The degree of interpolation type approximation processes for vector-valued functions

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THE DEGREE OF INTERPOLATION TYPE APPROXIMATION PROCESSES FOR VECTOR-VALUED FUNCTIONS

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ABSTRACT. We consider the convergence of equi-uniform approximation processes of interpolation type operators for vector-valued functions and give quantitative estimates of the rate of its convergence in terms of the modulus of continuity of functions to be approximated. Furthermore, applications are presented by Bernstein type operators and Hermite-Fejér type operators.

1. Introduction

Let $(E, \|\cdot\|)$ be a normed linear space and let (X, d) be a metric space. Let $B(X, E)$ denote the normed linear space of all E-valued bounded functions on X with the supremum norm. Also, we denote by $C(X, E)$ the linear space consisting of all E-valued continuous functions on X and set $BC(X, E) = B(X, E) \cap C(X, E)$. Let X_0 be a subset of X. Let $\mathfrak{K} = \{K_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of operators of $BC(X, E)$ into $B(X_0, E)$, where D is a directed set and Λ is an index set. Then $\mathfrak{K}$ is called an equi-uniform approximation process on $BC(X, E)$ if for all $F \in BC(X, E)$,

$$\lim_{\alpha} \|K_{\alpha,\lambda}(F)(x) - F(x)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda,\ x \in X_0 \quad (1)$$

([8]). We here consider a family $\mathfrak{K}$ of interpolation type operators on $BC(X, E)$ defined as follows:

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Let $\{Y_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of finite sets. Let $\{\xi_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of mappings of $Y_{\alpha,\lambda}$ into X and let $\{\chi_{\alpha,\lambda}(\cdot; k) : \alpha \in D, \lambda \in \Lambda, k \in Y_{\alpha,\lambda}\}$ be a family of real-valued functions on X which satisfy

$$\chi_{\alpha,\lambda}(x;\cdot) \geq 0, \qquad \sum_{k\in Y_{\alpha,\lambda}} \chi_{\alpha,\lambda}(x;k) = 1 \qquad (x \in X_0).$$

Then we define an interpolation type operator by the form

$$K_{\alpha,\lambda}(F)(x) = \sum_{k\in Y_{\alpha,\lambda}} \chi_{\alpha,\lambda}(x;k)F(\xi_{\alpha,\lambda}(k)) \tag{2}$$

$$(F \in BC(X,E), \; x \in X).$$

The purpose of this paper is to consider the convergence behavior of (1), where X_0 is assumed to be compact and give quantitative estimate of the rate of its convergence in terms of the modulus of continuity of F under certain appropriate conditions.

2. A convergence theorem

Let X_0 be a compact subset of X. Here we suppose that there exists an open subset O_{X_0} of X and a compact subset Z_{X_0} of X such that

$$X_0 \;\subseteq\; O_{X_0} \;\subseteq\; Z_{X_0}. \tag{3}$$

Note that if X is locally compact, then (3) holds. In particular, if X is a locally closed subset of the r-dimensional Euclidean space $\mathbb{R}^r$, then (3) holds.

Let Φ be a nonnegative real-valued function on $X_0 \times X$ which satisfies

$$\inf\{\Phi(x,y) : (x,y) \in X_0 \times X, d(x,y) \geq \delta\} > 0 \tag{4}$$

for every $\delta > 0$. We set

$$\tau_{\alpha,\lambda}(x;\Phi) = \sum_{k\in Y_{\alpha,\lambda}} \chi_{\alpha,\lambda}(x;k)\Phi(x,\xi_{\alpha,\lambda}(k)) \qquad (\alpha \in D, \;\lambda \in \Lambda, \; x \in X_0),$$

which is called the Φ-moment of $\chi_{\alpha,\lambda}$ at x.

Theorem 1. *If*

$$\lim_{\alpha} \tau_{\alpha,\lambda}(x;\Phi) = 0 \qquad \textit{uniformly in } \lambda \in \Lambda, x \in X_0, \tag{5}$$

then $\mathfrak{K}$ is an equi-uniform approximation process on $BC(X,E)$.

Proof. This follows from [8, Corollary 1] (cf. [9, Theorem 1]), which remains true without the completeness of E for interpolation type operators.

In particular, if the function Φ is given by the following special form, then we are able to establish Korovkin type results for interpolation type operators (cf. [8]). For an excellent source for references and a systematic treatment of Korovkin type approximation theory, we refer to the book of Altomare and Campiti [1] (cf. [3]): Let

$$\Phi(x,y)=\sum_{i=1}^{m} a_i(x)g_i(y)\geq 0 \qquad ((x,y)\in X_0\times X)$$

and

$$\Phi(x,x)=0 \qquad (x\in X_0),$$

where $a_i, 1\leq i\leq m$, are real-valued bounded functions on X_0 and $g_i, 1\leq i\leq m$, are real-valued continuous functions on X. In this case, we have

$$\tau_{\alpha,\lambda}(x;\Phi)=\sum_{i=1}^{m} a_i(x)\Big(\sum_{k\in Y_{\alpha,\lambda}}\chi_{\alpha,\lambda}(x;k)g_i(\xi_{\alpha,\lambda}(k))-g_i(x)\Big).$$

Also, if the function Φ is defined by

$$\Phi(x,y)=\sum_{i=1}^{r}(h_i(x)-h_i(y))^s \quad ((x,y)\in X_0\times X),$$

where s is an even positive integer and $h_i, 1\leq i\leq r$, are real-valued bounded continuous functions on X, then we have

$$\begin{aligned}\tau_{\alpha,\lambda}(x;\Phi)&=\sum_{i=1}^{r}\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}h_i^{s-j}(x)\\ &\times\Big(\sum_{k\in Y_{\alpha,\lambda}}\chi_{\alpha,\lambda}(x;k)h_i^j(\xi_{\alpha,\lambda}(k))-h_i^j(x)\Big)\\ &=\sum_{i=1}^{r}\sum_{j=0}^{s-1}(-1)^{s-j}\binom{s}{j}h_i^{s-j}(x)\\ &\times\Big(\sum_{k\in Y_{\alpha,\lambda}}\chi_{\alpha,\lambda}(x;k)h_i^j(\xi_{\alpha,\lambda}(k))-h_i^j(x)\Big)\\ &+\sum_{k\in Y_{\alpha,\lambda}}\chi_{\alpha,\lambda}(x;k)\sum_{i=1}^{r}h_i^s(\xi_{\alpha,\lambda}(k))\;-\;\sum_{i=1}^{r}h_i^s(x).\end{aligned}$$

Also, if

$$\Phi(x,y) = \varphi(d(x,y)) \qquad ((x,y) \in X_0 \times X),$$

where φ is a strictly increasing function on $[0,\infty)$ with $\varphi(0) = 0$, then Condition (4) is satisfied. In particular, if

$$q > 0, \quad \varphi(t) = t^q \qquad (t \geq 0),$$

then

$$\tau_{\alpha,\lambda}(x; d^q) = \sum_{k \in Y_{\alpha,\lambda}} \chi_{\alpha,\lambda}(x;k) d^q(x, \xi_{\alpha,\lambda}(k))$$

is called the qth moment of $\chi_{\alpha,\lambda}$ at x. Using this quantity, it follows from Theorem 1 that if for some $q > 0$,

$$\lim_{\alpha} \tau_{\alpha,\lambda}(x; d^q) = 0 \qquad \text{uniformly in } \lambda \in \Lambda, x \in X_0,$$

then $\mathfrak{K}$ is an approximation process on $BC(X,E)$.

3. Estimates of the rate of convergence

Let $F \in B(X,E)$ and let $\delta \geq 0$. Then we define

$$\omega_d(F,\delta) = \sup\{\|F(x) - F(y)\| : x,y \in X, d(x,y) \leq \delta\},$$

which is called the modulus of continuity of F. Obviously, $\omega_d(F,\cdot)$ is a monotone increasing function on $[0,\infty)$ and

$$\omega_d(F,0) = 0, \qquad \omega_d(F,\delta) \leq 2\sup\{\|F(x)\| : x \in X\} \qquad (\delta \geq 0).$$

Note that if X is bounded, then

$$\omega_d(F,\delta) = \omega_d(F,\delta(X))$$

for all $\delta \geq \delta(X)$, where $\delta(X)$ denotes the diameter of X, and F is uniformly continuous on X if and only if

$$\lim_{\delta \to +0} \omega_d(F,\delta) = 0.$$

For $\beta > 0$, a function $F \in B(X,E)$ is said to satisfy a Lipschitz condition of order β with constant $M > 0$, or to belong to the class $Lip_d(\beta, M)$ if

$$\omega_d(F,\delta) \leq M\delta^\beta$$

for all $\delta \geq 0$. Also, the class $Lip_d\,\beta$ consists of all $F \in Lip_d(\beta, M)$ for some constant $M > 0$. That is, we set

$$Lip_d\,\beta = \bigcup_{M>0} Lip_d(\beta, M),$$

which is called the Lipschitz class of order β. It is easy to see that

$$Lip_d\,\gamma \quad \subseteq \quad Lip_d\,\beta \qquad (0 < \beta \leq \gamma).$$

We give here quantitative versions of Theorem 1, in which we estimate the rate of convergence in terms of the modulus of continuity of functions to be approximated. For this it is always supposed that the following conditions (6) and (7) are satisfied:

There exist constants $C \geq 1$ and $K > 0$ such that

$$\omega_d(F, \xi\delta) \leq (C + K\xi)\omega_d(F, \delta) \tag{6}$$

for all $F \in B(X, E)$ and all $\delta, \xi \geq 0$.

There exist constants $q \geq 1$ and $\kappa > 0$ such that

$$d^q(x, y) \leq \kappa\Phi(x, y) \tag{7}$$

for all $(x, y) \in X_0 \times X$.

Note that Condition (7) implies (4).

d is said to be convex if $d(x, y) = a + b, a, b > 0$, then there exists a point $z \in X$ such that $d(x, z) = a$ and $d(z, y) = b$. Let (V, ρ) be a metric linear space. If $\rho(x, y) = \rho(x+z, y+z)$ for all $x, y, z \in V$, then ρ is called a translation invariant metric function. A real-valued function φ on a linear space S is said to be starshaped if $\varphi(\beta x) \leq \beta\varphi(x)$ for all $x \in S$ and all $\beta \in [0, 1]$.

The following lemma follows from [10, Lemma 2.4], which generalizes [5, Lemma 3], and it gives sufficient condition such that (6) holds with $C = K = 1$, which can be more convenient for later applications:

Lemma 1. *(a) If d is convex, then (6) holds with $C = K = 1$.*

(b) If X is a convex subset of a metric linear space with the translation invariant metric function d and if $d(\cdot, 0)$ is starshaped, then (6) holds with $C = K = 1$. In particular, if X is a convex subset of a normed linear space, then (6) holds with $C = K = 1$.

Let Ω be a strictly increasing continuous, subadditive function on $[0, \infty)$ with $\Omega(0) = 0$. Then we define

$$d_\Omega(x, y) = \Omega(d(x, y)) \qquad ((x, y) \in X \times X),$$

for which the following lemma holds (cf. [6, Lemma 3]):

Lemma 2. *(X, d_Ω) is a metric space and d_Ω is uniformly equivalent to d. Furthermore, we have*

$$\omega_d(F, \delta) = \omega_{d_\Omega}(F, \Omega(\delta))$$

for all $F \in B(X, E)$ and all $\delta \geq 0$.

A typical example of the above function Ω is given as follows:

Let $a > 0, b \geq 0, 0 < \beta \leq 1$ and let w be a nonnegative, increasing continuous function on $[0, \infty)$ such that the function $w(t)/t^\beta$ is decreasing on $(0, \infty)$. Then we define

$$\Omega(t) = \frac{t^\beta}{a + bw(t)} \qquad (t \geq 0).$$

For any $\alpha \in D, F \in BC(X, E)$ we define

$$E_\alpha(F) = \sup\{\|K_{\alpha,\lambda}(F)(x) - F(x)\| : \lambda \in \Lambda, x \in X_0\}.$$

Note that $\mathfrak{K}$ is an equi-uniform approximation process on $BC(X, E)$ if and only if

$$\lim_\alpha E_\alpha(F) = 0$$

for every $F \in BC(X, E)$. Now, we recast Theorem 1 in the following quantitative form.

Theorem 2. *Let $\{\epsilon_\alpha\}_{\alpha \in D}$ be a net of positive real numbers. Then for all $F \in BC(X, E)$ and all $\alpha \in D$,*

$$E_\alpha(F) \leq (C + K \min\{\kappa^{1/q}\epsilon_\alpha^{-1}, \kappa\epsilon_\alpha^{-q}\})\omega_{d_\Omega}(F, \Omega(\epsilon_\alpha \tau_\alpha(\Phi; q))), \qquad (8)$$

where

$$\tau_\alpha(\Phi; q) = (\sup\{\tau_{\alpha,\lambda}(x; \Phi) : \lambda \in \Lambda, x \in X_0\})^{1/q}.$$

Proof. This follows from Lemma 2 and [9,Theorem 2] (cf. [10, Theorem 3.2]), which remains true without of the completeness of E for interpolation type operators and for D instead of $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ denotes the set of all positive integers.

Corollary 1. *For all $F \in Lip_{d_\Omega}(\beta, M)$ and all $\alpha \in D$,*

$$E_\alpha(F) \leq M(C + K \min\{\kappa^{1/q}\epsilon_\alpha^{-1}, \ \kappa\epsilon_\alpha^{-q}\})\Omega^\beta(\epsilon_\alpha \tau_\alpha(\Phi; q)).$$

In the rest of this section, we restrict the interpolation type operators $K_{\alpha,\lambda}$ defined by (2) to the subclass of $BC(X, E)$ as follows:

Let E_0 be a subset of E and let $\mathfrak{T} = \{T(x) : x \in X\}$ be a family of mappings of E_0 into E such that for each $f \in E_0$, the mapping $x \mapsto T(x)(f)$ is strongly continuous and bounded on X. Let $L_{\alpha,\lambda}$ denote the restriction of $K_{\alpha,\lambda}$ to the set $\{T(\cdot)(f) : f \in E_0\}$, i.e.,

$$L_{\alpha,\lambda}(x)(f) = \sum_{k \in Y_{\alpha,\lambda}} \chi_{\alpha,\lambda}(x;k) T(\xi_{\alpha,\lambda}(k))(f) \qquad (f \in E_0). \tag{9}$$

Then the family $\mathfrak{L} = \{L_{\alpha,\lambda}(x) : \alpha \in D, \lambda \in \Lambda, x \in X\}$ is called an equi-uniform $\mathfrak{T}$- approximation process on E_0 if for every $f \in E_0$,

$$\lim_{\alpha} \|L_{\alpha,\lambda}(x)(f) - T(x)(f)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda, x \in X_0. \tag{10}$$

By Theorem 1, if (5) holds, then $\mathfrak{L}$ is an equi-uniform $\mathfrak{T}$-approximation process on E_0.

Concerning the rate of convergence behavior (10) for the family $\mathfrak{L}$, we define

$$\omega_{d,\mathfrak{T}}(f, \delta) = \sup\{\|T(x)(f) - T(y)(f)\| : x, y \in X, d(x, y) \leq \delta\}$$
$$(f \in E_0, \ \delta \geq 0),$$

which is called the modulus of continuity of f associated with $\mathfrak{T}$, and

$$e_\alpha(f) = \sup\{\|L_{\alpha,\lambda}(x)(f) - T(x)(f)\| : \lambda \in \Lambda, x \in X_0\}.$$

Note that $\mathfrak{L}$ is an equi-uniform $\mathfrak{T}$-approximation process on E_0 if and only if

$$\lim_{\alpha} e_\alpha(f) = 0$$

for every $f \in E_0$.

For $\beta > 0$, an element $f \in E_0$ is said to satisfy a Lipschitz condition of order β with constant $M > 0$ with respect to the family $\mathfrak{T}$, or to belong to the class $Lip_{d,\mathfrak{T}}(\beta, M)$ if

$$\omega_{d,\mathfrak{T}}(f, \delta) \leq M\delta^\beta$$

for all $\delta \geq 0$. Also, we set

$$Lip_{d,\mathfrak{T}}\,\beta = \bigcup_{M>0} Lip_{d,\mathfrak{T}}(\beta, M),$$

which is called the Lipschitz class of order β with respect to the family $\mathfrak{T}$. It is easy to see that

$$Lip_{d,\mathfrak{T}}\,\gamma \quad \subseteq \quad Lip_{d,\mathfrak{T}}\,\beta \qquad (0 < \beta \leq \gamma).$$

Theorem 3. *Let $\{\epsilon_\alpha\}_{\alpha \in D}$ be a net of positive real numbers. Then for all $f \in E_0$ and all $\alpha \in D$,*

$$e_\alpha(f) \leq (C + K \min\{\kappa^{1/q}\epsilon_\alpha^{-1}, \kappa\epsilon_\alpha^{-q}\})\omega_{d_\Omega,\mathfrak{T}}(f, \Omega(\epsilon_\alpha \tau_\alpha(\Phi; q))).$$

Proof. Since

$$\omega_{d_\Omega,\mathfrak{T}}(f, \delta) = \omega_{d_\Omega}(T(\cdot)(f), \delta), \quad e_\alpha(f) = E_\alpha(T(\cdot)(f))$$
$$(f \in E_0, \ \delta \geq 0, \ \alpha \in D),$$

we take $F(\cdot) = T(\cdot)(f)$ in (8).

Corollary 2. *For all $f \in Lip_{d_\Omega,\mathfrak{T}}(\beta, M)$ and all $\alpha \in D$,*

$$e_\alpha(f) \leq M(C + K \min\{\kappa^{1/q}\epsilon_\alpha^{-1}, \kappa\epsilon_\alpha^{-q}\})\Omega^\beta(\epsilon_\alpha \tau_\alpha(\Phi; q)).$$

4. Bernstein type operators

Let $1 \leq p \leq \infty$ be fixed and let X be a locally closed convex subset of the r-dimensional Euclidean space $\mathbb{R}^r$ with the metric

$$d(x,y) = d^{(p)}(x,y) := \begin{cases} \left(\sum_{i=1}^r |x_i - y_i|^p\right)^{1/p} & (1 \leq p < \infty) \\ \max\{|x_i - y_i| : 1 \leq i \leq r\} & (p = \infty), \end{cases}$$

where $x = (x_1, x_2, \dots, x_r), y = (y_1, y_2, \dots, y_r) \in \mathbb{R}^r$. Therefore, by Lemma 1 (b), Condition (6) holds with $C = K = 1$. For $i = 1, 2, \dots, r$, p_i denotes the ith coordinate function on $\mathbb{R}^r$ defined by $p_i(x) = x_i$ for all $x = (x_1, x_2, \dots, x_r) \in \mathbb{R}^r$. Then we have

$$(d^{(p)}(x,y))^q \leq c(p,q,r) \sum_{i=1}^r |p_i(x) - p_i(y)|^q \qquad (x, y \in \mathbb{R}^r, \ q > 0),$$

where

$$c(p,q,r) = \begin{cases} r^{q/p} & (1 \leq p < \infty, \ p \neq q) \\ 1 & (1 \leq p < \infty, \ p = q) \\ 1 & (p = \infty). \end{cases}$$

Therefore, (7) holds with

$$\kappa = c(p,q,r), \quad \Phi(x,y) = \sum_{i=1}^r (p_i(x) - p_i(y))^q \quad (q \geq 1),$$

and so all the estimates obtained in the previous section hold for

$$\tau_\alpha(\Phi; q) = (\sup\{\tau_{\alpha,\lambda}(x; d^{(q)^q}) : \lambda \in \Lambda, x \in X_0\})^{1/q} \qquad (q \geq 1).$$

Let

$$X = [0, \infty)^r = \{x = (x_1, x_2, \dots, x_r) \in \mathbb{R}^r : x_i \geq 0, i = 1, 2, \dots, r\}$$

be the region of the first hyperquadrant and let X_0 be a closed subset of $\mathbb{I}_r$, where

$$\mathbb{I}_r = \{x = (x_1, x_2, \dots, x_r) \in \mathbb{R}^r : 0 \leq x_i \leq 1, i = 1, 2, \dots, r\}$$

is the unit r-cube. Let

$$m_{\alpha,i} : \Lambda \to \mathbb{N}, \quad a_{\alpha,i} : \Lambda \to (0, \infty) \qquad (\alpha \in D, i = 1, 2, \dots, r)$$

and let

$$Y_{\alpha,\lambda} = \{k = (k_1, k_2, \dots, k_r) \in \mathbb{N}_0^r : 0 \leq k_i \leq m_{\alpha,i}(\lambda), i = 1, 2, \dots, r\}.$$

We define

$$\chi_{\alpha,\lambda}(x; k) = \prod_{i=1}^{r} \binom{m_{\alpha,i}(\lambda)}{k_i} x_i^{k_i} (1 - x_i)^{m_{\alpha,i}(\lambda) - k_i} \qquad (x \in X, \ k \in Y_{\alpha,\lambda})$$

and

$$\xi_{\alpha,\lambda}(k) = (a_{\alpha,1}(\lambda) k_1, a_{\alpha,2}(\lambda) k_2, \dots, a_{\alpha,r}(\lambda) k_r) \qquad (k \in Y_{\alpha,\lambda}).$$

Then the corresponding interpolation type operators $K_{\alpha,\lambda}$ defined by (2) and $L_{\alpha,\lambda}$ defined by (9) are called the Bernstein type operators. These generalize the r-dimensional Bernstein operators, which are defined as follows (cf. [4], [8]):

Let $\{\nu_{n,i}\}_{n \in \mathbb{N}}, i = 1, 2, \dots, r$, be strictly monotone increasing sequences of positive integers. Then we define

$$B_n(F)(x) = \sum_{k_1=0}^{\nu_{n,1}} \sum_{k_2=0}^{\nu_{n,2}} \cdots \sum_{k_r=0}^{\nu_{n,r}} F\Big(\frac{k_1}{\nu_{n,1}}, \frac{k_2}{\nu_{n,2}}, \dots, \frac{k_r}{\nu_{n,r}}\Big) \times \prod_{j=1}^{r} \binom{\nu_{n,j}}{k_j} x_j^{k_j} (1 - x_j)^{\nu_{n,j} - k_j}$$

$$(F \in C(\mathbb{I}_r, E), x \in \mathbb{I}_r)$$

and

$$C_n(x)(f) = \sum_{k_1=0}^{\nu_{n,1}} \sum_{k_2=0}^{\nu_{n,2}} \cdots \sum_{k_r=0}^{\nu_{n,r}} T\Big(\frac{k_1}{\nu_{n,1}}, \frac{k_2}{\nu_{n,2}}, \dots, \frac{k_r}{\nu_{n,r}}\Big)(f)$$

$$\times \prod_{j=1}^{r} \binom{\nu_{n,j}}{k_j} x_j^{k_j}(1-x_j)^{\nu_{n,j}-k_j}$$

$$(f \in E_0, x \in \mathbb{I}_r).$$

Now, we have

$$\zeta_\alpha := \tau_\alpha(\Phi; 2) = \left(\sup\left\{\sum_{i=1}^{r} \zeta_{\alpha,i}(\lambda, x) : \lambda \in \Lambda, x \in X_0\right\}\right)^{1/2},$$

where

$$\zeta_{\alpha,i}(\lambda, x) = (a_{\alpha,i}(\lambda)m_{\alpha,i}(\lambda) - 1)^2 p_i^2(x)$$
$$+ a_{\alpha,i}(\lambda)^2 m_{\alpha,i}(\lambda) p_i(x)(1 - p_i(x)).$$

Therefore, Theorems 2 and 3 establish the following result for the Bernstein type operators.

Theorem 4. *(a) For all $F \in BC(X, E)$ and all $\alpha \in D$,*

$$E_\alpha(F) \leq \gamma_\alpha(p, r)\omega_{d^{(p)}{}_\Omega}(F, \Omega(\epsilon_\alpha \zeta_\alpha)). \tag{11}$$

(b) For all $f \in E_0$ and all $\alpha \in D$,

$$e_\alpha(f) \leq \gamma_\alpha(p, r)\omega_{d^{(p)}{}_\Omega, \mathfrak{T}}(f, \Omega(\epsilon_\alpha \zeta_\alpha)). \tag{12}$$

Here

$$\gamma_\alpha(p, r) = 1 + \min\left\{\frac{\sqrt{c(p, r)}}{\epsilon_\alpha}, \frac{c(p, r)}{\epsilon_\alpha^2}\right\}$$

and

$$c(p, r) = \begin{cases} r^{2/p} & (1 \leq p < \infty, p \neq 2) \\ 1 & (p = 2, \infty). \end{cases}$$

In particular, if

$$m_{\alpha,i}(\lambda)a_{\alpha,i}(\lambda) = 1$$

for all $\alpha \in D, \lambda \in \Lambda$ and for $i = 1, 2, \ldots, r$, then (11) and (12) hold with

$$\zeta_\alpha = \left(\sup\left\{\sum_{i=1}^{r} \frac{1}{m_{\alpha,i}(\lambda)}(p_i(x) - p_i^2(x)) : \lambda \in \Lambda, x \in X_0\right\}\right)^{1/2}.$$

We define

$$\|G\|_S = \sup\{\|G(x)\| : x \in S\} \qquad (G \in B(X,E),\ S \subseteq X).$$

Let $\{\epsilon_n\}_{n\in\mathbb{N}}$ be a sequence of positive integers. Then in view of (11) and (12), we have the following estimates for the Bernstein operators:

$$\|B_n(F) - F\|_{X_0} \le \gamma_n(p,r)\omega_{d^{(p)}\Omega}(F, \Omega(\epsilon_n\theta_n)) \tag{13}$$

$$(F \in C(\mathbb{I}_r, E));$$

$$\|C_n(\cdot)(f) - T(\cdot)(f)\|_{X_0} \le \gamma_n(p,r)\omega_{d^{(p)}\Omega,\mathfrak{T}}(f, \Omega(\epsilon_n, \theta_n)) \tag{14}$$

$$(f \in E_0).$$

Here

$$\gamma_n(p,r) = 1 + \min\Big\{\frac{\sqrt{c(p,r)}}{\epsilon_n}, \frac{c(p,r)}{\epsilon_n^2}\Big\}$$

and

$$\theta_n = \Big(\max\Big\{\sum_{i=1}^{r}\frac{1}{\nu_{n,i}}(p_i(x) - p_i^2(x)) : x \in X_0\Big\}\Big)^{1/2}.$$

Therefore, (13) and (14) yield the following estimates:

$$\|B_n(F) - F\|_{\mathbb{I}_r} \le \theta_n(p,r)\omega_{d^{(p)}\Omega}\Big(F, \Omega\Big(\epsilon_n\sqrt{\sum_{i=1}^{r}\frac{1}{\nu_{n,i}}}\Big)\Big) \tag{15}$$

$$(F \in C(\mathbb{I}_r, E));$$

$$\|C_n(\cdot)(f) - T(\cdot)(f)\|_{\mathbb{I}_r} \le \theta_n(p,r)\omega_{d^{(p)}\Omega,\mathfrak{T}}\Big(f, \Omega\Big(\epsilon_n\sqrt{\sum_{i=1}^{r}\frac{1}{\nu_{n,i}}}\Big)\Big) \tag{16}$$

$$(f \in E_0).$$

Here

$$\theta_n(p,r) = 1 + \min\Big\{\frac{\sqrt{c(p,r)}}{2\epsilon_n}, \frac{c(p,r)}{4\epsilon_n^2}\Big\}.$$

Let $\{T_i(t) : 0 \le t \le 1, i = 1,2,\dots,r\}$ be a family of strongly continuous mappings of E_0 into itself such that for every $t,u \in [0,1]$, $tT_i(u)$ commutes with $(1-t)I$, where I is the identity operator on E and $T_i(v)^n = T_i(nv)$ whenever $v \in [0,1], n \in \mathbb{N}_0$ and $nv \in [0,1]$. If

$$T(x) = \prod_{i=1}^{r} T_i(x_i)$$

for all $x = (x_1, x_2, \dots, x_r) \in \mathbb{I}_r$, then we have

$$C_n(x)(f) = \prod_{i=1}^{r}\Big((1-x_i)I + x_i T_i\Big(\frac{1}{\nu_{n,i}}\Big)\Big)^{\nu_{n,i}}(f)$$
$$= \prod_{i=1}^{r}\Big(I + x_i\Big(T_i\Big(\frac{1}{\nu_{n,i}}\Big) - I\Big)\Big)^{\nu_{n,i}}(f).$$

Therefore, the inequality (16) estimates the rate of convergence in [7, Theorem 5] for $r = 1$, which improve the estimate in [2, Proposition 1.2.9].

Let $\{n_\alpha\}_{\alpha\in D}$ be a net of positive integers. If

$$m_{\alpha,i}(\lambda) = n_\alpha + [\lambda] + i \qquad (\alpha \in D, \lambda \in \Lambda \subseteq [0,\infty), i = 1,2,\dots,r)$$

and

$$a_{\alpha,i}(\lambda) = \frac{1}{n_\alpha + [\lambda] + i} \qquad (\alpha \in D, \lambda \in \Lambda \subseteq [0,\infty), i = 1,2,\dots,r),$$

where $[\lambda]$ denotes the largest integer not exceeding λ, then we have

$$E_\alpha(F) \le \theta(p,r)\omega_{d^{(p)}\Omega}\Big(F, \Omega\Big(\frac{1}{\sqrt{n_\alpha+1}}\Big)\Big) \qquad (F \in BC(X,E))$$

and

$$e_\alpha(f) \le \theta(p,r)\omega_{d^{(p)}\Omega,\mathfrak{T}}\Big(f, \Omega\Big(\frac{1}{\sqrt{n_\alpha+1}}\Big)\Big) \qquad (f \in E_0),$$

where

$$\theta(p,r) = 1 + \min\Big\{\frac{\sqrt{rc(p,r)}}{2}, \frac{rc(p,r)}{4}\Big\}.$$

Also, for the Bernstein operators, (15) and (16) establish

$$\|B_n(F) - F\|_{\mathbb{I}_r} \le \theta(p,r)\omega_{d^{(p)}\Omega}\Big(F, \Omega\Big(\frac{1}{\sqrt{\nu_n}}\Big)\Big) \qquad (F \in C(\mathbb{I}_r, E))$$

and

$$\|C_n(\cdot)(f) - T(\cdot)(f)\|_{\mathbb{I}_r} \le \theta(p,r)\omega_{d^{(p)}\Omega,\mathfrak{T}}\Big(f, \Omega\Big(\frac{1}{\sqrt{\nu_n}}\Big)\Big) \qquad (f \in E_0),$$

where

$$\nu_n = \min\{\nu_{n,i} : i = 1,2,\dots,r\}.$$

We can consider the statements analogous to the above results for the following setting:

Let X_0 be a closed subset of Δ_r, where

$$\Delta_r = \{x = (x_1, x_2, \dots, x_r) \in \mathbb{R}^r : x_i \geq 0, i = 1, 2, \dots, r, \sum_{i=1}^{r} x_i \leq 1\}$$

is the standard r-simplex. Let

$$m_\alpha : \Lambda \to \mathbb{N}, \quad a_{\alpha,i} : \Lambda \to (0, \infty) \qquad (\alpha \in D, i = 1, 2, \dots, r)$$

and let

$$Y_{\alpha,\lambda} = \{k = (k_1, k_2, \dots, k_r) \in \mathbb{N}_0^r : k_1 + k_2 + \cdots + k_r \leq m_\alpha(\lambda)\}.$$

We define

$$\chi_{\alpha,\lambda}(x; k) = \binom{m_\alpha(\lambda)}{k} \prod_{i=1}^{r} x_i^{k_i} \Big(1 - \sum_{j=1}^{r} x_j\Big)^{m_\alpha(\lambda) - \sum_{j=1}^{r} k_j}$$
$$(x \in X,\ k \in Y_{\alpha,\lambda}),$$

where

$$\binom{m_\alpha(\lambda)}{k} = \frac{m_\alpha(\lambda)!}{k_1! k_2! \cdots k_r! (m_\alpha(\lambda) - k_1 - k_2 - \cdots - k_r)!},$$

and

$$\xi_{\alpha,\lambda}(k) = (a_{\alpha,1}(\lambda) k_1, a_{\alpha,2}(\lambda) k_2, \dots, a_{\alpha,r}(\lambda) k_r) \qquad (k \in Y_{\alpha,\lambda}).$$

Then we have the following result:

Theorem 5. *(a) For all $F \in BC(X, E)$ and all $\alpha \in D$,*

$$E_\alpha(F) \leq \gamma_\alpha(p, r) \omega_{{d^{(p)}}_\Omega}(F, \Omega(\epsilon_\alpha \delta_\alpha)). \tag{17}$$

(b) For all $f \in E_0$ and all $\alpha \in D$,

$$e_\alpha(f) \leq \gamma_\alpha(p, r) \omega_{{d^{(p)}}_\Omega, \mathfrak{T}}(f, \Omega(\epsilon_\alpha \delta_\alpha)). \tag{18}$$

Here

$$\delta_\alpha = \Big(\sup\Big\{\sum_{i=1}^{r} \delta_{\alpha,i}(\lambda, x) : \lambda \in \Lambda, x \in X_0\Big\}\Big)^{1/2},$$

and

$$\delta_{\alpha,i}(\lambda, x) = (a_{\alpha,i}(\lambda) m_\alpha(\lambda) - 1)^2 p_i^2(x) + a_{\alpha,i}(\lambda)^2 m_\alpha(\lambda) p_i(x)(1 - p_i(x)).$$

In pariticular, if

$$m_\alpha(\lambda) a_{\alpha,i}(\lambda) = 1$$

for all $\alpha \in D, \lambda \in \Lambda$ and for $i = 1, 2, \dots, r$, then (17) and (18) yield

$$E_\alpha(F) \leq \eta_{X_0}(p, r) \omega_{{d^{(p)}}_\Omega}(F, \Omega(\mu_\alpha))$$

and

$$e_\alpha(f) \le \eta_{X_0}(p,r)\omega_{d^{(p)}{}_{\Omega},\mathfrak{T}}(f, \Omega(\mu_\alpha)),$$

where

$$\mu_\alpha = \sqrt{\sup\left\{\frac{1}{m_\alpha(\lambda)} : \lambda \in \Lambda\right\}}$$

and

$$\eta_{X_0}(p,r)$$
$$= 1 + \min\left\{\sqrt{c(p,r}\left\|\sum_{i=1}^{r}(p_i - p_i^2)\right\|_{X_0}^{1/2},\ c(p,r)\left\|\sum_{i=1}^{r}(p_i - p_i^2)\right\|_{X_0}\right\}.$$

If we define

$$m_\alpha(\lambda) = n_\alpha + [\lambda] \qquad (\alpha \in D, \lambda \in \Lambda \subseteq [0,\infty))$$

and

$$a_{\alpha,i}(\lambda) = \frac{1}{n_\alpha + [\lambda]} \qquad (\alpha \in D, \lambda \in \Lambda \subseteq [0,\infty), i = 1,2,\dots,r),$$

then the following estimates hold:

$$E_\alpha(F) \le \theta(p,r)\omega_{d^{(p)}{}_{\Omega}}\left(F, \Omega\left(\frac{1}{\sqrt{n_\alpha}}\right)\right) \qquad (F \in BC(X,E));$$

$$e_\alpha(f) \le \theta(p,r)\omega_{d^{(p)}{}_{\Omega},\mathfrak{T}}\left(f, \Omega\left(\frac{1}{\sqrt{n_\alpha}}\right)\right) \qquad (f \in E_0).$$

5. Hermite-Fejér type operators

Let $(\mathbb{R}^r, d^{(p)}), 1 \le p \le \infty$, be as in Section 4. Let $X = [-1,1]^r$ and let X_0 be a closed subset of X. Let $Q_n(t) = \cos(n \arccos t)$ be the Chebyshev polynomial of degree n, and let $t_{n,j}, j = 1,2,\dots,n$ be zeros of $Q_n(t)$, i.e.,

$$t_{n,j} = \cos\left(\frac{2j-1}{2n}\pi\right), \qquad j = 1,2,\dots,n.$$

Let

$$m_{\alpha,i} : \Lambda \to \mathbb{N}, \quad a_{\alpha,i} : \Lambda \to [-1,1] \qquad (\alpha \in D,\ i = 1,2,\dots,r)$$

and let

$$Y_{\alpha,\lambda} = \{k = (k_1,k_2,\dots,k_r) \in \mathbb{N}^r : 1 \le k_i \le m_{\alpha,i}(\lambda),\ i = 1,2,\dots,r\}.$$

We define

$$\chi_{\alpha,\lambda}(x;k) = \prod_{i=1}^{r} \chi_{m_{\alpha,i}(\lambda)}(x_i;k_i) \qquad (x \in X,\ k \in Y_{\alpha,\lambda}),$$

where

$$\chi_{m_{\alpha,i}(\lambda)}(x_i;k_i) = (1 - x_i t_{m_{\alpha,i}(\lambda),k_i})\left\{\frac{Q_{m_{\alpha,i}(\lambda)}(x_i)}{m_{\alpha,i}(\lambda)(x_i - t_{m_{\alpha,i}(\lambda),k_i})}\right\}^2$$

and

$$\xi_{\alpha,\lambda}(k) = (a_{\alpha,1}(\lambda)t_{m_{\alpha,1}(\lambda),k_1}, \dots, a_{\alpha,r}(\lambda)t_{m_{\alpha,r}(\lambda),k_r}) \qquad (k \in Y_{\alpha,\lambda}).$$

Then the corresponding interpolation type operators $K_{\alpha,\lambda}$ defined by (2) and $L_{\alpha,\lambda}$ defined by (9) are called the Hermite-Fejér type operators. These generalize the r-dimensional Hermite-Fejér operators, which are defined as follows (cf. [3], [8]):

Let $\{\nu_{n,i}\}_{n\in\mathbb{N}}, i = 1, 2, \dots, r$, be strictly monotone increasing sequences of positive integers. Then we define

$$H_n(F)(x) = \sum_{k_1=1}^{\nu_{n,1}} \sum_{k_2=1}^{\nu_{n,2}} \cdots \sum_{k_r=1}^{\nu_{n,r}} F(t_{\nu_{n,1},k_1}, t_{\nu_{n,2},k_2}, \dots, t_{\nu_{n,r},k_r})$$
$$\times \prod_{i=1}^{r}(1 - x_i t_{\nu_{n,i},k_i})\left\{\frac{Q_{\nu_{n,i}}(x_i)}{\nu_{n,i}(x_i - t_{\nu_{n,i},k_i})}\right\}^2$$
$$(F \in C(X,E), x \in X)$$

and

$$G_n(x)(f) = \sum_{k_1=1}^{\nu_{n,1}} \sum_{k_2=1}^{\nu_{n,2}} \cdots \sum_{k_r=1}^{\nu_{n,r}} T(t_{\nu_{n,1},k_1}, t_{\nu_{n,2},k_2}, \dots, t_{\nu_{n,r},k_r})(f)$$
$$\times \prod_{i=1}^{r}(1 - x_i t_{\nu_{n,i},k_i})\left\{\frac{Q_{\nu_{n,i}}(x_i)}{\nu_{n,i}(x_i - t_{\nu_{n,i},k_i})}\right\}^2$$
$$(f \in E_0,\ x \in X).$$

Now, we have

$$\eta_\alpha := \tau_\alpha(\Phi;2) = \left(\sup\left\{\sum_{i=1}^{r} \eta_{\alpha,i}(\lambda,x) : \lambda \in \Lambda, x \in X_0\right\}\right)^{1/2},$$

where

$$\eta_{\alpha,i}(\lambda,x) = \frac{Q^2_{m_{\alpha,i}(\lambda)}(x_i)}{m_{\alpha,i}(\lambda)} - 2x_i(a_{\alpha,i}(\lambda)-1)\sum_{k_i=1}^{m_{\alpha,i}(\lambda)} t_{m_{\alpha,i}(\lambda),k_i}\chi_{m_{\alpha,i}(\lambda)}(x_i;k_i)$$

$$+\ (a^2_{\alpha,i}(\lambda)-1)\sum_{k_i=1}^{m_{\alpha,i}(\lambda)} t_{m_{\alpha,i}(\lambda),k_i}{}^2\chi_{m_{\alpha,i}(\lambda)}(x_i;k_i).$$

Therefore, Theorems 2 and 3 yield the following result for the Hermite-Fejér type operators.

Theorem 6. *(a) For all $F \in C(X,E)$ and all $\alpha \in D$,*

$$E_\alpha(F) \le \gamma_\alpha(p,r)\omega_{d^{(p)}{}_\Omega}(F,\Omega(\epsilon_\alpha\eta_\alpha)). \tag{19}$$

(b) For all $f \in E_0$ and all $\alpha \in D$,

$$e_\alpha(f) \le \gamma_\alpha(p,r)\omega_{d^{(p)}{}_\Omega,\mathfrak{T}}(f,\Omega(\epsilon_\alpha\eta_\alpha)). \tag{20}$$

In pariticular, if

$$a_{\alpha,i}(\lambda) = 1$$

for all $\alpha \in D, \lambda \in \Lambda$ and for $i = 1,2,\ldots,r$, then (19) and (20) hold wth

$$\eta_\alpha = \left(\sup\left\{\sum_{i=1}^{r}\frac{(Q_{m_{\alpha,i}(\lambda)}\circ p_i)^2(x)}{m_{\alpha,i}(\lambda)} : \lambda \in \Lambda, x \in X_0\right\}\right)^{1/2}.$$

In view of (19) and (20), we have the following estimates for the Hermite-Fejér operators:

$$\|H_n(F)-F\|_{X_0} \le \gamma_n(p,r)\omega_{d^{(p)}{}_\Omega}(F,\Omega(\epsilon_n\tau_n))$$
$$(F \in C(X,E));$$
$$\|G_n(\cdot)(f)-T(\cdot)(f)\|_{X_0} \le \gamma_n(p,r)\omega_{d^{(p)}{}_\Omega,\mathfrak{T}}(f,\Omega(\epsilon_n\tau_n))$$
$$(f \in E_0).$$

Here

$$\tau_n = \left(\max\left\{\sum_{i=1}^{r}\frac{(Q_{\nu_{n,i}}\circ p_i)^2(x)}{\nu_{n,i}} : x \in X_0\right\}\right)^{1/2}.$$

In particular, the following estimates hold:

$$\|H_n(F)-F\|_X \le \gamma_n(p,r)\omega_{d^{(p)}{}_\Omega}\left(F,\Omega\left(\epsilon_n\sqrt{\sum_{i=1}^{r}\frac{1}{\nu_{n,i}}}\right)\right)$$
$$(F \in C(X,E));$$
$$\|G_n(\cdot)(f)-T(\cdot)(f)\|_X \le \gamma_n(p,r)\omega_{d^{(p)}{}_\Omega,\mathfrak{T}}\left(f,\Omega\left(\epsilon_n\sqrt{\sum_{i=1}^{r}\frac{1}{\nu_{n,i}}}\right)\right)$$
$$(f \in E_0).$$

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