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THE DEGREE OF CONVERGENCE OF EQUI-UNIFORM SUMMATION PROCESSES OF INTERPOLATION TYPE OPERATORS IN BANACH SPACES

TOSHIHIKO NISHISHIRAHO

ABSTRACT. We give quantitative estimates of the rate of convergence of equi-uniform summation processes of interpolation type operators in Banach spaces in terms of the modulus of continuity of functions to be approximated. Moreover, applications are presented by various equi-uniform summation processes of Bernstein type and Hermite-Fejér type operators.

1. Introduction

Let $(E, \|\cdot\|)$ be a Banach space and let (X, d) be a metric space. Let $B(X, E)$ denote the Banach space of all E-valued bounded functions on X with the supremum norm. $BC(X, E)$ stands for the closed linear subspace of $B(X, E)$ consisting of all E-valued bounded continuous functions on X. Also, we denote by $C(X, E)$ the linear space consisting of all E-valued continuous functions on X. Let $\{Y_{m,\gamma} : m \in \mathbb{N}_0, \gamma \in \Gamma\}$ be a family of finite sets, where $\mathbb{N}_0$ is the set of all nonnegative integers and Γ is an index set.

Let $\mathcal{A} = \{a_{\alpha,m}^{(\lambda)} : \alpha \in D, m \in \mathbb{N}_0, \lambda \in \Lambda\}$ be a family of scalars, where D is a directed set and Λ is an index set. Let $\mathfrak{A} = \{\chi_{m,\gamma}(\cdot; k) : m \in \mathbb{N}_0, \gamma \in \Gamma, k \in Y_{m,\gamma}\}$ be a family of scalar-valued functions on X

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such that

$$(1) \qquad g_{\alpha,\lambda,\gamma}(x) := \sum_{m=0}^{\infty} \sum_{k \in Y_{m,\gamma}} |a_{\alpha,m}^{(\lambda)} \chi_{m,\gamma}(x;k)| < \infty$$

for each $\alpha \in D, \lambda \in \Lambda, \gamma \in \Gamma, x \in X$ and let $\{\xi_{m,\gamma} : m \in \mathbb{N}_0, \gamma \in \Gamma\}$ be a family of mappings from $Y_{m,\gamma}$ to X. Then we define an interpolation type operator by the form

$$(2) \qquad K_{m,\gamma}(F)(x) = \sum_{k \in Y_{m,\gamma}} \chi_{m,\gamma}(x;k) F(\xi_{m,\gamma}(k))$$

$$(m \in \mathbb{N}_0,\ \gamma \in \Gamma,\ F \in BC(X,E),\ x \in X)$$

(cf. [11], [12]). Furthermore, we define

$$(3) \qquad K_{\alpha,\lambda,\gamma}(F)(x) = \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} K_{m,\gamma}(F)(x)$$

$$(\alpha \in D,\ \lambda \in \Lambda,\ \gamma \in \Gamma,\ F \in BC(X,E),\ x \in X),$$

which converges in E because of (1). Let X_0 be a subset of X. Then the family $\mathfrak{K} = \{K_{m,\gamma} : m \in \mathbb{N}_0, \gamma \in \Gamma\}$ is called an equi-uniform $\mathcal{A}$-summation process on $BC(X,E)$ if for every $F \in BC(X,E)$,

$$(4) \quad \lim_{\alpha} \|K_{\alpha,\lambda,\gamma}(F)(x) - F(x)\| = 0 \text{ uniformly in } \lambda \in \Lambda, \gamma \in \Gamma, x \in X_0.$$

The purpose of this paper is to give quantitative estimates of the rate of convergence behavior (4) in terms of the modulus of continuity of F under certain appropriate conditions (cf. [13], [14], [15]). Besides, applications are presented by the equi-uniform $\mathcal{A}$-summation processes of Bernstein type and Hermite-Fejér type operators.

2. $\mathcal{A}$-summability methods

$\mathcal{A}$ is said to be regular if it satisfies the following conditions:

(A-1) For each $m \in \mathbb{N}_0$,

$$\lim_{\alpha} a_{\alpha,m}^{(\lambda)} = 0 \qquad \text{uniformly in } \lambda \in \Lambda.$$

(A-2) $\lim_{\alpha} \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} = 1$ uniformly in $\lambda \in \Lambda$.

(A-3) For each $\alpha \in D, \lambda \in \Lambda$,

$$a_{\alpha}^{(\lambda)} := \sum_{m=0}^{\infty} |a_{\alpha,m}^{(\lambda)}| < \infty,$$

and there exists an element $\alpha_0 \in D$ such that

$$\sup\{a_\alpha^{(\lambda)} : \alpha \geq \alpha_0,\ \alpha \in D,\ \lambda \in \Lambda\} < \infty.$$

$\mathcal{A}$ is said to be positive if

$$a_{\alpha,m}^{(\lambda)} \geq 0 \qquad \text{for all } \alpha \in D, m \in \mathbb{N}_0 \text{ and all } \lambda \in \Lambda.$$

Also, $\mathcal{A}$ is said to be stochastic if it is positive and

$$\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} = 1 \qquad \text{for all } \alpha \in D \text{ and all } \lambda \in \Lambda.$$

Obviously, if $\mathcal{A}$ is positive, then (A-2) already implies (A-3) and if $\mathcal{A}$ is stochastic, then Conditions (A-2) and (A-3) are automatically satisfied.

A sequence $\{f_m\}_{m \in \mathbb{N}_0}$ of elements in E is said to be $\mathcal{A}$-summable to f if

$$\text{(5)} \qquad \lim_\alpha \Big\| \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} f_m - f \Big\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda,$$

where it is assumed that the series in (5) converges for each $\alpha \in D$ and each $\lambda \in \Lambda$.

Concerning the relation between the regularity of $\mathcal{A}$ and the $\mathcal{A}$-summability, $\mathcal{A}$ is regular if and only if every convergent sequence in E is $\mathcal{A}$-summable to its limit (cf. [1], [8]).

As the following examples with $D = \mathbb{N}_0$ show, there is a wide variety of families $\mathcal{A}$ of particular interest which cover many important summation methods scattered in the literature.

(1°) Given a matrix $A = (a_{nm})_{n,m \in \mathbb{N}_0}$, if $a_{n,m}^{(\lambda)} = a_{nm}$ for all $n, m \in \mathbb{N}_0$ and all $\lambda \in \Lambda$, then we obtain the usual matrix summability by A.

(2°) If $\Lambda = \mathbb{N}_0$, then we obtain the summation method by introduced by Petersen [16] (cf. [1]). In particular, if

$$a_{n,m}^{(\lambda)} = \begin{cases} \frac{1}{n+1} & \text{if } \lambda \leq m \leq \lambda + n, \\ 0 & \text{otherwise,} \end{cases}$$

then we obtain the notion of almost convergence method introduced by Lorentz [5].

(3°) Let $Q = \{q^{(\lambda)} : \lambda \in \Lambda\}$ be a familiy of sequences $q^{(\lambda)} = \{q_m^{(\lambda)}\}_{m \in \mathbb{N}_0}$ of nonnegative real numbers such that

$$Q_n^{(\lambda)} := q_0^{(\lambda)} + q_1^{(\lambda)} + \cdots + q_n^{(\lambda)} > 0 \qquad (n \in \mathbb{N}_0,\ \lambda \in \Lambda).$$

We define

$$a_{n,m}^{(\lambda)} = \begin{cases} \dfrac{q_{n-m}^{(\lambda)}}{Q_n^{(\lambda)}} & \text{if } m \leq n, \\ 0 & \text{if } m > n. \end{cases}$$

Then $\mathcal{A}$-summability method is called (N, Q)-summability method and in particular, if $q^{(\lambda)} = \{q_m\}_{m \in \mathbb{N}_0}$ is a fixed sequence of nonnegative real numbers satisfying $q_0 > 0$, then this reduces to the Nörlund summability method. Another special case of interest is the following:

Let $\Lambda \subseteq [0, \infty)$, $\beta > 0$ and

$$q_m^{(\lambda)} = C_m^{(\lambda+\beta-1)} \qquad (\lambda \in \Lambda, \ m \in \mathbb{N}_0),$$

where $\tau > -1$ and

$$C_0^{(\tau)} = 1, \quad C_m^{(\tau)} = \binom{m+\tau}{m} = \frac{(\tau+1)(\tau+2)\cdots(\tau+m)}{m!} \qquad (m \in \mathbb{N}).$$

In particular, if $\Lambda = \{0\}$, then we obtain the Cesàro summability of order β.

(4°) *Cesàro type* : Let $\Lambda \subseteq (0, \infty)$, $\beta > -1$ and define

$$a_{n,m}^{(\lambda)} = \begin{cases} C_{n-m}^{(\lambda-1)} C_m^{(\beta)} / C_n^{(\beta+\lambda)} & \text{if } m \leq n, \\ 0 & \text{if } m > n. \end{cases}$$

(5°) *Euler-Knopp-Bernstein type* : Let $\Lambda \subseteq [0, 1]$ and define

$$a_{n,m}^{(\lambda)} = \begin{cases} \binom{n}{m} \lambda^m (1-\lambda)^{n-m} & \text{if } m \leq n, \\ 0 & \text{if } m > n. \end{cases}$$

(6°) *Meyer-König-Vermes-Zeller type* : Let $\Lambda \subseteq [0, 1)$ and define

$$a_{n,m}^{(\lambda)} = \binom{n+m}{m} \lambda^m (1-\lambda)^{n+1}.$$

(7°) *Borel-Szász type* : Let $\Lambda \subseteq [0, \infty)$ and define

$$a_{n,m}^{(\lambda)} = \exp(-n\lambda) \frac{(n\lambda)^m}{m!}.$$

(8°) *Baskakov type* : Let $\Lambda \subseteq [0, \infty)$ and define

$$a_{n,m}^{(\lambda)} = \binom{n+m-1}{m} \lambda^m (1+\lambda)^{-n-m}.$$

Note that all the families $\mathcal{A}$ of the generic entories $a_{n,m}^{\lambda)}$ given in the above Examples (2°)-(8°) are stochastic and all the families $\mathcal{A}$ of the generic entories $a_{n,m}^{(\lambda)}$ given in the above Examples (4°)-(8°) are regular for any finite interval Λ.

3. Convergence rates

Let $F \in B(X, E)$ and let $\delta \geq 0$. Then we define

$$\omega_d(F, \delta) = \sup\{\|F(x) - F(y)\| : x, y \in X, d(x, y) \leq \delta\},$$

which is called the modulus of continuity of F with respect to d. Evidently, $\omega_d(F, \cdot)$ is a monotone increasing function on $[0, \infty)$ and

$$\omega_d(F, 0) = 0, \quad \omega_d(F, \delta) \leq 2 \sup\{\|F(x)\| : x \in X\} \qquad (\delta \geq 0).$$

Note that if X is bounded, then

$$\omega_d(F, \delta) = \omega_d(F, \delta(X)) \qquad (F \in B(X, E),\ \delta \geq \delta(X)),$$

where $\delta(X)$ denotes the diameter of X, and F is uniformly continuous on X if and only if

$$\lim_{\delta \to +0} \omega_d(F, \delta) = 0.$$

For $\beta > 0$, a function $F \in B(X, E)$ is said to satisfy a Lipschitz condition of order β with constant $M > 0$ with respect to d, or to belong to the class $Lip_d(\beta, M)$ if

$$\omega_d(F, \delta) \leq M\delta^\beta$$

for all $\delta \geq 0$. Also, we set

$$Lip_d\, \beta = \bigcup_{M>0} Lip_d(\beta, M),$$

which is called the Lipschitz class of order β with respect to d.

From now on, we suppose that there exist constants $C \geq 1$ and $K > 0$ such that

$$\omega_d(F, \xi\delta) \leq (C + K\xi)\omega_d(F, \delta) \tag{6}$$

for all $\delta, \xi \geq 0$ and all $F \in B(X, E)$.

Lemma 1. *Let Y be a finite set and $p \geq 1$. Let $\{\chi(x; \cdot) : x \in X\}$ be a family of scalar-valued functions on Y and let τ be a mapping from Y to X. Then for all $F \in BC(X, E), x \in X$ and all $\delta > 0$,*

$$\Big\|\sum_{k \in Y} \chi(x; k)(F(\tau(k)) - F(x))\Big\| \leq \Big(C \sum_{k \in Y} |\chi(x; k)| + Kc(x; p, \delta)\Big)\omega_d(F, \delta),$$

where

$$c(x;p,\delta) = \min\Big\{\delta^{-p}\sum_{k\in Y}|\chi(x;k)d^p(x,\tau(k))|,$$

$$\delta^{-1}\Big(\sum_{k\in Y}|\chi(x;k)|\Big)^{1-1/p}\Big(\sum_{k\in Y}|\chi(x;k)d^p(x,\tau(k))|\Big)^{1/p}\Big\}.$$

Proof. This follows from [15, Lemma 2.7].

Let Ω be a strictly increasing continuous, subadditive function on $[0,\infty)$ with $\Omega(0)=0$. Then we define

$$d_\Omega(x,y) = \Omega(d(x,y)) \qquad ((x,y)\in X\times X),$$

which becomes a metric function on $X\times X$. d_Ω is uniformly equivalent to d and

$$\omega_d(F,\delta) = \omega_{d_\Omega}(F,\Omega(\delta)) \tag{7}$$

for all $F\in B(X,E)$ and all $\delta\geq 0$ ([14, Lamma 2], cf. [9, Lemma 3]).

If $\chi_{m,\gamma}(x;k)\geq 0$ for all $m\in\mathbb{N}_0,\gamma\in\Gamma,k\in Y_{m,\gamma}$ and all $x\in X_0$, then $\mathfrak{A}$ is said to be positive. Also, if

$$\sum_{k\in Y_{m,\gamma}}\chi_{m,\gamma}(x;k) = 1$$

for all $m\in\mathbb{N}_0,\gamma\in\Gamma$ and all $x\in X_0$, then $\mathfrak{A}$ is said to be normal.

Now, let $K_{\alpha,\lambda,\gamma}$ be defined by (3) and for each $\alpha\in D$, $F\in BC(X,E)$ we define

$$E_\alpha(F) = \sup\{\|K_{\alpha,\lambda,\gamma}(F)(x)-F(x)\| : \lambda\in\Lambda,\gamma\in\Gamma,x\in X_0\}$$

and

$$\|F\|_{X_0} = \sup\{\|F(x)\| : x\in X_0\}.$$

Then $\mathfrak{K}$ is an equi-uniform $\mathcal{A}$-summation process on $BC(X,E)$ if and only if

$$\lim_\alpha E_\alpha(F) = 0$$

for every $F\in BC(X,E)$.

Let $p\geq 1$ be any fixed real number and let $\{\epsilon_\alpha\}_{\alpha\in D}$ be a net of positive real numbers.

Theorem 1. *For all $F\in BC(X,E)$ and all $\alpha\in D$,*

$$E_\alpha(F) \leq \|F\|_{X_0}\tau_\alpha + \tau_\alpha(p)\omega_{d_\Omega}(F,\Omega(\epsilon_\alpha\nu_\alpha(p))), \tag{8}$$

where

$$\tau_\alpha = \sup\Big\{\Big|\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} \sum_{k\in Y_{m,\gamma}} \chi_{m,\gamma}(x;k) - 1\Big| : \lambda\in\Lambda, \gamma\in\Gamma, x\in X_0\Big\},$$

$$\tau_\alpha(p) = \sup\{C g_{\alpha,\lambda,\gamma}(x) + K\min\{\epsilon_\alpha^{-p},\ \epsilon_\alpha^{-1} g_{\alpha,\lambda,\gamma}(x)^{1-1/p}\} \\ : \lambda\in\Lambda,\ \gamma\in\Gamma,\ x\in X_0\}$$

and

$$\nu_\alpha(p) = \Big(\sup\Big\{\sum_{m=0}^{\infty} |a_{\alpha,m}^{(\lambda)}| \sum_{k\in Y_{m,\gamma}} |\chi_{m,\gamma}(x;k) d^p(x,\xi_{m,\gamma}(k))| \\ : \lambda\in\Lambda,\ \gamma\in\Gamma,\ x\in X_0\Big\}\Big)^{1/p}.$$

Proof. In view of Lemma 1, we carry out the process as in the proof of [15, Theorem 4.1] and use the equality (7).

Corollary 1. *For all* $F\in Lip_{d_\Omega}(\beta, M)$ *and all* $\alpha\in D$,

$$E_\alpha(F) \le \|F\|_{X_0}\tau_\alpha + M\tau_\alpha(p)\Omega^\beta(\epsilon_\alpha\nu_\alpha(p)).$$

Theorem 2. *If* $\mathfrak{A}$ *is positive and normal and if* $\mathcal{A}$ *is stochastic, then for all* $F\in BC(X,E)$ *and all* $\alpha\in D$,

$$E_\alpha(F) \le (C + K\min\{\epsilon_\alpha^{-1},\ \epsilon_\alpha^{-p}\})\omega_{d_\Omega}(F, \Omega(\epsilon_\alpha\nu_\alpha(p))). \tag{9}$$

Proof. Since $\tau_\alpha = 0$ and $g_{\alpha,\lambda,\gamma}(x) = 1$ for all $\alpha\in D, \lambda\in\Lambda, \gamma\in\Gamma$ and all $x\in X_0$, (9) immediately follows from (8).

Corollary 2. *Suppose that* $\mathfrak{A}$ *is positive and normal and that* $\mathcal{A}$ *is stochastic. Then for all* $F\in Lip_{d_\Omega}(\beta, M)$ *and all* $\alpha\in D$,

$$E_\alpha(F) \le M(C + K\min\{\epsilon_\alpha^{-1},\ \epsilon_\alpha^{-p}\})\Omega^\beta(\epsilon_\alpha\nu_\alpha(p)).$$

Let E_0 be a subset of E. Let $\mathfrak{T} = \{T(x) : x\in X\}$ be a family of mappings from E_0 to E such that for each $f\in E_0$, the mapping $x\mapsto T(x)(f)$ is strongly continuous and bounded on X and let $L_{m,\gamma}$ denote the restriction of $K_{m,\gamma}$ to the set $\{T(\cdot)(f) : f\in E_0\}$, i.e.,

$$L_{m,\gamma}(x)(f) = \sum_{k\in Y_{m,\gamma}} \chi_{m,\gamma}(x;k) T(\xi_{m,\gamma}(k))(f) \tag{10}$$

$$(m\in\mathbb{N}_0,\ \gamma\in\Gamma,\ f\in E_0,\ x\in X).$$

We define

$$L_{\alpha,\lambda,\gamma}(x)(f) = \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} L_{m,\gamma}(x)(f) \qquad (f\in E_0), \tag{11}$$

which converges in E because of (1). Then the family $\mathfrak{L} = \{L_{m,\gamma}(x) : m \in \mathbb{N}_0, \gamma \in \Gamma, x \in X\}$ is called an equi-uniform $\mathfrak{T}$-$\mathcal{A}$-summation process on E_0 if for every $f \in E_0$,

(12)
$$\lim_{\alpha} \|L_{\alpha,\lambda,\gamma}(x)(f) - T(x)(f)\| = 0 \text{ uniformly in } \lambda \in \Lambda, \gamma \in \Gamma, x \in X_0.$$

Concerning the rate of convergence behavior (12), we define

$$\omega_{d,\mathfrak{T}}(f,\delta) = \sup\{\|T(x)(f) - T(y)(f)\| : x, y \in X, d(x,y) \leq \delta\}$$
$$(f \in E_0,\ \delta \geq 0),$$

which is called the modulus of continuity of f associated with $\mathfrak{T}$ with respect to d, and

$$e_\alpha(f) = \sup\{\|L_{\alpha,\lambda,\gamma}(x)(f) - T(x)(f)\| : \lambda \in \Lambda, \gamma \in \Gamma, x \in X_0\}.$$

Evidently, $\mathfrak{L}$ is an equi-uniform $\mathfrak{T}$-$\mathcal{A}$-summation process on E_0 if and only if

$$\lim_{\alpha} e_\alpha(f) = 0$$

for every $f \in E_0$.

For $\beta > 0$, an element $f \in E_0$ is said to satisfy a Lipschitz condition of order β with constant $M > 0$ associated with $\mathfrak{T}$ with respect to d, or to belong to the class $Lip_{d,\mathfrak{T}}(\beta, M)$ if

$$\omega_{d,\mathfrak{T}}(f,\delta) \leq M\delta^\beta$$

for all $\delta \geq 0$. Also, we set

$$Lip_{d,\mathfrak{T}}\,\beta = \bigcup_{M>0} Lip_{d,\mathfrak{T}}(\beta, M),$$

which is called the Lipschitz class of order β associated with $\mathfrak{T}$ with respect to d.

Let $\tau_\alpha, \tau_\alpha(p)$ and $\nu_\alpha(p)$ be as in Theorem 1. Then we have the following result which estimates the rate of convergence of the equi-uniform $\mathfrak{T}$-$\mathcal{A}$-summation process $\mathfrak{L}$ on E_0.

Theorem 3. *For all $f \in E_0$ and all $\alpha \in D$,*

$$e_\alpha(f) \leq \|T(\cdot)(f)\|_{X_0}\tau_\alpha + \tau_\alpha(p)\omega_{d_\Omega,\mathfrak{T}}(f, \Omega(\epsilon_\alpha\nu_\alpha(p))). \tag{13}$$

Proof. Since

$$\omega_{d_\Omega,\mathfrak{T}}(f,\delta) = \omega_{d_\Omega}(T(\cdot)(f),\delta), \quad e_\alpha(f) = E_\alpha(T(\cdot)(f)) \tag{14}$$
$$(f \in E_0,\ \delta \geq 0,\ \alpha \in D),$$

taking $F(\cdot) = T(\cdot)(f)$ in (8) we have the desired inequality (13).

Corollary 3. *For all $f \in Lip_{d_\Omega,\mathfrak{T}}(\beta, M)$ and all $\alpha \in D$,*

$$e_\alpha(f) \le \|T(\cdot)(f)\|_{X_0}\tau_\alpha + M\tau_\alpha(p)\Omega^\beta(\epsilon_\alpha\nu_\alpha(p)).$$

Theorem 4. *If $\mathfrak{A}$ is positive and normal and if $\mathcal{A}$ is stochastic, then for all $f \in E_0$ and all $\alpha \in D$,*

$$e_\alpha(f) \le (C + K\min\{\epsilon_\alpha^{-1},\ \epsilon_\alpha^{-p}\})\omega_{d_\Omega,\mathfrak{T}}(f, \Omega(\epsilon_\alpha\nu_\alpha(p))). \tag{15}$$

Proof. In view of (14), the inequality (15) immediately follows from (9).

Corollary 4. *Suppose that $\mathfrak{A}$ is positive and normal and that $\mathcal{A}$ is stochastic. Then for all $f \in Lip_{d_\Omega,\mathfrak{T}}(\beta, M)$ and all $\alpha \in D$,*

$$e_\alpha(f) \le M(C + K\min\{\epsilon_\alpha^{-1},\ \epsilon_\alpha^{-p}\})\Omega^\beta(\epsilon_\alpha\nu_\alpha(p)).$$

Let Φ be a nonnegative real-valued function on $X \times X$ and suppose that there exists a constant $\kappa > 0$ such that

$$d^p(x,y) \le \kappa\Phi(x,y) \tag{16}$$

for all $(x,y) \in X_0 \times X$. We define

$$\mu_\alpha(\Phi;p) = \Big(\sup\Big\{\sum_{m=0}^{\infty}|a_{\alpha,m}^{(\lambda)}| \sum_{k\in Y_{m,\gamma}} |\chi_{m,\gamma}(x;k)|\Phi(x,\xi_{m,\gamma}(k)) \\ : \lambda \in \Lambda,\ \gamma \in \Gamma,\ x \in X_0\Big\}\Big)^{1/p}.$$

Then we have

$$\nu_\alpha(p) \le \kappa^{1/p}\mu_\alpha(\Phi;p)$$

for all $\alpha \in D$. Therefore, all the above results hold with $\kappa^{-1/p}\epsilon_\alpha$ instead of ϵ_α and with $\mu_\alpha(\Phi;p)$ instead of $\nu_\alpha(p)$. In particular, in view of Theorems 2 and 4, we obtain the following result which can be more convenient for later applications.

Theorem 5. *Suppose that $\mathfrak{A}$ is positive and normal and that $\mathcal{A}$ is stochastic. Then we have:*

(a) For all $F \in BC(X, E)$ and all $\alpha \in D$,

$$E_\alpha(F) \le (C + K\min\{\kappa^{1/p}\epsilon_\alpha^{-1},\ \kappa\epsilon_\alpha^{-p}\})\omega_{d_\Omega}(F, \Omega(\epsilon_\alpha\mu_\alpha(\Phi;p))).$$

(b) For all $f \in E_0$ and all $\alpha \in D$,

$$e_\alpha(f) \le (C + K\min\{\kappa^{1/p}\epsilon_\alpha^{-1},\ \kappa\epsilon_\alpha^{-p}\})\omega_{d_\Omega,\mathfrak{T}}(f, \Omega(\epsilon_\alpha\mu_\alpha(\Phi;p))).$$

Corollary 5. *Let $\mathfrak{A}$ and $\mathcal{A}$ be as in Theorem 5.*

(a) For all $F \in Lip_{d_\Omega}(\beta, M)$ and all $\alpha \in D$,

$$E_\alpha(F) \leq M(C + K\min\{\kappa^{1/p}\epsilon_\alpha^{-1},\ \kappa\epsilon_\alpha^{-p}\})\Omega^\beta(\epsilon_\alpha\mu_\alpha(\Phi; p)).$$

(b) For all $f \in Lip_{d_\Omega, \mathfrak{X}}(\beta, M)$ and all $\alpha \in D$,

$$e_\alpha(f) \leq M(C + K\min\{\kappa^{1/p}\epsilon_\alpha^{-1},\ \kappa\epsilon_\alpha^{-p}\})\Omega^\beta(\epsilon_\alpha\mu_\alpha(\Phi; p)).$$

4. Summation process of Bernstein type operators

Let X be a convex subset of a metric linear space Z with the translation invariant metric function d, i.e.,

$$d(x, y) = d(x + z, y + z)$$

for all $x, y, z \in Z$ and with $d(\cdot, 0)$ being starshaped, i.e.,

$$d(\beta x, 0) \leq \beta d(x, 0)$$

for all $x \in Z$ and all $\beta \in [0, 1]$. Then, in view of [14, Lemma 1 (b)] (cf. [15, Lemma 2.4 (b)], [10, Lemma 3 (ii)]), all the results obtained in the preceding section hold with $C = K = 1$. Here we restrict ourselves to the following situation:

Let $1 \leq q \leq \infty$ be fixed and let X be a convex subset of the r-dimensional Euclidean space $\mathbb{R}^r$ with the usual metric

$$d(x, y) = d^{(q)}(x, y) := \begin{cases} \left(\sum_{i=1}^r |x_i - y_i|^q\right)^{1/q} & (1 \leq q < \infty) \\ \max\{|x_i - y_i| : 1 \leq i \leq r\} & (q = \infty), \end{cases}$$

where $x = (x_1, x_2, \ldots, x_r), y = (y_1, y_2, \ldots, y_r) \in \mathbb{R}^r$. For $i = 1, 2, \ldots, r$, p_i denotes the ith coordinate function defined by $p_i(x) = x_i$ for all $x = (x_1, x_2, \ldots, x_r) \in \mathbb{R}^r$. Then we have

$$(17)\quad (d^{(q)}(x, y))^p \leq c(p, q, r)\sum_{i=1}^r |p_i(x) - p_i(y)|^p \qquad (x, y \in \mathbb{R}^r,\ p > 0),$$

where

$$c(p, q, r) = \begin{cases} r^{p/q} & (1 \leq q < \infty, q \neq p) \\ 1 & (1 \leq q < \infty, q = p) \\ 1 & (q = \infty). \end{cases}$$

Therefore, (16) holds with

$$\kappa = c(p,q,r), \quad \Phi(x,y) = \sum_{i=1}^{r} |p_i(x) - p_i(y)|^p. \tag{18}$$

Let

$$X = [0,\infty)^r := \{x = (x_1, x_2, \dots, x_r) \in \mathbb{R}^r : x_i \geq 0, \ i = 1,2,\dots,r\}$$

be the region of the first hyperquadrant and let

$$n_{m,i} : \Gamma \to \mathbb{N}, \quad b_{m,i} : \Gamma \to (0,\infty) \qquad (m \in \mathbb{N}_0, \ i = 1,2,\dots,r),$$

where $\mathbb{N}$ denotes the set of all positive integers.

Let X_0 be a subset of $\mathbb{I}_r$, where

$$\mathbb{I}_r := \{x = (x_1, x_2, \dots, x_r) \in X : 0 \leq x_i \leq 1, \ i = 1,2,\dots,r\}$$

is the unit r-cube and

$$I_{m,\gamma} := \{k = (k_1, k_2, \dots, k_r) \in \mathbb{N}_0^r : 0 \leq k_i \leq n_{m,i}(\gamma), \ 1 \leq i \leq r\}$$
$$(m \in \mathbb{N}_0, \ \gamma \in \Gamma).$$

Then we define the corresponding interpolation type operators (2) and (10) by

$$B_{m,\gamma}(F)(x) = \sum_{k \in I_{m,\gamma}} \prod_{i=1}^{r} \binom{n_{m,i}(\gamma)}{k_i} x_i^{k_i} (1 - x_i)^{n_{m,i}(\gamma) - k_i} \tag{19}$$
$$\times F(b_{m,1}(\gamma)k_1, b_{m,2}(\gamma)k_2, \dots, b_{m,r}(\gamma)k_r)$$
$$(m \in \mathbb{N}_0, \ \gamma \in \Gamma, \ F \in BC(X,E), \ x \in X)$$

and

$$C_{m,\gamma}(x)(f) = \sum_{k \in I_{m,\gamma}} \prod_{i=1}^{r} \binom{n_{m,i}(\gamma)}{k_i} x_i^{k_i} (1 - x_i)^{n_{m,i}(\gamma) - k_i} \tag{20}$$
$$\times T(b_{m,1}(\gamma)k_1, b_{m,2}(\gamma)k_2, \dots, b_{m,r}(\gamma)k_r)(f)$$
$$(m \in \mathbb{N}_0, \ \gamma \in \Gamma, \ f \in E_0, \ x \in X),$$

respectively (cf. [12]). These are called the Bernstein type operators associated with the unit r-cube $\mathbb{I}_r$.

Now, we assume that $\mathcal{A}$ is stochastic. Let $\{\epsilon_\alpha\}_{\alpha \in D}$ be a net of positive real numbers and we define

$$c_\alpha(q,r) = 1 + \min\left\{\frac{\sqrt{c(q,r)}}{\epsilon_\alpha}, \ \frac{c(q,r)}{\epsilon_\alpha^2}\right\},$$

where

$$c(q,r) = \begin{cases} r^{2/q} & (1 \le q < \infty,\ q \ne 2) \\ 1 & (q = 2, \infty). \end{cases}$$

We take

(21) $$K_{m,\gamma}(F) = B_{m,\gamma}(F) \qquad (m \in \mathbb{N}_0,\ \gamma \in \Gamma,\ F \in BC(X,E))$$

and

(22) $$L_{m,\gamma}(\cdot)(f) = C_{m,\gamma}(\cdot)(f) \qquad (m \in \mathbb{N}_0,\ \gamma \in \Gamma,\ f \in E_0).$$

Theorem 6. *(a) For all $F \in BC(X,E)$ and all $\alpha \in D$,*

(23) $$E_\alpha(F) \le c_\alpha(q,r)\omega_{d^{(q)}{}_\Omega}(F, \Omega(\epsilon_\alpha \zeta_\alpha)).$$

(b) For all $f \in E_0$ and all $\alpha \in D$,

(24) $$e_\alpha(f) \le c_\alpha(q,r)\omega_{d^{(q)}{}_\Omega, \mathfrak{T}}(f, \Omega(\epsilon_\alpha \zeta_\alpha)).$$

Here

$$\zeta_\alpha = \left(\sup\left\{\sum_{i=1}^{r}\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} \zeta_{m,i}(\gamma,x) : \lambda \in \Lambda, \gamma \in \Gamma, x \in X_0\right\}\right)^{1/2}$$

and

$$\zeta_{m,i}(\gamma,x) = (n_{m,i}(\gamma)b_{m.i}(\gamma) - 1)^2 p_i^2(x) + n_{m,i}(\gamma)b_{m,i}^2(\gamma)(p_i(x) - p_i^2(x)).$$

(c) If

(25) $$n_{m,i}(\gamma)b_{m,i}(\gamma) = 1 \qquad (m \in \mathbb{N}_0,\ i = 1, 2, \ldots, r)$$

for all $\gamma \in \Gamma$, then (23) and (24) hold with

$$\zeta_\alpha = \left(\sup\left\{\sum_{i=1}^{r}(p_i(x) - p_i^2(x)) \sum_{m=0}^{\infty} \frac{a_{\alpha,m}^{(\lambda)}}{n_{m,i}(\gamma)} : \lambda \in \Lambda, \gamma \in \Gamma, x \in X_0\right\}\right)^{1/2}.$$

Proof. We define

$$\chi_{m,\gamma}(x;k) = \prod_{j=1}^{r} \binom{n_{m,j}(\gamma)}{k_j} x_j^{k_j}(1 - x_j)^{n_{m,j}(\gamma) - k_j} \qquad (x \in X,\ k \in I_{m,\gamma})$$

and

$$\xi_{m,\gamma}(k) = (b_{m,1}(\gamma)k_1, b_{m,2}(\gamma)k_2, \ldots, b_{m,r}(\gamma)k_r) \qquad (k \in I_{m,\gamma}).$$

Then $\mathfrak{A}$ is positive and normal. Furthermore, we have

$$\sum_{k \in I_{m,\gamma}} \chi_{m,\gamma}(x;k)|p_i(x) - p_i(\xi_{m,\gamma}(k))|^2 = \zeta_{m,i}(\gamma,x) \qquad (i = 1, 2, \ldots, r)$$

for all $m \in \mathbb{N}_0, \gamma \in \Gamma$ and all $x \in X_0$. Therefore, in view of (17) and (18), the desired result follows from Theorem 5.

Corollary 6. *(a) For all $F \in Lip_{d^{(q)}{}_{\Omega}}(\beta, M)$ and all $\alpha \in D$,*

$$E_\alpha(F) \leq Mc_\alpha(q,r)\Omega^\beta(\epsilon_\alpha\zeta_\alpha).$$

(b) For all $f \in Lip_{d^{(q)}{}_{\Omega},\mathfrak{T}}(\beta, M)$ and all $\alpha \in D$,

$$e_\alpha(f) \leq Mc_\alpha(q,r)\Omega^\beta(\epsilon_\alpha\zeta_\alpha).$$

We assume that (25) holds for all $\gamma \in \Gamma$. Then we can reduce (19) and (20) to

$$(26)\quad B_{m,\gamma}(F)(x) = \sum_{k_1=0}^{n_{m,1}(\gamma)} \sum_{k_2=0}^{n_{m,2}(\gamma)} \cdots \sum_{k_r=0}^{n_{m,r}(\gamma)} F\Big(\frac{k_1}{n_{m,1}(\gamma)}, \dots, \frac{k_r}{n_{m,r}(\gamma)}\Big) \times \prod_{i=1}^{r} \binom{n_{m,i}(\gamma)}{k_i} x_i^{k_i}(1-x_i)^{n_{m,i}(\gamma)-k_i}$$

$$(m \in \mathbb{N}_0,\ \gamma \in \Gamma,\ F \in C(\mathbb{I}_r, E),\ x \in \mathbb{I}_r)$$

and

$$C_{m,\gamma}(x)(f) = \sum_{k_1=0}^{n_{m,1}(\gamma)} \sum_{k_2=0}^{n_{m,2}(\gamma)} \cdots \sum_{k_r=0}^{n_{m,r}(\gamma)} T\Big(\frac{k_1}{n_{m,1}(\gamma)}, \dots, \frac{k_r}{n_{m,r}(\gamma)}\Big)(f) \times \prod_{i=1}^{r} \binom{n_{m,i}(\gamma)}{k_i} x_i^{k_i}(1-x_i)^{n_{m,i}(\gamma)-k_i}$$

$$(m \in \mathbb{N}_0,\ \gamma \in \Gamma,\ f \in E_0,\ x \in \mathbb{I}_r),$$

respectively (cf. [12], [14]).

Let $\{n_m\}_{m \in \mathbb{N}_0}$ be a strictly monotone increasing sequence of positive integers and let $v : \Gamma \to [0, \infty)$. We define

$$n_{m.i}(\gamma) = n_m + [v(\gamma)] + i \qquad (m \in \mathbb{N}_0,\ \gamma \in \Gamma,\ i = 1, 2, \dots, r)$$

and

$$b_{m,i}(\gamma) = \frac{1}{n_m + [v(\gamma)] + i} \qquad (m \in \mathbb{N}_0,\ \gamma \in \Gamma,\ i = 1, 2, \dots, r),$$

where $[v(\gamma)]$ denotes the largest integer not exceeding $v(\gamma)$. Then, in view of Theorem 6 (c), for all $F \in C(\mathbb{I}_r, E), f \in E_0$ and all $\alpha \in D$, (23) and (24) hold with

$$\zeta_\alpha = \Big(\sup\Big\{\sum_{i=1}^{r}(p_i(x) - p_i^2(x)) \sum_{m=0}^{\infty} \frac{a_{\alpha,m}^{(\lambda)}}{n_m + [v(\gamma)] + i}$$

$$: \lambda \in \varLambda, \ \gamma \in \varGamma, \ x \in X_0 \Big\}\Big)^{1/2}$$

$$\leq \Big(\sup\Big\{\sum_{i=1}^{r}(p_i(x) - p_i^2(x)) \sum_{m=0}^{\infty} \frac{a_{\alpha,m}^{(\lambda)}}{n_m + i} : \lambda \in \varLambda, \ x \in X_0\Big\}\Big)^{1/2}.$$

Let $\{\nu_{m,i}\}_{m\in\mathbb{N}_0}, i = 1, 2, \ldots, r$, be strictly monotone increasing sequences of positive integers. We define

$$n_{m,i}(\gamma) = \nu_{m,i}, \quad b_{m,i}(\gamma) = \frac{1}{\nu_{m,i}} \qquad (m \in \mathbb{N}_0, \ \gamma \in \varGamma, \ i = 1, 2, \ldots, r).$$

Then (26) reduces to the r-dimensional Bernstein polynomial operators on $C(\mathbb{I}_r, E)$ for $E = \mathbb{R}$ ([6], cf. [2], [3]), and for all $F \in C(\mathbb{I}_r, E), f \in E_0$ and all $\alpha \in D$, (23) and (24) hold with

$$\zeta_\alpha = \Big(\sup\Big\{\sum_{i=1}^{r}(p_i(x) - p_i^2(x)) \sum_{m=0}^{\infty} \frac{a_{\alpha,m}^{(\lambda)}}{\nu_{m,i}} : \lambda \in \varLambda, \ x \in X_0\Big\}\Big)^{1/2}.$$

Next, let X_0 be a subset of $\varDelta_r$, where

$$\varDelta_r := \Big\{x = (x_1, x_2, \ldots, x_r) \in \mathbb{R}^r : x_i \geq 0, \ 1 \leq i \leq r, \ \sum_{i=1}^{r} x_i \leq 1\Big\}$$

is the standard r-simplex. Let

$$n_m : \varGamma \to \mathbb{N}, \quad b_{m,i} : \varGamma \to (0, \infty) \qquad (m \in \mathbb{N}_0, \ i = 1, 2, \ldots, r)$$

and

$$J_{m,\gamma} := \{k = (k_1, k_2, \ldots, k_r) \in \mathbb{N}_0^r : k_1 + k_2 + \cdots + k_r \leq n_m(\gamma)\}$$
$$(m \in \mathbb{N}_0, \ \gamma \in \varGamma).$$

Now we define the corresponding interpolation type operators (2) and (10) by

$$(27) \quad B_{m,\gamma}(F)(x) = \sum_{k\in J_{m,\gamma}} \binom{n_m(\gamma)}{k} \prod_{i=1}^{r} x_i^{k_i} \Big(1 - \sum_{j=1}^{r} x_j\Big)^{n_m(\gamma) - \sum_{j=1}^{r} k_j}$$
$$\times F(b_{m,1}(\gamma)k_1, b_{m,2}(\gamma)k_2, \ldots, b_{m,r}(\gamma)k_r)$$
$$(m \in \mathbb{N}_0, \ \gamma \in \varGamma, \ F \in BC(X, E), \ x \in X)$$

and

$$(28) \quad C_{m,\gamma}(x)(f) = \sum_{k\in J_{m,\gamma}} \binom{n_m(\gamma)}{k} \prod_{i=1}^{r} x_i^{k_i} \Big(1 - \sum_{j=1}^{r} x_j\Big)^{n_m(\gamma) - \sum_{j=1}^{r} k_j}$$
$$\times T(b_{m,1}(\gamma)k_1, b_{m,2}(\gamma)k_2, \ldots, b_{m,r}(\gamma)k_r)(f)$$

$$(m \in \mathbb{N}_0,\ \gamma \in \Gamma,\ f \in E_0,\ x \in X),$$

where

$$\binom{n_m(\gamma)}{k} = \frac{n_m(\gamma)!}{k_1!k_2!\cdots k_r!(n_m(\gamma) - k_1 - k_2 - \cdots - k_r)!},$$

respectively (cf. [12]). These are called the Bernstein type operators associated with the standard r-simplex Δ_r. Let $K_{m,\gamma}$ and $L_{m,\gamma}$ be as in (21) with (27) and (22) with (28), respectively. Then the similar argument as in the proof of Theorem 6 yields the following result.

Theorem 7. *(a) For all $F \in BC(X, E)$ and all $\alpha \in D$,*

$$E_\alpha(F) \le c_\alpha(q, r)\omega_{d^{(q)}{}_\Omega}(F, \Omega(\epsilon_\alpha\delta_\alpha)). \tag{29}$$

(b) For all $f \in E_0$ and all $\alpha \in D$,

$$e_\alpha(f) \le c_\alpha(q, r)\omega_{d^{(q)}{}_\Omega,\mathfrak{I}}(f, \Omega(\epsilon_\alpha\delta_\alpha)). \tag{30}$$

Here

$$\delta_\alpha = \left(\sup\left\{\sum_{i=1}^{r}\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)}\delta_{m,i}(\gamma, x) : \lambda \in \Lambda, \gamma \in \Gamma, x \in X_0\right\}\right)^{1/2}$$

and

$$\delta_{m,i}(\gamma, x) = (n_m(\gamma)b_{m,i}(\gamma) - 1)^2 p_i^2(x) + n_m(\gamma)b_{m,i}^2(\gamma)(p_i(x) - p_i^2(x)).$$

(c) If

$$n_m(\gamma)b_{m,i}(\gamma) = 1 \qquad (m \in \mathbb{N}_0,\ i = 1, 2, \dots, r) \tag{31}$$

for all $\gamma \in \Gamma$, then (29) and (30) hold with

$$\delta_\alpha = \left(\sup\left\{\sum_{i=1}^{r}(p_i(x) - p_i^2(x))\sum_{m=0}^{\infty}\frac{a_{\alpha,m}^{(\lambda)}}{n_m(\gamma)} : \lambda \in \Lambda, \gamma \in \Gamma, x \in X_0\right\}\right)^{1/2}.$$

Corollary 7. *(a) For all $F \in Lip_{d^{(q)}{}_\Omega}(\beta, M)$ and all $\alpha \in D$,*

$$E_\alpha(F) \le Mc_\alpha(q, r)\Omega^\beta(\epsilon_\alpha\delta_\alpha).$$

(b) For all $f \in Lip_{d^{(q)}{}_\Omega,\mathfrak{I}}(\beta, M)$ and all $\alpha \in D$,

$$e_\alpha(f) \le Mc_\alpha(q, r)\Omega^\beta(\epsilon_\alpha\delta_\alpha).$$

We suppose that (31) holds for all $\gamma \in \Gamma$. Then we can reduce (27) and (28) to

$$(32) \quad B_{m,\gamma}(F)(x) = \sum_{k \in J_{m,\gamma}} \binom{n_m(\gamma)}{k} \prod_{i=1}^{r} x_i^{k_i} \Big(1 - \sum_{j=1}^{r} x_j\Big)^{n_m(\gamma) - \sum_{j=1}^{r} k_j} \times F\Big(\frac{k_1}{n_m(\gamma)}, \frac{k_2}{n_m(\gamma)}, \dots, \frac{k_r}{n_m(\gamma)}\Big)$$

$$(m \in \mathbb{N}_0,\ \gamma \in \Gamma,\ F \in C(\Delta_r, E),\ x \in \Delta_r)$$

and

$$C_{m,\gamma}(x)(f) = \sum_{k \in J_{m,\gamma}} \binom{n_m(\gamma)}{k} \prod_{i=1}^{r} x_i^{k_i} \Big(1 - \sum_{j=1}^{r} x_j\Big)^{n_m(\gamma) - \sum_{j=1}^{r} k_j} \times T\Big(\frac{k_1}{n_m(\gamma)}, \frac{k_2}{n_m(\gamma)}, \dots, \frac{k_r}{n_m(\gamma)}\Big)(f)$$

$$(m \in \mathbb{N}_0,\ \gamma \in \Gamma,\ f \in E_0,\ x \in \Delta_r),$$

respectively (cf. [12]; [14]).

Let $\{\nu_m\}_{m \in \mathbb{N}_0}$ be a strictly monotone increasing sequence of positive integers and let $v : \Gamma \to [0, \infty)$. We define

$$n_m(\gamma) = \nu_m + [v(\gamma)] \qquad (m \in \mathbb{N}_0,\ \gamma \in \Gamma)$$

and

$$b_{m,i}(\gamma) = \frac{1}{\nu_m + [v(\gamma)]} \qquad (m \in \mathbb{N}_0,\ \gamma \in \Gamma,\ i = 1, 2, \dots, r).$$

Then, in view of Theorem 7 (c), for all $F \in C(\Delta_r, E)$, $f \in E_0$ and all $\alpha \in D$, (29) and (30) hold with

$$\delta_\alpha = \Big(\sup\Big\{\sum_{i=1}^{r} (p_i(x) - p_i^2(x)) \sum_{m=0}^{\infty} \frac{a_{\alpha,m}^{(\lambda)}}{\nu_m + [v(\gamma)]} : \lambda \in \Lambda, \gamma \in \Gamma, x \in X_0\Big\}\Big)^{1/2}.$$

Also, we define

$$n_m(\gamma) = \nu_m, \quad b_{m,i}(\gamma) = \frac{1}{\nu_m} \qquad (m \in \mathbb{N}_0,\ \gamma \in \Gamma,\ i = 1, 2, \dots, r).$$

Then (32) reduces to the r-dimensional Bernstein polynomial operators on $C(\Delta_r, E)$ for $E = \mathbb{R}$ (cf. [6]), and for all $F \in C(\Delta_r, E)$, $f \in E_0$ and all $\alpha \in D$, (29) and (30) hold with

$$\delta_\alpha = \Big(\sup\Big\{\sum_{i=1}^{r} (p_i(x) - p_i^2(x)) \sum_{m=0}^{\infty} \frac{a_{\alpha,m}^{(\lambda)}}{\nu_m} : \lambda \in \Lambda, x \in X_0\Big\}\Big)^{1/2}.$$

5. Summation process of Hermite-Fejér type operators

Let $X = \mathbb{R}^r$ and let X_0 be a subset of $X_r := [-1, 1]^r$. Let

$$n_{m,i} : \Gamma \to \mathbb{N}, \quad b_{m,i} : \Gamma \to \mathbb{R} \qquad (m \in \mathbb{N}_0,\ i = 1, 2, \ldots, r)$$

and

$$N_{m,\gamma} := \{k = (k_1, k_2, \ldots, k_r) \in \mathbb{N}^r : 1 \leq k_i \leq n_{m,i}(\gamma), 1 \leq i \leq r\}.$$

Let $Q_n(t) = \cos(n \arccos t)$ be the Chebyshev polynomial of degree n and let $t_{n,j}$, $j = 1, 2, \ldots, n$, be zeros of $Q_n(t)$, i.e.,

$$t_{n,j} = \cos\left(\frac{2j-1}{2n}\pi\right) \qquad (j = 1, 2, \ldots, n).$$

Then we define the corresponding interpolation type operators (2) and (10) by

$$(33) \quad H_{m,\gamma}(F)(x) = \sum_{k \in N_{m,\gamma}} F(b_{m,1}(\gamma) t_{n_{m,1}(\gamma),k_1}, \ldots, b_{m,r}(\gamma) t_{n_{m,r}(\gamma),k_r})$$

$$\times \prod_{i=1}^{r} (1 - x_i t_{n_{m,i}(\gamma),k_i}) \left\{ \frac{Q_{n_{m,i}(\gamma)}(x_i)}{n_{m,i}(\gamma)(x_i - t_{n_{m,i}(\gamma),k_i})} \right\}^2$$

$$(m \in \mathbb{N}_0,\ \gamma \in \Gamma,\ F \in BC(X, E),\ x \in X)$$

and

(34)

$$G_{m,\gamma}(x)(f) = \sum_{k \in N_{m,\gamma}} T(b_{m,1}(\gamma) t_{n_{m,1}(\gamma),k_1}, \ldots, b_{m,r}(\gamma) t_{n_{m,r}(\gamma),k_r})(f)$$

$$\times \prod_{i=1}^{r} (1 - x_i t_{n_{m,i}(\gamma),k_i}) \left\{ \frac{Q_{n_{m,i}(\gamma)}(x_i)}{n_{m,i}(\gamma)(x_i - t_{n_{m,i}(\gamma),k_i})} \right\}^2$$

$$(m \in \mathbb{N},\ \gamma \in \Gamma,\ f \in E_0,\ x \in X),$$

respectively (cf. [12]). These are called the Hermite-Fejér type operators. We take

$$K_{m,\gamma}(F) = H_{m,\gamma}(F) \qquad (m \in \mathbb{N}_0,\ \gamma \in \Gamma,\ F \in BC(X, E)),$$

$$L_{m,\gamma}(\cdot)(f) = G_{m,\gamma}(\cdot)(f) \qquad (m \in \mathbb{N}_0,\ \gamma \in \Gamma,\ f \in E_0)$$

and suppose that $\mathcal{A}$ is stochastic. Then the similar argument as in the proof of Theorem 6 establishes the following result.

Theorem 8. *(a) For all $F \in BC(X, E)$ and all $\alpha \in D$,*

$$E_\alpha(F) \leq c_\alpha(q, r)\omega_{d^{(q)}_\Omega}(F, \Omega(\epsilon_\alpha \eta_\alpha)). \tag{35}$$

(b) For all $f \in E_0$ and all $\alpha \in D$,

$$e_\alpha(f) \leq c_\alpha(q, r)\omega_{d^{(q)}_\Omega, \mathfrak{T}}(f, \Omega(\epsilon_\alpha \eta_\alpha)). \tag{36}$$

Here

$$\eta_\alpha = \left(\sup\left\{\sum_{i=1}^{r} \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} \eta_{m,i}(\gamma, x) : \lambda \in \Lambda, \gamma \in \Gamma, x \in X_0\right\}\right)^{1/2},$$

$$\eta_{m,i}(\gamma, x) = \frac{Q^2_{n_{m,i}(\gamma)}(x_i)}{n_{m,i}(\gamma)} - 2x_i(b_{m,i}(\gamma) - 1) \sum_{k_i=1}^{n_{m,i}(\gamma)} t_{n_{m,i}(\gamma),k_i} \chi_{n_{m,i}(\gamma)}(x_i; k_i)$$

$$+ (b^2_{m,i}(\gamma) - 1) \sum_{k_i=1}^{n_{m,i}(\gamma)} {t_{n_{m,i}(\gamma),k_i}}^2 \chi_{n_{m,i}(\gamma)}(x_i; k_i)$$

and

$$\chi_{n_{m,i}(\gamma)}(x_i; k_i) = (1 - x_i t_{n_{m,i}(\gamma),k_i})\left\{\frac{Q_{n_{m,i}(\gamma)}(x_i)}{n_{m,i}(\gamma)(x_i - t_{n_{m,i}(\gamma),k_i})}\right\}^2.$$

(c) If

$$b_{m,i}(\gamma) = 1 \qquad (m \in \mathbb{N}_0,\ i = 1, 2, \ldots, r) \tag{37}$$

for all $\gamma \in \Gamma$, then (35) and (36) hold with

$$\eta_\alpha = \left(\sup\left\{\sum_{i=1}^{r} \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} \frac{(Q_{n_{m,i}(\gamma)} \circ p_i)^2(x)}{n_{m,i}(\gamma)} : \lambda \in \Lambda, \gamma \in \Gamma, x \in X_0\right\}\right)^{1/2}.$$

Corollary 8. *(a) For all $F \in Lip_{d^{(q)}_\Omega}(\beta, M)$ and all $\alpha \in D$,*

$$E_\alpha(F) \leq M c_\alpha(q, r) \Omega^\beta(\epsilon_\alpha \eta_\alpha).$$

(b) For all $f \in Lip_{d^{(q)}_\Omega, \mathfrak{T}}(\beta, M)$ and all $\alpha \in D$,

$$e_\alpha(f) \leq M c_\alpha(q, r) \Omega^\beta(\epsilon_\alpha \eta_\alpha).$$

We assume that (37) holds for all $\gamma \in \Gamma$. Then we can reduce to (33) and (34) to

$$H_{m,\gamma}(F)(x) = \sum_{k_1=1}^{n_{m,1}(\gamma)} \sum_{k_2=1}^{n_{m,2}(\gamma)} \cdots \sum_{k_r=1}^{n_{m,r}(\gamma)} \prod_{i=1}^{r} \chi_{n_{m,i}(\gamma)}(x_i; k_i) \tag{38}$$

$$\times F(t_{n_{m,1}(\gamma),k_1}, t_{n_{m,2}(\gamma),k_2}, \ldots, t_{n_{m,r}(\gamma),k_r})$$

$$(m \in \mathbb{N}_0,\ \gamma \in \Gamma,\ F \in C(X_r, E),\ x \in X_r)$$

and

$$G_{m,\gamma}(x)(f) = \sum_{k_1=1}^{n_{m,1}(\gamma)} \sum_{k_2=1}^{n_{m,2}(\gamma)} \cdots \sum_{k_r=1}^{n_{m,r}(\gamma)} \prod_{i=1}^{r} \chi_{n_{m,i}(\gamma)}(x_i; k_i)$$

$$\times T(t_{n_{m,1}(\gamma),k_1}, t_{n_{m,2}(\gamma),k_2}, \ldots , t_{n_{m,r}(\gamma),k_r})(f)$$

$$(m \in \mathbb{N}_0,\ \gamma \in \Gamma,\ f \in E_0,\ x \in X_r),$$

respectively (cf. [12], [14]).

Let $\{\nu_{m,i}\}_{m\in\mathbb{N}_0}, i = 1, 2, \ldots , r$, be strictly monotone increasing sequences of positive integers and let $v : \Gamma \to [0, \infty)$. We define

$$n_{m,i}(\gamma) = \nu_{m,i} + [v(\gamma)], \quad b_{m,i}(\gamma) = 1 \quad (m \in \mathbb{N}_0, \gamma \in \Gamma, i = 1, 2, \ldots , r)$$

Then, in view of Theorem 8 (c), for all $F \in C(X_r, E), f \in E_0$ and all $\alpha \in D$, (35) and (36) hold with

$$\eta_\alpha = \left(\sup\left\{\sum_{i=1}^{r}\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} \frac{(Q_{\nu_{m,i}+[v(\gamma)]} \circ p_i)^2(x)}{\nu_{m,i} + [v(\gamma)]} : \lambda \in \Lambda, \gamma \in \Gamma, x \in X_0\right\}\right)^{1/2}.$$

Also, we define

$$n_{m,i}(\gamma) = \nu_{m,i}, \quad b_{m,i}(\gamma) = 1 \qquad (m \in \mathbb{N}_0,\ \gamma \in \Gamma,\ i = 1, 2, \ldots , r).$$

Then (38) generalizes the classical Hermite-Fejér interpolating polynomial operators on $C(X_1, \mathbb{R})$ (cf. [4], [7]), and for all $F \in C(X_r, E), f \in E_0$ and all $\alpha \in D$, (35) and (36) hold with

$$\eta_\alpha = \left(\sup\left\{\sum_{i=1}^{r}\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} \frac{(Q_{\nu_{m,i}} \circ p_i)^2(x)}{\nu_{m,i}} : \lambda \in \Lambda,\ x \in X_0\right\}\right)^{1/2}.$$

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Department of Mathematical Sciences
Faculty of Science
University of the Ryukyus
Nishihara-cho, Okinawa 903-0213
JAPAN