ring of Q in P and C the center of Q. Furthermore, let \mathcal{B}_{r}° be the set of subrings

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B of P such that $Q \subset B$, ${}_QB_B \langle \oplus {}_QP_B$ and the multiplication map $P \otimes_Q B \to P$ splits as a P - B-homomorphism, and $\mathcal{D}^{\circ}_{\tau}$ the set of C-subalgebras D of Δ° such that $D_D \langle \oplus \Delta^{\circ}_D$ and the multiplication map $\Delta^{\circ} \otimes_C D \to \Delta^{\circ}$ splits as a $\Delta^{\circ} - D$ -homomorphism.

LEMMA. Let P be Q-centrally projective, that is

$$P \left(\oplus \left(Q \oplus \cdots \oplus Q \right) \right)$$

as Q-Q-bimodules. Then we have one to one correspondence between \mathcal{B}°_{r} and \mathcal{D}°_{r} , by taking $B \cap \Delta^{\circ}$ for $B \in \mathcal{B}^{\circ}_{r}$ and QD for $D \in \mathcal{D}^{\circ}_{r}$.

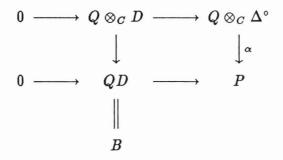
PROOF: Let $B \in \mathcal{B}^{\circ}_{r}$ and p the Q - B-projection of P to B. Since, for any $q \in Q$ and $\delta \in \Delta^{\circ}$,

$$qp(\delta) = p(q\delta) = p(\delta q) = p(\delta)q,$$

we have $D \subset \Delta^{\circ}$ and $D_D \langle \oplus \Delta_D^{\circ} \rangle$, where $D = p(\Delta^{\circ})$. By [2, Proposition 5.5], $P \cong Q \otimes_C \Delta^{\circ}$. Then we have

$$B=p(P)=p(Q\Delta^\circ)=Qp(\Delta^\circ)=QD.$$

Since $D \ (\oplus \Delta^\circ)$, we have the commutative diagram



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where two rows are exact and α is an isomorphism. Therefore $B \cong Q \otimes_C D$. On the other hand, it is easy to check that $D = B \cap \Delta^\circ$. Now, for any Q - Q-bimodule M, we denote by M^Q the subset $\{m \in M; qm = mq \text{ for all } q \in Q\}$ of M. Then we have the following isomorphisms

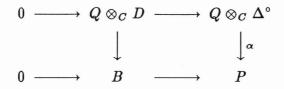
$$\begin{split} &Hom(_{Q-\Delta^{\circ}}Q\otimes_{C}\Delta^{\circ}_{Q-D},_{Q-\Delta^{\circ}}Q\otimes_{C}(\Delta^{\circ}\otimes_{C}D)_{Q-D})\\ &\cong Hom(_{\Delta^{\circ}}\Delta^{\circ}_{D},_{\Delta^{\circ}}Hom(_{Q}Q_{Q},_{Q}Q\otimes_{C}(\Delta^{\circ}\otimes_{C}D)_{Q})_{D})\\ &\cong Hom(_{\Delta^{\circ}}\Delta^{\circ}_{D},_{\Delta^{\circ}}(Q\otimes_{C}(\Delta^{\circ}\otimes_{C}D))_{D}^{Q})\\ &\cong Hom(_{\Delta^{\circ}}\Delta^{\circ}_{D},_{\Delta^{\circ}}Q^{Q}\otimes_{C}(\Delta^{\circ}\otimes_{C}D)_{D})\\ &= Hom(_{\Delta^{\circ}}\Delta^{\circ}_{D},_{\Delta^{\circ}}C\otimes_{C}(\Delta^{\circ}\otimes_{C}D)_{D})\\ &\cong Hom(_{\Delta^{\circ}}\Delta^{\circ}_{D},_{\Delta^{\circ}}\Delta^{\circ}\otimes_{C}D_{D}). \end{split}$$

The inverse map of the composition of these isomorphism is given by $f \mapsto [q \otimes \delta \mapsto q \otimes f(\delta)]$ where $f \in Hom(\Delta \circ \Delta_D^\circ, \Delta \circ \otimes_C D_D), q \in Q$ and $\delta \in \Delta^\circ$. Furthermore, since

$$P \otimes_{\boldsymbol{Q}} B \cong (\boldsymbol{Q} \otimes_{\boldsymbol{C}} \Delta^{\circ}) \otimes_{\boldsymbol{Q}} (\boldsymbol{Q} \otimes_{\boldsymbol{C}} D) \cong \boldsymbol{Q} \otimes_{\boldsymbol{C}} (\Delta^{\circ} \otimes_{\boldsymbol{C}} D)$$

and the multiplication map $P \otimes_Q B \longrightarrow P$ splits as a P - B-homomorphism, there exist a splitting map of the multiplication map $\Delta^{\circ} \otimes_C D \longrightarrow \Delta^{\circ}$ as a $\Delta^{\circ} - D$ -homomorphism. Therefore $D \in \mathcal{D}^{\circ}_{r}$.

Next let $D \in \mathcal{D}^{\circ}_{r}$ and B = QD. Since $D \langle \oplus \Delta^{\circ},$ we have the commutative diagram



where two rows are exact and α is an isomorphism. Therefore $B \cong Q \otimes_C D$. Then we have

$${}_{Q}B_{B}\cong_{Q}(Q\otimes_{C}D)_{Q-D} \langle \oplus_{Q}(Q\otimes_{C}\Delta^{\circ})_{Q-D}\cong_{Q}P_{B}$$

and

$${}_{P}P \otimes_{Q} B_{B} \cong_{Q-\Delta^{\circ}} (Q \otimes_{C} \Delta^{\circ}) \otimes_{Q} (Q \otimes_{C} D)_{Q-D}$$
$$\cong_{Q-\Delta^{\circ}} Q \otimes_{C} (\Delta^{\circ} \otimes_{C} D)_{Q-D} \oplus \rangle_{Q-\Delta^{\circ}} Q \otimes_{C} \Delta^{\circ}_{Q-D}$$
$$\cong_{P} P_{B}.$$

Therefore $B \in \mathcal{B}^{\circ}_{r}$. If $D' = B \cap \Delta^{\circ}$ then $D' \in \mathcal{D}^{\circ}_{r}$, for the result of the first half. Since $D \langle \oplus \Delta^{\circ} \text{ and } D \subset D', D \langle \oplus D'.$ So, tensoring Q over C to the short exact sequence

$$0 \longrightarrow D \longrightarrow D' \longrightarrow D'/D \longrightarrow 0,$$

we have the short exact sequence

$$0 \longrightarrow Q \otimes_C D \longrightarrow Q \otimes_C D' \longrightarrow Q \otimes_C D'/D \longrightarrow 0.$$

But $Q \otimes_C D \cong B \cong Q \otimes_C D'$. Then $Q \otimes_C D'/D = 0$. On the other hand, D'/D is C-finitely generated projective, for so are D' and D and $D \langle \oplus D'$. Hence D'/D is C-flat and

Therefore D'/D = 0 and D' = D. This completes the proof.

Next theorem is the purpose of this paper. Let $R \supset S$ be a ring extension, Δ the commutor ring of S in R and C the center of R. Furthermore, let \mathcal{B}_l be the set of subrings B of R such that $S \subset B$, $_BB_S \langle \oplus_B R_S$ and the multiplication map $B \otimes_S R \to R$ splits as a B-R-homomorphism, and \mathcal{D}_l the set of C-subalgebras D of Δ such that $_DD \langle \oplus_D \Delta$ and the multiplication map $D \otimes_C \Delta \to \Delta$ splits as a $D - \Delta$ -homomorphism.

THEOREM. Let $R \supset S$ be an *H*-separable extension and *M* a right *R*-module such that faithfully balanced. And let $P = Hom(M_S, M_S)$, $Q = Hom(M_R, M_R)$. Then we have one to one correspondence between \mathcal{B}_l and \mathcal{B}_r° , by taking the endomorphism ring.

PROOF: First we have

$$P^{Q} = Hom(M_{S}, M_{S})^{Q} = Hom(_{Q}M_{,Q}M)^{S} \cong R^{S} = \Delta.$$

Considering the ring structure, we have that $P^Q \cong \Delta^\circ$ (it means that the opposite ring of Δ). By the same way, we have that $Q^Q = C$, that

is the center of Q. By [3, proposition 2.1], P is Q-centrally projective. By [2, Proposition 5.6], $P \cong Q \otimes_C \Delta^{\circ}$.

Let $B \in \mathcal{B}_l$ and $D = R^B$. By [4, (1.3)], $D \in \mathcal{D}_l$ and $D^\circ \in \mathcal{D}^\circ_r$. Then we have

$$ilde{B}; = Hom(M_B, M_B) = Hom(M_S, M_S)^B = P^B$$

 $\cong (Q \otimes_C \Delta^\circ)^B = Q \otimes_C \Delta^{\circ B} = Q \otimes_C D^\circ.$

By Lemma, $\tilde{B} \in \mathcal{B}_{r}^{\circ}$. By [4, (1,3)],

$$Hom(_{\tilde{B}}M,_{\tilde{B}}M) = Hom(_{Q-D}\circ M,_{Q-D}\circ M)$$
$$Hom(_{Q}M,_{Q}M)^{D^{\circ}} \cong R^{D^{\circ}} = R^{D}$$
$$= B.$$

Next let $B \in \mathcal{B}^{\circ}_{r}$ and $D = \Delta^{\circ} \cap B$. By Lemma, $D \in \mathcal{D}^{\circ}_{r}$ and $D^{\circ} \in \mathcal{D}_{l}$. Then, by [4, (1.3)], we have

$$\ddot{B} := Hom(_B M,_B M) = Hom(_Q M,_Q M)^B$$
$$= Hom(_Q M,_Q M)^{Q-D} = Hom(_Q M,_Q M)^D$$
$$\cong R^D = R^{D^\circ} \in \mathcal{B}_I$$

By [4, (1.3)], $R^{\check{B}} = D^{\circ}$, and by the above result,

$$Hom(M_{\check{B}}, M_{\check{B}}) \cong Q \otimes_C D^{\circ \circ}$$

= $Q \otimes_C D \cong B.$

This completes the proof.

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