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THE CONVERGENCE OF EQUI-UNIFORM APPROXIMATION PROCESSES OF INTEGRAL OPERATORS IN BANACH SPACES

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ABSTRACT. We consider the convergence of equi-uniform approximation processes of integral operators in Banach spaces. Applications are presented by various summation processes, interpolation type operators and convolution type operators, and furthermore several concrete examples of approximating operators are also provided.

1. Introduction

Let $\mathbb{N}$ be the set of all natural numbers, and put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let ℓ^∞ denote the Banach space of all bounded sequences $\{s_n\}_{n\in\mathbb{N}_0}$ of real numbers with the usual supremum norm. A sequence $\{s_n\} \in \ell^\infty$ is said to be almost convergent to s if $\varphi(\{s_n\}) = s$ for every Banach limit φ on ℓ^∞ ([9]). If $\{s_n\}$ converges to s, then it is almost convergent to s, but not conversely. Also, $\{s_n\}$ is almost convergent to s if and only if

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=m}^{n+m} s_k = s \qquad \text{uniformly in } m \in \mathbb{N}_0$$

(cf. [9, Theorem 1]).

Let $\{B_n\}$ be the sequence of Bernstein operators defined by

$$B_n(f)(x) = \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big)\binom{n}{k} x^k (1-x)^{n-k}$$

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$$(f \in C[0,1],\ x \in [0,1],\ n \in \mathbb{N}).$$

Then for all $f \in C[0,1]$, $\{B_n(f)(x)\}$ is almost convergent to $f(x)$ uniformly on $[0,1]$ (cf. [7])

Let $\{\sigma_n\}$ be the sequence of Fejér operators defined by

$$\sigma_n(f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(x-t) f(t)\, dt$$

$$(f \in C_{2\pi},\ x \in \mathbb{R},\ n \in \mathbb{N}_0),$$

where

$$F_n(u) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) e^{iju} \qquad (u \in \mathbb{R})$$

is the nth Fejér kernel. Then for every $f \in C_{2\pi}$, $\{\sigma_n(f)(x)\}$ is almost convergent to $f(x)$ uniformly on the real line $\mathbb{R}$ (cf. [7])

In view of these results, we generally make the following definition:

Let $(E, \|\cdot\|)$ be a Banach space and (X, d) a metric space. Let $B(X, E)$ denote the Banach space of all E-valued bounded functions on X with the supremum norm. $BC(X, E)$ stands for the closed linear subspace of $B(X, E)$ consisting of all E-valued bounded continuous functions on X. Also, we denote by $C(X, E)$ the linear space consisting of all E-valued continuous functions on X. Let X_0 be a subset of X. Let $\mathfrak{K} = \{K_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of operators of $BC(X, E)$ into $B(X, E)$, where D is a directed set and Λ is an index set. Then $\mathfrak{K}$ is called an equi-uniform approximation process on $BC(X, E$ if for every $F \in BC(X, E)$,

$$\lim_{\alpha} \|K_{\alpha,\lambda}(F)(x) - F(x)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda,\ x \in X_0 \quad (1)$$

(cf. [12], [13], [14]). We here consider a family of $\mathfrak{K}$ of integral operators on $BC(X, E)$ defined as follows (cf. [17]):

Let Y be a separable topological space and let μ be a Borel measure on Y. Let $\{\xi_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of continuous mappings of Y into X and $\mathfrak{A} = \{\chi_{\alpha,\lambda}(x;\cdot) : \alpha \in D, \lambda \in \Lambda, x \in X\}$ a family of functions in $L^1(Y, \mu)$, which denotes the Banach space of all μ-integrable functions χ on Y with the norm

$$\|\chi\|_1 = \int_Y |\chi(y)|\, d\mu(y).$$

Then we define integral operators by the form

$$K_{\alpha,\lambda}(F)(x) = \int_Y \chi_{\alpha,\lambda}(x;y) F(\xi_{\alpha,\lambda}(y))\, d\mu(y) \quad (F \in BC(X,E)), \quad (2)$$

which exists as a Bochner integral.

The purpose of this paper is to consider the convergence behavior of (1) for the family $\mathfrak{K}$ of integral operators defined by (2) under certain appropriate conditions. Furthermore, applications are presented by various summation processes of interpolation type operators and convolution type operators, and several concrete examples of approximating operators are also provided. These treatments will be carried out by developing our techniques of [17].

2. Convergence theorems

Let X_0 be a compact subset of X. Here we suppose that there exists an open subset O_{X_0} of X and a compact subset Z_{X_0} of X such that

$$X_0 \subseteq O_{X_0} \subseteq Z_{X_0}, \tag{3}$$

under which in [16], we considered the refinement of Korovkin type approximation processes of positive linear operators.

Remark 1. If X is locally compact, then (3) holds. In particular, if X is a locally closed subset of the r-dimensional Euclidean space $\mathbb{R}^r$, then (3) holds.

Let $\mathfrak{A} = \{\chi_{\alpha,\lambda}(x;\cdot) : \alpha \in D, \lambda \in \Lambda, x \in X\}$ be a family of functions in $L^1(Y,\mu)$. Then $\mathfrak{A}$ is called an equi-uniform approximate kernel if it satisfies the conditions

$$\limsup_{\alpha}(\sup\{\|\chi_{\alpha,\lambda}(x;\cdot)\|_1 : \lambda \in \Lambda, x \in X_0\}) < \infty, \tag{4}$$

$$\lim_{\alpha} \int_Y \chi_{\alpha,\lambda}(x;y)\,d\mu(y) = 1 \qquad \text{uniformly in } \lambda \in \Lambda, x \in X_0, \tag{5}$$

and for any fixed $\delta > 0$,

$$\lim_{\alpha} \int_{d(x,\xi_{\alpha,\lambda}(y))\geq\delta} |\chi_{\alpha,\lambda}(x;y)|\,d\mu(y) = 0 \quad \text{uniformly in } \lambda \in \Lambda, x \in X_0. \tag{6}$$

If for all $\alpha \in D, \lambda \in \Lambda$ and all $x \in X_0$,

$$\chi_{\alpha,\lambda}(x;y) \geq 0 \quad (\mu\text{-a.e. } y \in Y),$$

then $\mathfrak{A}$ is said to be positive.

If for all $\alpha \in D, \lambda \in \Lambda$ and all $x \in X_0$,

$$\int_Y \chi_{\alpha,\lambda}(x;y)\,d\mu(y) = 1,$$

then $\mathfrak{A}$ is said to be normal.

Remark 2. If $\mathfrak{A}$ is positive, then (5) already implies (4). Also, if $\mathfrak{A}$ is positive and normal and if (6) is satisfied for each $\delta > 0$, then it becomes an equi-uniform approximate kernel.

Lemma 1. *Let Ψ be a nonnegative real-valued continuous function on $X^2 := X \times X$ such that $\Psi(x,x) = 0$ for all $x \in X_0$ and*

$$A := \sup\{\Psi(x,t) : (x,t) \in X_0 \times X\} < \infty. \tag{7}$$

If $\mathfrak{A}$ satisfies (4) and (6), then

$$\lim_{\alpha} \int_Y |\chi_{\alpha,\lambda}(x;y)| \Psi(x, \xi_{\alpha,\lambda}(y))\, d\mu(y) = 0 \quad \textit{uniformly in } \lambda \in \Lambda, x \in X_0. \tag{8}$$

Proof. Let $\epsilon > 0$ be given. Since Ψ is uniformly continuous on $Z_{X_0} \times Z_{X_0}$, there exists a $\delta_0 > 0$ such that $\Psi(x,t) < \epsilon$ for all $(x,t) \in X_0 \times Z_{X_0}$ with $d(x,t) < \delta_0$. Let $\delta_1 = d(X_0, X \setminus O_{X_0})$ be the distance between X_0 and $X \setminus O_{X_0}$, which is positive because of (3), and let $0 < \delta < \min\{\delta_0, \delta_1\}$. Now let $t \in X$ and $x \in X_0$. If $d(x,t) < \delta$, then $t \in Z_{X_0}$ and so $\Psi(x,t) < \epsilon$. Therefore, by (7) we have

$$\begin{aligned} A_{\alpha,\lambda}(x) &:= \int_Y |\chi_{\alpha,\lambda}(x;y)| \Psi(x, \xi_{\alpha,\lambda}(y))\, d\mu(y) \\ &= \int_{d(x,\xi_{\alpha,\lambda}(y))<\delta} + \int_{d(x,\xi_{\alpha,\lambda}(y))\geq\delta} \\ &\leq \epsilon \|\chi_{\alpha,\lambda}(x;\cdot)\|_1 + A \int_{d(x,\xi_{\alpha,\lambda}(y))\geq\delta} |\chi_{\alpha,\lambda}(x;y)|\, d\mu(y). \end{aligned}$$

By (4) and (6), there exists a positive constant B and an element $\alpha_0 \in D$ such that

$$\|\chi_{\alpha,\lambda}(x;\cdot)\|_1 \leq B \quad \text{and} \quad \int_{d(x,\xi_{\alpha,\lambda}(y))\geq\delta} |\chi_{\alpha,\lambda}(x;y)|\, d\mu(y) < \epsilon$$

for all $\lambda \in \Lambda, x \in X_0$ and all $\alpha \in D, \alpha \geq \alpha_0$. Thus, we obtain

$$A_{\alpha,\lambda}(x) < (A+B)\epsilon \qquad (\lambda \in \Lambda, x \in X_0,\ \alpha \in D, \alpha \geq \alpha_0),$$

which implies (8).

Theorem 1. *If $\mathfrak{A}$ is an equi-uniform approximate kernel, then $\mathfrak{K}$ is an equi-uniform approximation process on $BC(X,E)$. In particular, if $\mathfrak{A}$ is positive, and if (5) and (6) are satisfied for each $\delta > 0$, then $\mathfrak{K}$ is an equi-uniform approximation process on $BC(X,E)$.*

Proof. Let $F \in BC(X, E)$. Then we have

$$\left\| \int_Y \chi_{\alpha,\lambda}(x;y) F(\xi_{\alpha,\lambda}(y))\, d\mu(y) - F(x) \right\|$$

$$\leq \left| \int_Y \chi_{\alpha,\lambda}(x;y) d\mu(y) - 1 \right| \|F(x)\|$$

$$+ \int_Y |\chi_{\alpha,\lambda}(x;y)| \|F(\xi_{\alpha,\lambda}(y)) - F(x)\| d\mu(y) = A^{(1)}_{\alpha,\lambda}(x) + A^{(2)}_{\alpha,\lambda}(x), \quad (9)$$

say. By (5) and the boundedness of F, $\lim_\alpha A^{(1)}_{\alpha,\lambda}(x) = 0$ uniformly in $\lambda \in \Lambda, x \in X_0$. Taking $\Psi(u,v) = \|F(u) - F(v)\|$ for all $(u,v) \in X^2$ in Lemma 1, (8) implies $\lim_\alpha A^{(2)}_{\alpha,\lambda}(x) = 0$ uniformly in $\lambda \in \Lambda, x \in X_0$. Thus, (9) yields (1).

Let Φ be a nonnegative real-valued function on $X_0 \times X$ which satisfies $\chi_{\alpha,\lambda}(x;\cdot)\Phi(x, \xi_{\alpha,\lambda}(\cdot)) \in L^1(Y,\mu)$ for each $\alpha \in D, \lambda \in \Lambda,\ x \in X_0$ and

$$M := \inf\{\Phi(x,t) : (x,t) \in X_0 \times X, d(x,t) \geq \delta\} > 0 \qquad (10)$$

for every $\delta > 0$.

Remark 3. If there exist positive numbers q and C such that

$$d^q(x,t) \leq C\Phi(x,t) \quad \text{for all } (x,t) \in X_0 \times X,$$

then (10) is automatically fulfilled.

We set

$$\tau_{\alpha,\lambda}(x;\Phi) = \|\chi_{\alpha,\lambda}(x;\cdot)\Phi(x,\xi_{\alpha,\lambda}(\cdot))\|_1 \qquad (\alpha \in D,\ \lambda \in \Lambda,\ x \in X_0),$$

which is called the absolute Φ-moment of $\chi_{\alpha,\lambda}$ at x. In paricular, if $q > 0,\ x \in X$ and $\chi_{\alpha,\lambda}(x;\cdot)d^q(x,\xi_{\alpha,\lambda}(\cdot)) \in L^1(Y,\mu)$, then the quantity

$$\mu_{\alpha,\lambda}(x;q) := \tau_{\alpha,\lambda}(x;d^q) = \|\chi_{\alpha,\lambda}(x;\cdot)d^q(x,\xi_{\alpha,\lambda}(\cdot))\|_1$$

is called the qth absolute moment of $\chi_{\alpha,\lambda}$ at x.

Lemma 2. *If*

$$\lim_\alpha \tau_{\alpha,\lambda}(x;\Phi) = 0 \qquad \textit{uniformly in } \lambda \in \Lambda, x \in X_0, \qquad (11)$$

then (6) holds for every $\delta > 0$.

Proof. Let $\delta > 0$ and $x \in X_0$. Then by (10), we have

$$\int_{d(x,\xi_{\alpha,\lambda}(y)) \geq \delta} |\chi_{\alpha,\lambda}(x;y)|\, d\mu(y) \leq \frac{1}{M}\tau_{\alpha,\lambda}(x;\Phi),$$

which together with (11) implies (6).

Theorem 2. *If (4), (5) and (11) hold, then $\mathfrak{K}$ is an equi-uniform approximation process on $BC(X, E)$.*

Proof. This immediately follows from Lemma 2 and Theorem 1.

Corollary 1. *If $\mathfrak{A}$ is positive, and if (5) and (11) hold, then $\mathfrak{K}$ is an equi-uniform approximation process on $BC(X, E)$.*

Corollary 2. *Let $\{\Phi_1, \Phi_2, \dots, \Phi_r\}$ be a finite subset of nonnegative real-valued functions on $X_0 \times X$ such that $\chi_{\alpha,\lambda}(x; \cdot)\Phi_i(x, \xi_{\alpha,\lambda}(\cdot)) \in L^1(Y, \mu)$ for each $\alpha \in D, \lambda \in \Lambda, x \in X_0$ and for $i = 1, 2, \dots, r$. Suppose that there exist positive constants q and C such that*

$$d^q(x, t) \le C \sum_{i=1}^{r} \Phi_i(x, t) \qquad \text{for all } (x, t) \in X_0 \times X.$$

If $\mathfrak{A}$ is positive and (5) holds, and if

$$\lim_{\alpha} \tau_{\alpha,\lambda}(x; \Phi_i) = 0 \qquad \text{uniformly in } \lambda \in \Lambda,\ x \in X_0$$

for $i = 1, 2, \dots, r$, then $\mathfrak{K}$ is an equi-uniform approximation process on $BC(X, E)$.

Theorem 3. *If (4) and (5) hold, and if for some $q > 0$,*

$$\lim_{\alpha} \mu_{\alpha,\lambda}(x; q) = 0 \qquad \text{uniformly in } \lambda \in \Lambda, x \in X_0, \tag{12}$$

then $\mathfrak{K}$ is an equi-uniform approximation process on $BC(X, E)$.

Proof. By taking $\Phi = d^q$, this follows from Theorem 2, immediately.

Corollary 3. *If $\mathfrak{A}$ is positive, and if (5) and (12) hold, then $\mathfrak{K}$ is an equi-uniform approximation process on $BC(X, E)$.*

Next we consider the function Φ which has the following special form:

$$\Phi(x, t) = \sum_{i=1}^{m} a_i(x) w_i(t) \ge 0 \ \ ((x, t) \in X_0 \times X), \ \ \Phi(x, x) = 0 \ \ (x \in X_0),$$

where $a_i, 1 \le i \le m$, are bounded functions on X_0 and $w_i, 1 \le i \le m$, are continuous functions on X such that $\chi_{\alpha,\lambda}(x; \cdot) w_i(\xi_{\alpha,\lambda}(\cdot)) \in L^1(Y, \mu)$ for each $\alpha \in D, \lambda \in \Lambda, x \in X_0$ and for $i = 1, 2, \dots, m$. If w is a continuous function on X such that $\chi_{\alpha,\lambda}(x; \cdot) w(\xi_{\alpha,\lambda}(\cdot)) \in L^1(Y, \mu)$ for each $\alpha \in D, \lambda \in \Lambda$ and $x \in X_0$, then we define

$$\nu_{\alpha,\lambda}(w)(x) = \int_Y \chi_{\alpha,\lambda}(x; y) w(\xi_{\alpha,\lambda}(y))\, d\mu(y).$$

1_X denotes the unit function defined by $1_X(t) = 1$ for all $t \in X$. Then we have the following Korovkin type results for integral operators. For an excellent source for references and a systematic treatment of Korovkin type approximation theory, we refer to the book of Altomare and Campiti [1] (cf. [8]).

Theorem 4. *Let $W = \{1_X, w_1, w_2, \dots, w_m\}$. Suppose that (10) is satisfied for any $\delta > 0$ and that $\mathfrak{A}$ is positive. If for all $w \in W$,*

$$\lim_{\alpha} \nu_{\alpha,\lambda}(w)(x) = w(x) \quad \textit{uniformly in } \lambda \in \Lambda, x \in X_0,$$

then $\mathfrak{K}$ is an equi-uniform approximation process on $BC(X, E)$.

Proof. Since

$$\tau_{\alpha,\lambda}(x;\Phi) \le \sum_{i=1}^{m} |a_i(x)||\nu_{\alpha,\lambda}(w_i)(x) - w_i(x)| \qquad (\alpha \in D, \lambda \in \Lambda, x \in X_0),$$

the desired result follows from Corollary 1.

Let s be any fixed even positive integer, and let

$$H_s = \{h_i^j : i = 1, 2, \dots, r, j = 1, 2, \dots, s\},$$

where $\{h_1, h_2, \dots, h_r\}$ is a finite subset of $C(X, \mathbb{R})$ such that

$$\chi_{\alpha,\lambda}(x;\cdot)h(\xi_{\alpha,\lambda}(\cdot)) \in L^1(Y, \mu)$$

for each $\alpha \in D, \lambda \in \Lambda, x \in X_0$ and $h \in H_s$. Let

$$W_s = \{1_X, h_1^s + h_2^s + \dots + h_r^s\} \cup H_{s-1}.$$

Suppose that the function Φ defined by

$$\Phi(x, t) = \sum_{i=1}^{r} (h_i(x) - h_i(t))^s \quad ((x, t) \in X_0 \times X)$$

satisfies (10) for any $\delta > 0$.

Theorem 5. *If $\mathfrak{A}$ is positive and if for all $h \in W_s$,*

$$\lim_{\alpha} \nu_{\alpha,\lambda}(h)(x) = h(x) \quad \textit{uniformly in } \lambda \in \Lambda, x \in X_0, \qquad (13)$$

then $\mathfrak{K}$ is an equi-uniform approximation process on $BC(X, E)$.

Proof. For all $(x,t) \in X_0 \times X$, We have

$$\Phi(x,t) = \sum_{i=1}^{r}\Big(\sum_{j=0}^{s-1}\binom{s}{j}(-h_i(x))^{s-j}h_i(t)^j\Big) + \sum_{i=1}^{r} h_i(t)^s.$$

Therefore, the desired result follows from Theorem 4.

In order to refine Theorem 5, we make the following definition:

Let E_0 be a subset of E. Let $\mathfrak{T} = \{T(x) : x \in X\}$ and $\mathfrak{L} = \{L_{\alpha,\lambda}(x) : \alpha \in D, \lambda \in \Lambda, x \in X\}$ be families of mappings of E_0 into E. Then $\mathfrak{L}$ is called an equi-uniform $\mathfrak{T}$-approximation process on E_0 if for every $f \in E_0$,

$$\lim_{\alpha} \|L_{\alpha,\lambda}(x)(f) - T(x)(f)\| = 0 \qquad \text{uniformly in } \lambda \in \Lambda,\ x \in X_0. \quad (14)$$

Now, we suppose that for each $f \in E_0$, the orbit mapping $x \mapsto T(x)(f)$ is strongly continuous and bounded on X and define

$$L_{\alpha,\lambda}(x)(f) = \int_Y \chi_{\alpha,\lambda}(x;y)T(\xi_{\alpha,\lambda}(y))(f)\,d\mu(y) \qquad (f \in E_0) \qquad (15)$$

(cf. [17]). Shaw [19] considered the special case of (15) in the setting of certain spaces of operator-valued functions and obtained several representation formulas for strongly continuous semigroups of bounded linear operators on Banach spaces. For the families $\mathfrak{K}$ and $\mathfrak{L}$ of integral operators defined by (2) and (15), the limit-relation (1) is the same as (14) for each function $F(\cdot) = T(\cdot)(f), f \in E_0$.

Theorem 6. *Assume that $\mathfrak{A}$ is positive, and consider the following assertions :*

(a) (13) holds for every $h \in \{1_X\} \cup H_s$.

(b) (13) holds for every $h \in W_s$.

(c) $\mathfrak{K}$ is an equi-uniform approximation process on $BC(X,E)$.

(d) $\mathfrak{L}$ is an equi-uniform $\mathfrak{T}$-approximation process on E_0 for every $\mathfrak{T}$.

(e) The family $\{\nu_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ is an equi-uniform approximation process on $BC(X,\mathbb{R})$.

Then the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) hold. In particular, if $\{h_1, h_2, \cdots, h_r\} \subseteq BC(X,\mathbb{R})$, then the statements (a) $-$ (e) are equivalent.

Proof. In view of Theorem 5, it will be sufficient to show that (d) implies (e). Let $w \in BC(X,\mathbb{R})$ and we take the special family

$\mathfrak{T} = \{w(x)I : x \in X\}$, where I denotes the identity operator on E. Let $f \in E_0$ and $f \neq 0$. Then (14) and (15) yield

$$\lim_{\alpha} \|f\| |\nu_{\alpha,\lambda}(w)(x) - w(x)| = 0 \qquad \text{uniformly in } \lambda \in \Lambda, x \in X_0,$$

and so $\{\nu_{\alpha,\lambda}\}$ becomes an equi-uniform approximation process on $BC(X, \mathbb{R})$.

In the special case where $q = s = 2$, there is a wide variety of applications of Corollary 2, Theorems 5 and 6 to equi-uniform approximation processes of integral operators associated with positive and normal kernels.

3. Equi-uniform $\mathcal{A}$-summation processes

Let $\mathcal{A} = \{a_{\alpha,m}^{(\lambda)} : \alpha \in D, m \in \mathbb{N}_0, \lambda \in \Lambda\}$ be a family of scalars and let $\{\chi_m(x; \cdot) : m \in \mathbb{N}_0, x \in X\}$ be a family of functions in $L^1(Y, \mu)$ such that

$$\sum_{m=0}^{\infty} \int_Y |a_{\alpha,m}^{(\lambda)} \chi_m(x; y)| \, d\mu(y) < \infty$$

for each $\alpha \in D, \lambda \in \Lambda$ and $x \in X$. We define

$$\chi_{\alpha,\lambda}(x; \cdot) = \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} \chi_m(x; \cdot) \qquad (\alpha \in D, \lambda \in \Lambda, x \in X),$$

which belongs to $L^1(Y, \mu)$. Then by (2) and (15), we have

$$K_{\alpha,\lambda}(F)(x) = \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} \int_Y \chi_m(x; y) F(\xi_{\alpha,\lambda}(y)) \, d\mu(y)$$

$$(F \in BC(X, E),\ \alpha \in D,\ \lambda \in \Lambda,\ x \in X)$$

and

$$L_{\alpha,\lambda}(x)(f) = \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} \int_Y \chi_m(x; y) T(\xi_{\alpha,\lambda}(y))(f) \, d\mu(y)$$

$$(f \in E_0,\ \alpha \in D,\ \lambda \in \Lambda,\ x \in X).$$

In particular, if $Y = X$ and $\xi_{\alpha,\lambda}(y) = y$ for all $y \in Y, \alpha \in D$ and $\lambda \in \Lambda$, then we have

$$K_{\alpha,\lambda}(F)(x) = \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} K_m(F)(x)$$

$$(F \in BC(X, E),\ \alpha \in D,\ \lambda \in \Lambda,\ x \in X),$$

where

$$K_n(F)(x) = \int_X \chi_n(x;y)F(y)\,d\mu(y) \qquad (n \in \mathbb{N}_0) \tag{16}$$

and

$$L_{\alpha,\lambda}(x)(f) = \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} T_m(x)(f)$$

$$(f \in E_0,\ \alpha \in D,\ \lambda \in \Lambda,\ x \in X),$$

where

$$T_n(x)(f) = \int_X \chi_n(x;y)T(y)(f)\,d\mu(y) \qquad (n \in \mathbb{N}_0). \tag{17}$$

Consequently, all the results obtained in the preceding section are applied in the above setting, and these approximation processes can be concerned with the summability methods by the family $\mathcal{A}$ defined as follows (cf. [13], [14]):

$\mathcal{A}$ is said to be regular if it satisfies the following conditions:

(A-1) For each $m \in \mathbb{N}_0$,

$$\lim_{\alpha} a_{\alpha,m}^{(\lambda)} = 0 \qquad \text{uniformly in } \lambda \in \Lambda.$$

(A-2) $\lim_\alpha \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} = 1$ uniformly in$\lambda \in \Lambda$.

(A-3) For each $\alpha \in D, \lambda \in \Lambda$,

$$a_{\alpha}^{(\lambda)} := \sum_{m=0}^{\infty} |a_{\alpha,m}^{(\lambda)}| < \infty,$$

and there exists $\alpha_0 \in D$ such that

$$\sup\{a_{\alpha}^{(\lambda)} : \alpha \geq \alpha_0, \alpha \in D,\ \lambda \in \Lambda\} < \infty.$$

$\mathcal{A}$ is said to be positive if

$$a_{\alpha,m}^{(\lambda)} \geq 0 \qquad \text{for all } \alpha \in D, m \in \mathbb{N}_0 \text{ and } \lambda \in \Lambda.$$

Also, $\mathcal{A}$ is said to be stochastic if it is positive and

$$\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} = 1 \qquad \text{for all } \alpha \in D \text{ and } \lambda \in \Lambda.$$

Obviously, if $\mathcal{A}$ is positive, then (A-2) already implies (A-3) and if $\mathcal{A}$ is stochastic, then Conditions (A-2) and (A-3) are automatically satisfied.

A sequence $\{f_m\}_{m\in\mathbb{N}_0}$ of elements in E is said to be $\mathcal{A}$-summable to f if

$$\lim_{\alpha}\Big\|\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} f_m - f\Big\| = 0 \qquad \text{uniformly in } \lambda\in\Lambda, \tag{18}$$

where it is assumed that the series in (18) converges for each $\alpha\in D$ and $\lambda\in\Lambda$.

Concerning the relation between the regularity of $\mathcal{A}$ and the $\mathcal{A}$-summability, $\mathcal{A}$ is regular if and only if every convergent sequence in E is $\mathcal{A}$-summable to its limit (cf. [2], [13]).

As the following examples with $D = \mathbb{N}_0$ show, there is a wide variety of families $\mathcal{A}$ of particular interest which cover many important summation methods scattered in the literature.

(1°) Given a matrix $A = (a_{n,m})_{n,m\in\mathbb{N}_0}$, if $a_{n,m}^{(\lambda)} = a_{n,m}$ for all $n,m\in\mathbb{N}_0$ and $\lambda\in\Lambda$, then we obtain the usual matrix summability by A.

(2°) If $\Lambda = \mathbb{N}_0$, then we obtain the summation method by introduced by Petersen [18] (cf. [2]). In particular, if

$$a_{n,m}^{(\lambda)} = \begin{cases} \frac{1}{n}, & \text{if } \lambda\le m\le \lambda+n-1,\\ 0, & \text{otherwise,}\end{cases}$$

then we obtain the notion of almost convergence method introduced by Lorentz [9].

(3°) Let $Q = \{q^{(\lambda)} : \lambda\in\Lambda\}$ be a familiy of sequences $q^{(\lambda)} = \{q_m^{(\lambda)}\}_{m\in\mathbb{N}_0}$ of nonnegative real numbers such that

$$Q_n^{(\lambda)} = q_0^{(\lambda)} + q_1^{(\lambda)} + \cdots + q_n^{(\lambda)} > 0 \quad (n\in\mathbb{N}_0).$$

We define

$$a_{n,m}^{(\lambda)} = \begin{cases} \frac{q_{n-m}^{(\lambda)}}{Q_n^{(\lambda)}}, & \text{if } m\le n,\\ 0, & \text{if } m > n.\end{cases}$$

Then $\mathcal{A}$-summability method is called (N,Q)-summability method and in particular, if $q^{(\lambda)} = \{q_m\}_{m\in\mathbb{N}_0}$ is a fixed sequence of nonnegative real numbers satisfying $q_0 > 0$, this reduces to the classical Nörlund summability method. The special case of interest is the following:

Let $\Lambda\subseteq[0,\infty)$, $\beta > 0$ and

$$q_m^{(\lambda)} = C_m^{(\lambda+\beta-1)} \quad (\lambda\in\Lambda,\ m\in\mathbb{N}_0),$$

where $\tau > -1$ and

$$C_0^{(\tau)} = 1, \quad C_m^{(\tau)} = \binom{m+\tau}{m} = \frac{(\tau+1)(\tau+2)\cdots(\tau+m)}{m!} \qquad (m \in \mathbb{N}).$$

In particular, if $\Lambda = \{0\}$, then we have the Cesàro summability of order β.

(4°) Let $\Lambda \subseteq (0, \infty)$, $\beta > -1$ and define

$$a_{n,m}^{(\lambda)} = \begin{cases} C_{n-m}^{(\lambda-1)} C_m^{(\beta)} / C_n^{(\beta+\lambda)}, & \text{if } \ m \leq n, \\ 0, & \text{if } \ m > n. \end{cases}$$

(*Cesàro type*).

(5°) Let $\Lambda \subseteq [0, 1]$ and define

$$a_{n,m}^{(\lambda)} = \begin{cases} \binom{n}{m} \lambda^m (1-\lambda)^{n-m}, & \text{if } \ m \leq n, \\ 0, & \text{if } \ m > n \end{cases}$$

(*Euler-Knopp-Bernstein type*).

(6°) Let $\Lambda \subseteq [0, 1)$ and define

$$a_{n,m}^{(\lambda)} = \binom{n+m}{m} \lambda^m (1-\lambda)^{n+1}$$

(*Meyer-König-Vermes-Zeller type*).

(7°) Let $\Lambda \subseteq [0, \infty)$ and define

$$a_{n,m}^{(\lambda)} = \exp(-n\lambda) \frac{(n\lambda)^m}{m!}$$

(*Borel-Szász type*).

(8°) Let $\Lambda \subseteq [0, \infty)$ and define

$$a_{n,m}^{(\lambda)} = \binom{n+m-1}{m} \lambda^m (1+\lambda)^{-n-m}$$

(*Baskakov type*).

Note that all the families $\mathcal{A}$ of the generic entories $a_{n,m}^{\lambda)}$ given in the above Examples (2°)-(8°) are stochastic and all the families $\mathcal{A}$ of the generic entories $a_{n,m}^{(\lambda)}$ given in the above Examples (4°)-(8°)) are regular for any finite interval Λ.

Now, in view of the definition of $\mathcal{A}$-summability method, we say that a sequence $\{U_n : n \in \mathbb{N}_0\}$ of operators of $BC(X, E)$ into $B(X, E)$ is an equi-uniform $\mathcal{A}$-summation process on $BC(X, E)$ if the family

$\mathfrak{U} = \{U_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ is an equi-uniform approximation process on $BC(X, E)$, where each operator $U_{\alpha,\lambda}$ is defined by

$$U_{\alpha,\lambda}(F)(x) = \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} U_m(F)(x) \qquad (F \in BC(X,E),\ x \in X),$$

which is assumed to be convergent. Also, a family $\{V_n(x) : n \in \mathbb{N}_0, x \in X\}$ of operators of E_0 into E is called an equi-unifom $\mathfrak{T}$-$\mathcal{A}$-summation process on E_0 if the family $\mathfrak{V} = \{V_{\alpha,\lambda}(x) : \alpha \in D, \lambda \in \Lambda, x \in X\}$ is an equi-uniform $\mathfrak{T}$- approximation process on E_0, where each operator $V_{\alpha,\lambda}(x)$ is defined by

$$V_{\alpha,\lambda}(x)(f) = \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} V_m(x)(f) \qquad (f \in E_0,\ x \in X),$$

which is assumed to be convergent.

Let $\{\xi_n : n \in \mathbb{N}_0\}$ be a sequence of continuous functions of Y into X and let $\{\chi_n(x;\cdot) : n \in \mathbb{N}_0, x \in X\}$ be a family of functions in $L^1(Y,\mu)$. We define

$$U_n(F)(x) = \int_Y \chi_n(x;y) F(\xi_n(y))\, d\mu(y) \tag{19}$$
$$(F \in BC(X,E),\ n \in \mathbb{N}_0,\ x \in X)$$

and

$$V_n(x)(f) = \int_Y \chi_n(x;y) T(\xi_n(y))(f)\, d\mu(y) \tag{20}$$
$$(f \in E_0,\ n \in \mathbb{N}_0,\ x \in X).$$

Then we have the following:

Theorem 7. *Suppose that $\mathcal{A}$ is positive and that Conditions (A-1) and (A-2) are satisfied. Furthermore, suppose that $\{\chi_n(x;\cdot)\}$ is approximate kernel, i.e., Conditions (4), (5) and (6) hold uniformly in $x \in X_0$ with $\alpha = n \in \mathbb{N}_0, \chi_{\alpha,\lambda}(x;\cdot) = \chi_n(x;\cdot)$ and $\xi_{\alpha,\lambda} = \xi_n$. Then the following statements hold:*

(a) Let $\mathcal{U} = \{U_n : n \in \mathbb{N}_0\}$ be the sequence of integral operators defined by (19) and $\mathcal{V} = \{V_n(x) : n \in \mathbb{N}_0, x \in X\}$ the family of integral operators defined by (20). Then $\mathfrak{U}$ is an equi-uniform $\mathcal{A}$-summation process on $BC(X,E)$, and so $\mathfrak{V}$ is an equi-uniform $\mathfrak{T}$-$\mathcal{A}$-summation process on E_0. In particular, if $\{\chi_n(x;\cdot)\}$ is positive, and if

$$\lim_{n\to\infty} \int_Y \chi_n(x;y)\, d\mu(y) = 1 \qquad \textit{uniformly in } x \in X_0$$

and

$$\lim_{n\to\infty} \|\chi_n(x;\cdot)d^q(x,\xi_n(\cdot))\|_1 = 0 \qquad \textit{uniformly in } x \in X_0$$

for some $q > 0$, then $\mathfrak{U}$ is an equi-uniform $\mathcal{A}$-summation process on $BC(X,E)$, and so $\mathfrak{V}$ is an equi-uniform $\mathfrak{T}$-$\mathcal{A}$-summation process on E_0.

(b) Let $\mathcal{K} = \{K_n : n \in \mathbb{N}_0\}$ be the sequence of integral operators defined by (16) and $\mathcal{T} = \{T_n(x) : n \in \mathbb{N}_0, x \in X\}$ the family of integral operators defined by (17). Then $\mathcal{K}$ is an equi-uniform $\mathcal{A}$-summation process on $BC(X,E)$, and so $\mathcal{T}$ is an equi-uniform $\mathfrak{T}$-$\mathcal{A}$-summation process on E_0. In particular, if $\{\chi_n(x;\cdot)\}$ is positive, and if

$$\lim_{n\to\infty} \int_X \chi_n(x;y)\,d\mu(y) = 1 \qquad \textit{uniformly in } x \in X_0$$

and

$$\lim_{n\to\infty} \|\chi_n(x;\cdot)d^q(x,\cdot)\|_1 = 0 \qquad \textit{uniformly in } x \in X_0$$

for some $q > 0$, then $\mathcal{K}$ is an equi-uniform $\mathcal{A}$-summation process on $BC(X,E)$, and so $\mathcal{T}$ is an equi-uniform $\mathfrak{T}$-$\mathcal{A}$-summation process on E_0.

Proof. (a) By Theorem 1 (cf. [17, Theorem 1]), there holds

$$\lim_{n\to\infty} \|U_n(F)(x) - F(x)\| = 0 \qquad \text{uniformly in } x \in X_0,$$

and so the desired result follows from the regularity of $\mathcal{A}$.

(b) This immediately follows from Part (a).

4. Interpolation type operators

Let Y be a finite set. Then the integral operators defined by (2) and (15) reduce to

$$K_{\alpha,\lambda}(F)(x) = \sum_{k\in Y} \chi_{\alpha,\lambda}(x;k)F(\xi_{\alpha,\lambda}(k)) \qquad (F \in BC(X,E)) \tag{21}$$

and

$$L_{\alpha,\lambda}(x)(f) = \sum_{k\in Y} \chi_{\alpha,\lambda}(x;k)T(\xi_{\alpha,\lambda}(k))(f) \qquad (f \in E_0), \tag{22}$$

respectively. These are called interpolation type operators with the interpolation system $\{\chi_{\alpha,\lambda}(\cdot;k) : k \in Y\}$ and nodes $\{\xi_{\alpha,\lambda}(k) : k \in Y\}$. Also, we have

$$\mu_{\alpha,\lambda}(x;q) = \sum_{k\in Y} |\chi_{\alpha,\lambda}(x;k)| d^q(x, \xi_{\alpha,\lambda}(k)) \qquad (q > 0,\ x \in X).$$

Here we restrict ourselves to the following situation:

Let $1 \leq p \leq \infty$ be fixed and let X be a locally closed subset of the r-dimensional Euclidean space $\mathbb{R}^r$ with the metric

$$d(x,t) = d_p(x,t) := \begin{cases} \left(\sum_{i=1}^r |x_i - t_i|^p \right)^{1/p} & (1 \leq p < \infty) \\ \max\{|x_i - t_i| : 1 \leq i \leq r\} & (p = \infty), \end{cases}$$

where $x = (x_1, x_2, \ldots, x_r), t = (t_1, t_2, \ldots, t_r) \in \mathbb{R}^r$. Therefore, in view of Remark 1, Condition (3) holds. For $i = 1, 2, \ldots, r$, p_i denotes the ith coordinate function defined by $p_i(x) = x_i$ for all $x = (x_1, x_2, \ldots, x_r) \in \mathbb{R}^r$. Then we have

$$d_p^q(x,t) \leq c(p,q,r) \sum_{i=1}^r |p_i(x) - p_i(t)|^q \qquad (x, t \in \mathbb{R}^r,\ q > 0) \tag{23}$$

for some constant $c(p,q,r) > 0$. Consequently, in view of (23) and Remark 3, by taking

$$\Phi(x,t) = \sum_{i=1}^r (h_i(x) - h_i(t))^s, \quad h_i = p_i \qquad (i = 1, 2, \ldots, r),$$

Theorem 6 applied in the above setting and so we have a far-reaching generalization of [5, Theorem 1.2.6]. Also, as an immeditate consequence of Corollary 2 we have the following result which can be more convenient for later applications to the concrete examples of interpolation type operators.

Theorem 8. *Suppose that $\mathfrak{A}$ is positive and normal. If*

$$\lim_{\alpha} \sum_{k \in Y} \chi_{\alpha,\lambda}(x;k) |p_i(x) - p_i(\xi_{\alpha,\lambda}(k))|^q = 0 \quad \textit{uniformly in } \lambda \in \Lambda, x \in X_0$$

for some $q > 0$ and for $i = 1, 2, \ldots, r$, then $\mathfrak{K}$ is an equi-uniform approximation process on $BC(X, E)$, and so $\mathfrak{L}$ is an equi-uniform $\mathfrak{T}$-approximation process on E_0.

Let

$$X = [0, \infty)^r := \{x = (x_1, x_2, \dots, x_r) \in \mathbb{R}^r : x_i \ge 0, i = 1, 2, \dots, r\}$$

be the region of the first hyperquadrant and let

$$m_{\alpha,i} : \Lambda \to \mathbb{N}, \quad a_{\alpha,i} : \Lambda \to (0, \infty) \qquad (\alpha \in D, i = 1, 2, \dots, r). \quad (24)$$

Let X_0 be a closed subset of $\mathbb{I}_r$, where

$$\mathbb{I}_r := \{x = (x_1, x_2, \dots, x_r) \in X : 0 \le x_i \le 1, i = 1, 2, \dots, r\}$$

is the unit r-cube and

$$I_{\alpha,\lambda} := \{k = (k_1, k_2, \dots, k_r) \in \mathbb{N}_0^r : 0 \le k_i \le m_{\alpha,i}(\lambda), 1 \le i \le r\}.$$

Then we define the corresponding interpolation type operators (21) and (22) by

$$B_{\alpha,\lambda}(F)(x) = \sum_{k \in I_{\alpha,\lambda}} \prod_{i=1}^{r} \binom{m_{\alpha,i}(\lambda)}{k_i} x_i^{k_i} (1 - x_i)^{m_{\alpha,i}(\lambda) - k_i}$$

$$\times F(a_{\alpha,1}(\lambda)k_1, a_{\alpha,2}(\lambda)k_2, \dots, a_{\alpha,r}(\lambda)k_r) \qquad (F \in BC(X, E), x \in X)$$

and

$$C_{\alpha,\lambda}(x)(f) = \sum_{k \in I_{\alpha,\lambda}} \prod_{i=1}^{r} \binom{m_{\alpha,i}(\lambda)}{k_i} x_i^{k_i} (1 - x_i)^{m_{\alpha,i}(\lambda) - k_i}$$

$$\times T(a_{\alpha,1}(\lambda)k_1, a_{\alpha,2}(\lambda)k_2, \dots, a_{\alpha,r}(\lambda)k_r)(f) \qquad (f \in E_0, \ x \in X),$$

respectively.

Theorem 9. *If, for* $i = 1, 2, \dots, r$,

$$\lim_{\alpha} a_{\alpha,i}(\lambda) m_{\alpha,i}(\lambda) = 1 \qquad \textit{uniformly in } \lambda \in \Lambda$$

and

$$\lim_{\alpha} a_{\alpha,i}(\lambda)^2 m_{\alpha,i}(\lambda) = 0 \qquad \textit{uniformly in } \lambda \in \Lambda,$$

then $\{B_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ *is an equi-uniform approximation process on* $BC(X, E)$*, and so* $\{C_{\alpha,\lambda}(x) : \alpha \in D, \lambda \in \Lambda, x \in X\}$ *is an equi-uniform* $\mathfrak{T}$*-approximation process on* E_0.

Proof. We define

$$\chi_{\alpha,\lambda}(x; k) = \prod_{j=1}^{r} \binom{m_{\alpha,j}(\lambda)}{k_j} x_j^{k_j} (1 - x_j)^{m_{\alpha,j}(\lambda) - k_j} \qquad (x \in X, k \in I_{\alpha,\lambda})$$

and

$$\xi_{\alpha,\lambda}(k) = (a_{\alpha,1}(\lambda)k_1, a_{\alpha,2}(\lambda)k_2, \dots, a_{\alpha,r}(\lambda)k_r) \qquad (k \in I_{\alpha,\lambda}).$$

Then for all $\alpha \in D, \lambda \in \Lambda, x \in X_0$ and for $i = 1, 2, \ldots, r$ we have

$$\sum_{k \in I_{\alpha,\lambda}} \chi_{\alpha,\lambda}(x;k)|p_i(x) - p_i(\xi_{\alpha,\lambda}(k))|^2$$

$$= (a_{\alpha,i}(\lambda)m_{\alpha,i}(\lambda) - 1)^2 p_i(x)^2 + a_{\alpha,i}(\lambda)^2 m_{\alpha,i}(\lambda)(p_i(x)(1 - p_i(x))).$$

Therefore, applying Theorem 8 with $q = 2$ we obtain the desired result.

We assume that

$$a_{\alpha,i}(\lambda)m_{\alpha,i}(\lambda) = 1 \qquad (\alpha \in D, i = 1, 2, \ldots, r) \tag{25}$$

for all $\lambda \in \Lambda$ and we define $B_{\alpha,\lambda}(F)(x)$ and $C_{\alpha,\lambda}(x)(f)$ for all $F \in C(\mathbb{I}_r, E), f \in E_0$ and $x \in \mathbb{I}_r$. Then we have the following:

Corollary 4. *If, for* $i = 1, 2, \ldots, r$,

$$\lim_{\alpha} m_{\alpha,i}(\lambda) = +\infty \qquad \textit{uniformly in } \lambda \in \Lambda, \tag{26}$$

then $\{B_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ *is an equi-uniform approximation process on* $C(\mathbb{I}_r, E)$*, and so* $\{C_{\alpha,\lambda}(x) : \alpha \in D, \lambda \in \Lambda, x \in \mathbb{I}_r\}$ *is an equi-uniform* $\mathfrak{T}$*-approximation process on* E_0.

Let $D = \mathbb{N}_0$ and let $\{\nu_{n,i}\}_{n \in \mathbb{N}_0}, i = 1, 2, \ldots, r$, be strictly monotone increasing sequences of positive integers. We define

$$m_{n,i}(\lambda) = \nu_{n,i}, \quad a_{n,i}(\lambda) = \frac{1}{\nu_{n,i}} \qquad (n \in \mathbb{N}_0, i = 1, 2, \ldots, r)$$

for all $\lambda \in \Lambda$, and so (25) holds with $\alpha = n \in \mathbb{N}_0$. Then we write $B_{\nu_{n,1},\nu_{n,2},\cdots,\nu_{n,r}}$ and $C_{\nu_{n,1}\nu_{n,2},\cdots,\nu_{n,r}}$ instead of $B_{n,\lambda}$ and $C_{n,\lambda}$, respectively. Therefore, as an immediate consequence of Corollary 4 we have the following (cf. [15]):

$$\lim_{n \to \infty} \|B_{\nu_{n,1},\nu_{n,2},\cdots,\nu_{n,r}}(F)(x) - F(x)\| = 0 \qquad \text{uniformly in } x \in \mathbb{I}_r \tag{27}$$

for all $F \in C(\mathbb{I}_r, E)$, and so

$$\lim_{n \to \infty} \|C_{\nu_{n,1},\nu_{n,2},\cdots,\nu_{n,r}}(x)(f) - T(x)(f)\| = 0 \quad \text{uniformly in } x \in \mathbb{I}_r. \tag{28}$$

for all $f \in E_0$

The statement (27) generalizes the uniform convergence theorem [10] (cf. [4], [6]) for the r-dimensional Bernstein operators on $C(\mathbb{I}_r, \mathbb{R})$ and also the statement (28) generalizes [5, Corollary 1.2.8] to the multi-dimensionnal case. More generally, Theorem 7 (a) establishes that if $\mathcal{A}$ is stochastic with (A-1), then $\{B_{\nu_{n,1},\nu_{n,2},\cdots,\nu_{n,r}} : n \in \mathbb{N}_0\}$ is an equi-uniform $\mathcal{A}$-summation process on $C(\mathbb{I}_r, E)$, and so $\{C_{\nu_{n,1},\cdots,\nu_{n,r}}(x) : n \in \mathbb{N}_0, x \in \mathbb{I}_r\}$ is an equi-uniform $\mathfrak{T}$-$\mathcal{A}$-summation process on E_0.

Next, let X_0 be a closed subset of Δ_r, where

$$\Delta_r := \{x = (x_1, x_2, \dots, x_r) \in \mathbb{R}^r : x_i \geq 0, 1 \leq i \leq r, \sum_{i=1}^{r} x_i \leq 1\}$$

is the standard r-simplex. Let $m_\alpha : \Lambda \to \mathbb{N}, \alpha \in D$ and let $\{a_{\alpha,i}\}$ be as in (24) and

$$J_{\alpha,\lambda} := \{k = (k_1, k_2, \dots, k_r) \in \mathbb{N}_0^r : k_1 + k_2 + \dots + k_r \leq m_\alpha(\lambda)\}.$$

Now we define the corresponding interpolation type operators (21) and (22) by

$$B_{\alpha,\lambda}(F)(x) = \sum_{k \in J_{\alpha,\lambda}} \binom{m_\alpha(\lambda)}{k} \prod_{i=1}^{r} x_i^{k_i} \Big(1 - \sum_{j=1}^{r} x_j\Big)^{m_\alpha(\lambda) - \sum_{j=1}^{r} k_j}$$

$$\times F(a_{\alpha,1}(\lambda)k_1, a_{\alpha,2}(\lambda)k_2, \dots, a_{\alpha,r}(\lambda)k_r) \qquad (F \in BC(X,E), x \in X)$$

and

$$C_{\alpha,\lambda}(x)(f) = \sum_{k \in J_{\alpha,\lambda}} \binom{m_\alpha(\lambda)}{k} \prod_{i=1}^{r} x_i^{k_i} \Big(1 - \sum_{j=1}^{r} x_j\Big)^{m_\alpha(\lambda) - \sum_{j=1}^{r} k_j}$$

$$\times T(a_{\alpha,1}(\lambda)k_1, a_{\alpha,2}(\lambda)k_2, \dots, a_{\alpha,r}(\lambda)k_r)(f) \qquad (f \in E_0,\ x \in X),$$

where

$$\binom{m_\alpha(\lambda)}{k} = \frac{m_\alpha(\lambda)!}{k_1! k_2! \cdots k_r! (m_\alpha(\lambda) - k_1 - k_2 - \dots - k_r)!},$$

respectively.

Theorem 10. *If, for* $i = 1, 2, \dots, r$,

$$\lim_\alpha a_{\alpha,i}(\lambda) m_\alpha(\lambda) = 1 \qquad \textit{uniformly in } \lambda \in \Lambda$$

and

$$\lim_\alpha a_{\alpha,i}(\lambda)^2 m_\alpha(\lambda) = 0 \qquad \textit{uniformly in } \lambda \in \Lambda,$$

then $\{B_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ *is an equi-uniform approximation process on* $BC(X,E)$*, and so* $\{C_{\alpha,\lambda}(x) : \alpha \in D, \lambda \in \Lambda, x \in X\}$ *is an equi-uniform* $\mathfrak{T}$-$\mathcal{A}$*-summation process on* E_0.

Proof. This can be similar to the proof of Theorem 9.

Corollary 5. *Suppose that*

$$a_{\alpha,i}(\lambda)m_\alpha(\lambda) = 1 \qquad (\alpha \in D, i = 1, 2, \dots, r) \tag{29}$$

for all $\lambda \in \Lambda$ *and we define* $B_{\alpha,\lambda}(F)(x)$ *and* $C_{\alpha,\lambda}(x)(f)$ *for all* $F \in C(\Delta_r, E), f \in E_0$ *and* $x \in \Delta_r$. *If*

$$\lim_\alpha m_\alpha(\lambda) = +\infty \qquad \text{uniformly in } \lambda \in \Lambda,$$

then $\{B_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ *ia an equi-uniform approximation process on* $C(\Delta_r, E)$, *and so* $\{C_{\alpha,\lambda}(x) : \alpha \in D, \lambda \in \Lambda, x \in \Delta_r\}$ *is an equi-uniform* $\mathfrak{T}$*-approximation process on* E_0.

Let $D = \mathbb{N}_0$ and let $\{\nu_n\}_{n\in\mathbb{N}_0}$ be a strictly monotone increasing sequence of positive integers. We define

$$m_n(\lambda) = \nu_n, \quad a_{n,i}(\lambda) = \frac{1}{\nu_n} \qquad (\alpha \in D, i = 1, 2, \dots, r)$$

for all $\lambda \in \Lambda$, and so (29) holds with $\alpha = n \in \mathbb{N}_0$. Then we write B_{ν_n} and C_{ν_n} instead of $B_{n,\lambda}$ and $C_{n,\lambda}$, respectively. Therefore, as an immediate consequence of Corollary 5 we have the following (cf. [15]):

For all $F \in C(\Delta_r, E)$,

$$\lim_{n\to\infty} \|B_{\nu_n}(F)(x) - F(x)\| = 0 \qquad \text{uniformly in } x \in \Delta_r,$$

and so for all $f \in E_0$,

$$\lim_{n\to\infty} \|C_{\nu_n}(x)(f) - T(x)(f)\| = 0 \qquad \text{uniformly in } x \in \Delta_r.$$

This result generalizes the uniform convergence theorem [10] of the r-dimensional Bernstein operators on $C(\Delta_r, \mathbb{R})$. More generally, Theorem 7 (a) establishes that if $\mathcal{A}$ is stochastic with (A-1), then $\{B_{\nu_n} : n \in \mathbb{N}_0\}$ is an equi-uniform $\mathcal{A}$-summation process on on $C(\Delta_r, E)$, and so $\{C_{\nu_n}(x) : n \in \mathbb{N}_0, x \in \Delta_r\}$ is an equi-uniform $\mathfrak{T}$-$\mathcal{A}$-summation process on E_0

Let $X = \mathbb{R}^r$ and let X_0 be a closed subset of $X_r := [-1, 1]^r$. Let $\{m_{\alpha,i}\}, \{a_{\alpha,i}\}$ be as in (24) and

$$N_{\alpha,\lambda} := \{k = (k_1, k_2, \dots, k_r) \in \mathbb{N}^r : 1 \le k_i \le m_{\alpha,i}(\lambda), 1 \le i \le r\}.$$

Let $Q_n(t) = \cos(n \arccos t)$ be the Chebyshev polynomial of degree n and let $t_{n,j}, j = 1, 2, \dots, n$, be zeros of $Q_n(t)$, i.e.,

$$t_{n,j} = \cos\left(\frac{2j-1}{2n}\pi\right), \qquad (j = 1, 2, \dots, n).$$

Then we define the corresponding interpolation type operators (21) and (22) by

$$H_{\alpha,\lambda}(F)(x) = \sum_{k\in N_{\alpha,\lambda}} \prod_{i=1}^{r} (1 - x_i t_{m_{\alpha,i}(\lambda),k_i}) \left\{ \frac{Q_{m_{\alpha,i}(\lambda)}(x_i)}{m_{\alpha,i}(\lambda)(x_i - t_{m_{\alpha,i}(\lambda),k_i})} \right\}^2$$

$$\times F(a_{\alpha,1}(\lambda) t_{m_{\alpha,1}(\lambda),k_1}, \dots, a_{\alpha,r}(\lambda) t_{m_{\alpha,r}(\lambda),k_r})$$
$$(F \in BC(X,E),\ x \in X)$$

and

$$G_{\alpha,\lambda}(x)(f) = \sum_{k\in N_{\alpha,\lambda}} \prod_{i=1}^{r} (1 - x_i t_{m_{\alpha,i}(\lambda),k_i}) \left\{ \frac{Q_{m_{\alpha,i}(\lambda)}(x_i)}{m_{\alpha,i}(\lambda)(x_i - t_{m_{\alpha,i}(\lambda),k_i})} \right\}^2$$

$$\times T(a_{\alpha,1}(\lambda) t_{m_{\alpha,1}(\lambda),k_1}, \dots, a_{\alpha,r}(\lambda) t_{m_{\alpha,r}(\lambda),k_r})(f)$$
$$(f \in E_0,\ x \in X),$$

respectively.

Theorem 11. *If, for $i = 1, 2, \dots, r$, (26) holds and*

$$\lim_{\alpha} a_{\alpha,i}(\lambda) = 1 \qquad \text{uniformly in } \lambda \in \Lambda,$$

then $\{H_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ is an equi-uniform approximation process on $BC(X,E)$, and so $\{G_{\alpha,\lambda}(x) : \alpha \in D, \lambda \in \Lambda, x \in X\}$ is an equi-uniform $\mathfrak{T}$-approximation process on E_0.

Proof. We define

$$\chi_{\alpha,\lambda}(x;k) = \prod_{i=1}^{r} \chi_{m_{\alpha,i}(\lambda)}(x_i;k_i) \qquad (x \in X,\ k \in N_{\alpha,\lambda}),$$

where

$$\chi_{m_{\alpha,i}(\lambda)}(x_i;k_i) = (1 - x_i t_{m_{\alpha,i}(\lambda),k_i}) \left\{ \frac{Q_{m_{\alpha,i}(\lambda)}(x_i)}{m_{\alpha,i}(\lambda)(x_i - t_{m_{\alpha,i}(\lambda),k_i})} \right\}^2$$

and

$$\xi_{\alpha,\lambda}(k) = (a_{\alpha,1}(\lambda) t_{m_{\alpha,1}(\lambda),k_1}, \dots, a_{\alpha,r}(\lambda) t_{m_{\alpha,r}(\lambda),k_r}) \qquad (k \in N_{\alpha,\lambda}).$$

Then for all $\alpha \in D, \lambda \in \Lambda, x \in X_0$ and for $i = 1, 2, \dots, r$, we have

$$\sum_{k\in N_{\alpha,\lambda}} \chi_{\alpha,\lambda}(x;k) |p_i(x) - p_i(\xi_{\alpha,\lambda}(k))|^2$$

$$= \frac{Q_{m_{\alpha,i}(\lambda)}(x_i)^2}{m_{\alpha,i}(\lambda)} - 2x_i(a_{\alpha,i}(\lambda) - 1) \sum_{k_i=1}^{m_{\alpha,i}(\lambda)} t_{m_{\alpha,i}(\lambda),k_i} \chi_{m_{\alpha,i}(\lambda)}(x_i;k_i)$$

$$+ (a_{\alpha,i}(\lambda)^2 - 1) \sum_{k_i=1}^{m_{\alpha,i}(\lambda)} t_{m_{\alpha,i}(\lambda),k_i}{}^2 \chi_{m_{\alpha,i}(\lambda)}(x_i; k_i).$$

Therefore, applying Theorem 8 with $q = 2$ we obtain the desired result.

We assume that

$$a_{\alpha,i}(\lambda) = 1 \qquad (\alpha \in D, i = 1, 2, \dots, r) \tag{30}$$

for all $\lambda \in \Lambda$ and we define $H_{\alpha,\lambda}(F)(x)$ and $G_{\alpha,\lambda}(x)(f)$ for all $F \in C(X_r, E), f \in E_0$ and $x \in X_r$. Then it immediately follows from Theorem 11 that if (26) holds for $i = 1, 2, \dots, r$, then $\{H_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ is an equi-uniform approximation process on $C(X_r, E)$, and so $\{G_{\alpha,\lambda}(x) : \alpha \in D, \lambda \in \Lambda, x \in X_r\}$ is an equi-uniform $\mathfrak{T}$-approximation process on E_0. Also, let $D = \mathbb{N}_0$ and let $\{\nu_{n,i}\}_{n \in \mathbb{N}_0}, i = 1, 2, \dots, r$, be strictly monotone increasing sequences of positive integers. We define

$$m_{n,i}(\lambda) = \nu_{n,i}, \quad a_{n,i}(\lambda) = 1 \qquad (n \in \mathbb{N}_0, i = 1, 2, \dots, r)$$

for all $\lambda \in \Lambda$, and so (30) holds with $\alpha = n \in \mathbb{N}_0$. Then we write $H_{\nu_{n,1},\nu_{n,2},\dots,\nu_{n,r}}$ and $G_{\nu_{n,1},\nu_{n,2},\dots,\nu_{n,r}}$ instead of $H_{n,\lambda}$ and $G_{n,\lambda}$, respectively. Then we have the following:

$$\lim_{n \to \infty} \|H_{\nu_{n,1},\nu_{n,2},\dots,\nu_{n,r}}(F)(x) - F(x)\| = 0 \quad \text{uniformly in } x \in X_r \tag{31}$$

for all $F \in C(X_r, E)$, and so

$$\lim_{n \to \infty} \|G_{\nu_{n,1},\nu_{n,2},\dots,\nu_{n,r}}(x)(f) - T(x)(f)\| = 0 \qquad \text{uniformly in } x \in X_r$$

for all $f \in E_0$. The statement (31) generalizes the uniform convergence theorem (cf. [8], [11]) for the classical Hermite-Fejér operators on $C(X_1, \mathbb{R})$. More generally, Theorem 7 (a) yields that if $\mathcal{A}$ is stochastic with (A-1), then $\{H_{\nu_{n,1},\nu_{n,2},\dots,\nu_{n,r}} : n \in \mathbb{N}_0\}$ is an equi-uniform $\mathcal{A}$-summation process on $C(X_r, E)$, and so $\{G_{\nu_{n,1},\nu_{n,2},\dots,\nu_{n,r}} : n \in \mathbb{N}_0\}$ is an equi-uniform $\mathfrak{T}$-$\mathcal{A}$-summation process on E_0.

5. Convolution type operarors

Here, we treat equi-uniform $\mathcal{A}$-summation processes of convolution type operators defined as follows:

Let $(\mathbb{R}^r, d), d = d_p, 1 \leq p \leq \infty$, be as in the preceding section. Let $c > 0$ and let $\{g_n : n \in \mathbb{N}_0\}$ be a sequence of nonnegative even continuous functions on $[-c, c]$ such that

$$\int_{-c}^{c} g_n(t)\, dt = 1.$$

Let

$$X = \prod_{i=1}^{r} [a_i, b_i], \quad 0 < b_i - a_i \leq c, \quad i = 1, 2, \dots, r$$

and

$$X_0 = \prod_{i=1}^{r} [a_i + \delta_i, b_i - \delta_i], \quad 0 < \delta_i < \frac{1}{2}(b_i - a_i), \quad i = 1, 2, \dots, r.$$

Then we define

$$K_n(F)(x) := \int_X \prod_{i=1}^{r} (g_n \circ p_i)(x - y) F(y)\, dy \qquad (F \in C(X, E),\ x \in X)$$

and

$$T_n(x)(f) = \int_X \prod_{i=1}^{r} (g_n \circ p_i)(x - y) T(y)(f)\, dy \qquad (f \in E_0,\ x \in X),$$

which are called convolution type operators.

Theorem 12. *Suppose that $\mathcal{A}$ is positive and that Conditions (A-1) and (A-2) are satisfied. If*

$$\lim_{n \to \infty} \int_0^c t^2 g_n(t)\, dt = 0, \tag{32}$$

then $\mathcal{K} = \{K_n : n \in \mathbb{N}_0\}$ is an equi-uniform $\mathcal{A}$-summation process on $C(X, E)$, and so $\mathcal{T} = \{T_n(x) : n \in \mathbb{N}_0, x \in X\}$ is an equi-uniform $\mathfrak{T}$-$\mathcal{A}$-summation process on E_0.

Proof. We define

$$\chi_n(x; y) := \prod_{i=1}^{r} (g_n \circ p_i)(x - y) \qquad (x, y \in X,\ n \in \mathbb{N}_0).$$

Then by [17, Lemma 4] and (32), we have

$$0 \leq 1 - \int_X \chi_n(x; y)\, dy \leq \left(2 \sum_{i=1}^{r} \frac{1}{\delta_i^2} \right) \int_0^c t^2 g_n(t)\, dt$$

$$\to 0 \qquad \text{uniformly in } x \in X_0$$

and

$$\int_X \chi_n(x; y) d_2^2(x; y)\, dy \leq 2r \int_0^c t^2 g_n(t)\, dt \to 0 \quad \text{uniformly in } x \in X_0.$$

Therefore, applying (23) with $q = 2$, the desired result follows from Theorem 7 (b).

Let φ be a nonnegative, even continuous function on $[-c, c]$ such that φ is decreasing on $[0, c]$ and

$$\varphi(0) = 1, \quad 0 \leq \varphi(t) < 1 \qquad (0 < t \leq c).$$

We define

$$g_n(t) = \rho_n \varphi(t)^n \qquad (|t| \leq c,\ n \in \mathbb{N}_0),$$

where

$$\rho_n = \left(\int_{-c}^{c} \varphi(t)^n \, dt \right)^{-1} \qquad (n \in \mathbb{N}_0).$$

Then we have

$$K_n(F)(x) = \rho_n^r \int_X \prod_{i=1}^{r} \varphi^n \circ p_i(x - y) F(y) \, dy \qquad (F \in C(X, E),\ x \in X),$$

which reduce to the Korovkin operators in case $r = 1$ and $E = \mathbb{R}$, and

$$T_n(x)(f) = \rho_n^r \int_X \prod_{i=1}^{r} \varphi^n \circ p_i(x - y) T(y)(f) \, dy \qquad (f \in E_0,\ x \in X).$$

Note that

$$\lim_{n \to \infty} \int_{-c}^{c} t^2 g_n(t) \, dt = \lim_{n \to \infty} \frac{\int_0^c t^2 \varphi(t)^n \, dt}{\int_0^c \varphi(t)^n \, dt} = 0$$

(cf. the proof of [8, Theorem 5]), and so Theorem 12 holds. More precisely, if there exist constants $q, s > 0$ such that

$$\lim_{t \to +0} \frac{1 - \varphi(t)}{t^s} = q,$$

then

$$\int_0^c t^2 g_n(t) \, dt = O\left(\frac{1}{n^{2/s}} \right) \quad (n \to \infty)$$

(cf. [3, Lemma 2] and the proof of [3, Theorem 1]). Several important examples of φ are the following:

(1°) *Weierstrass*:

$$\varphi(t) = e^{-t^2}; \quad 0 < c < \infty,\ s = 2,\ q = 1.$$

(2°) *Picard*:

$$\varphi(t) = e^{-|t|}; \quad 0 < c < \infty,\ s = 1,\ q = 1.$$

(3°) *Bui-Fedorov-Cervakov*:

$$\varphi(t) = e^{-|t|^{1/\nu}}; \quad 0 < c < \infty,\ \nu > 0,\ s = 1/\nu,\ q = 1.$$

(4°) *Landau*:

$$\varphi(t) = 1 - t^2; \quad c = 1, \ s = 2, \ q = 1.$$

(5°) *Mamedov*:

$$\varphi(t) = 1 - t^{2m}; \quad c = 1, \ m \in \mathbb{N}, \ s = 2m, \ q = 1.$$

(6°) Let $\nu > 0$ and

$$\varphi(t) = 1 - |t|^{\nu}; \quad c = 1, \ s = \nu, \ q = 1.$$

(7°) *de la Vallée-Poussin*:

$$\varphi(t) = \cos^2 \frac{1}{2}t; \quad c = \pi, \ s = 2, \ q = 1/4.$$

(8°) Let $\nu > 0$ and

$$\varphi(t) = \left(\cos \frac{1}{2}t\right)^{\nu}; \quad c = \pi, \ s = 2, \ q = \nu/8.$$

Next, we consider the convolution operators for the whole space $\mathbb{R}^r$. Let $\{h_n : n \in \mathbb{N}_0\}$ be a sequence of nonnegative Lebesgue measurable functions on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} h_n(t)\,dt = 1 \qquad (n \in \mathbb{N}_0), \tag{33}$$

and for $q > 0$, we define

$$\mu_n(q) = \mu(q; h_n) := \int_{\mathbb{R}} |t|^q h_n(t)\,dt < \infty,$$

which is called the qth absolute moment of h_n. Let $X = \mathbb{R}^r$. We define

$$K_n(F)(x) = \int_X \prod_{i=1}^{r} (h_n \circ p_i)(x-y)F(y)\,dy \qquad (F \in BC(X, E), x \in X)$$

and

$$T_n(x)(f) = \int_X \prod_{i=1}^{r} (h_n \circ p_i)(x-y)T(y)(f)\,dy \qquad (f \in E_0, x \in X).$$

Then we have the following:

Theorem 13. *Suppose that $\mathcal{A}$ is positive and that Conditions (A-1) and (A-2) are satisfied. If*

$$\lim_{n\to\infty} \mu_n(q) = 0 \tag{34}$$

for some $q > 0$, *then* $\mathcal{K} = \{K_n : n \in \mathbb{N}_0\}$ *is an equi-uniform* $\mathcal{A}$-*summation process on* $BC(X, E)$, *and so* $\mathcal{T} = \{T_n(x) : n \in \mathbb{N}_0, x \in X\}$ *is an equi-uniform* $\mathfrak{T}$-$\mathcal{A}$-*summation process on* E_0.

Proof. We define

$$\chi_n(x; y) := \prod_{i=1}^{r} (h_n \circ p_i)(x - y)\, dy \qquad (x, y \in X, , n \in \mathbb{N}_0).$$

Then by (33), (34) and (23) we have

$$\int_X \chi_n(x; y)\, dy = 1 \qquad (x \in X,\ n \in \mathbb{N}_0)$$

and

$$\int_X \chi_n(x; y) d^q(x; y)\, dy \leq rc(p, q, r)\mu_n(q) \to 0 \quad \text{uniformly in } x \in X_0.$$

Therefore, the desired result follows from Theorem 7 (b).

Let $\{k_n : n \in \mathbb{N}_0\}$ be a sequence of nonnegative, even, 2π-periodic, Lebesgue measurable functions on $\mathbb{R}$ having Fourier series expansions

$$k_n(t) \quad \sim \quad \sum_{j=-\infty}^{\infty} \hat{k_n}(j) e^{ijt}, \qquad \hat{k_n}(j) := \frac{1}{2\pi} \int_{-\pi}^{\pi} k_n(t) e^{-ijt}\, dt$$

with

$$\hat{k_n}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} k_n(t) = 1 \qquad (n \in \mathbb{N}_0), \tag{35}$$

and we define

$$h_n(t) = \begin{cases} \frac{1}{2\pi} k_n(t) & (|t| \leq \pi) \\ 0 & (|t| > \pi). \end{cases} \tag{36}$$

Then we have the following.

Theorem 14. *Suppose that* $\mathcal{A}$ *is positive and that Conditions (A-1) and (A-2) are satisfied. If*

$$\lim_{n \to \infty} \hat{k_n}(1)) = 1, \tag{37}$$

then $\mathcal{K} = \{K_n : n \in \mathbb{N}_0\}$ *is an equi-uniform* $\mathcal{A}$-*summation process on* $BC(X, E)$, *and so* $\mathcal{T} = \{T_n(x) : n \in \mathbb{N}_0, x \in X\}$ *is an equi-uniform* $\mathfrak{T}$-$\mathcal{A}$- *summation process on* E_0.

Proof. By (35) and (36), h_n clearly satisfies (33). Also, by Jordan's inequality

$$\frac{2}{\pi}t \leq \sin t \leq t \qquad \left(0 \leq t \leq \frac{\pi}{2}\right), \tag{38}$$

we have

$$\int_{-\pi}^{\pi} t^2 k_n(t)\,dt \leq \pi^2 \int_{-\pi}^{\pi} k_n(t) \sin^2\frac{t}{2}\,dt = \frac{\pi^2}{2}\int_{-\pi}^{\pi}(1-\cos t)k_n(t)\,dt.$$

Therefore, the desired result follows from Theorem 13 with $q = 2$.

Let $(\lambda_n(j))$ $(n, j = 1, 2, \dots)$ be a lower triangular infinite matrix of real numbers and we define

$$k_0(t) = 1, \quad k_n(t) = 1 + 2\sum_{j=1}^{n} \lambda_n(j)\cos jt \qquad (n \in \mathbb{N},\ t \in \mathbb{R}).$$

Then applying the Abel's transformation twice to the function $k_n(t)$, we have

$$k_n(t) = \sum_{j=0}^{n-1}(j+1)F_j(t)\Delta^2\lambda_n(j) + (n+1)\lambda_n(n)F_n(t), \quad \lambda_n(0) = 1,$$

where $F_m(t)$ is the mth Fejér kernel and

$$\Delta^2\lambda_n(j) := \lambda_n(j) - 2\lambda_n(j+1) + \lambda_n(j+2).$$

Therefore, if $\lambda_n(n) \geq 0$ and $\{\lambda_n(j) : j \in \mathbb{N}_0\}$ is convex, i.e., $\Delta^2\lambda_n(j) \geq 0$ for all $j \in \mathbb{N}_0$, then $k_n(t)$ is a nonnegative, even trigonometric polynomial of degree at most n satisfying (35), and so Theorem 14 holds with $\hat{k}_n(1) = \lambda_n(1)$.

Several examples of $\lambda_n(j)$ produce important summability kernels as follows:

(1°) *Fejér*:

$$\lambda_n(j) = \begin{cases} 1 - \frac{j}{n+1} & (1 \leq j \leq n) \\ 0 & (j > n). \end{cases}$$

(2°) *de la Vallée-Poussin*:

$$\lambda_n(j) = \begin{cases} \frac{(n!)^2}{(n-j)!(n+j)!} & (1 \leq j \leq n) \\ 0 & (j > n). \end{cases}$$

(3°) *Fejér-Korovkin* :

$$\lambda_n(j) = \begin{cases} A_n \sum_{m=0}^{n-j} a_m a_{j+m} & (1 \le j \le n) \\ 0 & (j > n), \end{cases}$$

where

$$a_m = \sin\Big(\frac{m+1}{n+2}\Big)\pi \qquad (m = 0, 1, \ldots, n), \qquad A_n = \Big(\sum_{m=0}^{n} a_m^2\Big)^{-1}.$$

In this case, we have

$$k_n(t) = A_n \Big|\sum_{m=0}^{n} a_m e^{imt}\Big|^2, \qquad \lambda_n(1) = \cos\Big(\frac{\pi}{n+2}\Big).$$

(4°) *Nörlund*:

$$\lambda_n(j) = \begin{cases} \frac{Q_{n-j}}{Q_n} & (1 \le j \le n) \\ 0 & (j > n), \end{cases}$$

where

$$0 < q_0 \le q_n \le q_{n+1}, \qquad Q_n := \sum_{m=0}^{n} q_m \qquad (n \in \mathbb{N}_0).$$

Obviously, if $q_n = 1$ for all $n \in \mathbb{N}_0$, then the Nörlund kernel reduces to the Fejér kernel.

(5°) *Cesàro*:

$$\lambda_n(j) = \begin{cases} \frac{C_{n-j}^{(\beta)}}{C_n^{(\beta)}} & (1 \le j \le n) \\ 0 & (j > n), \end{cases} \qquad (\beta \ge 1)$$

where $C_n^{(\tau)}$ $(n \in \mathbb{N}_0, \tau > -1)$ is defined as in Example (3°) of Section 3. Note that if $q_n = C_n^{(\beta-1)}$ for all $n \in \mathbb{N}_0$, then Nörlund kernel reduces to the Cesàro kernel. In particular, if $\beta = 1$, then the Cesàro kernel turns out the Fejér kernel.

Other important examples of nonnegative, even, 2π-periodic continuous functions $k_n(t)$ on $\mathbb{R}$ satisfying (35) and (37) are the following:

(6°) *Jackson*:

$$k_n(t) = c_{n,s} \begin{cases} \left(\frac{\sin((n+1)t/2)}{\sin(t/2)}\right)^{2s}, & \text{if } t \text{ is not a multiple of } 2\pi \\ (n+1)^{2s}, & \text{if } t \text{ is a multiple of } 2\pi, \end{cases}$$

where $s \in \mathbb{N}$ and the normalizing constant $c_{n,s} > 0$ is taken in such a way that

$$\frac{1}{\pi}\int_0^\pi k_n(t)\,dt = 1.$$

Since $k_n(t) = c_{n,s}(n+1)^s F_n(t)^s$, $k_n(t)$ is a nonnegative, even trigonometric polynomial of degree ns and we have, for $s=1$, $c_{n,1} = 1/(n+1)$, and so $k_n(t)$ becomes the nth Fejér kernel. Also, we have, for $s=2$,

$$c_{n,2} = \frac{3}{(n+1)(2(n+1)^2+1)}, \qquad \hat{k_n}(1) = \frac{2((n+1)^2-1)}{2(n+1)^2+1}.$$

Furthermore, making use of Jordan's inequality (38) we have that for $s \geq 3$,

$$\left(\frac{\pi}{2}\right)^{1-2s}\frac{2s-1}{2s}(n+1)^{1-2s} < c_{n,s} \leq \left(\frac{\pi}{2}\right)^{2s}(n+1)^{1-2s}$$

and

$$0 < 1 - \hat{k_n}(1) < \left(\frac{\pi}{2}\right)^{2(2s-1)}\frac{8s}{3\pi(2s-3)}(n+1)^{-2} \to 0 \quad (n \to \infty).$$

(7°) *Abel-Poisson*:

$$k_n(t) = 1 + 2\sum_{m=1}^{\infty} r_n^m \cos mt \qquad (n \in \mathbb{N}_0,\ t \in \mathbb{R}),$$

where $\{r_n : n \in \mathbb{N}_0\}$ is a sequence of real numbers to one such that $0 \leq r_n < 1$ for all $n \in \mathbb{N}_0$. Thus, $\lim_{n\to\infty} \hat{k_n}(1) = \lim_{n\to\infty} r_n = 1$, and since

$$1 + 2\sum_{m=1}^{\infty} z^m = \frac{1+z}{1-z} = \frac{(1+re^{it})(1-re^{-it})}{|1-re^{it}|^2} = \frac{1-r^2+2ir\sin t}{1-2r\cos t+r^2}$$

for all $z = re^{it}, 0 \leq r < 1$, we have

$$k_n(t) = \frac{1-r_n^2}{1-2r_n\cos t + r_n^2} = \frac{1-r_n^2}{(1-r_n)^2+4r_n\sin^2(t/2)}.$$

(8°) *Gauss-Weierstrass*:

$$k_n(t) = \sqrt{\frac{\pi}{\tau_n}}\sum_{m=-\infty}^{\infty}\exp\left\{-\frac{(t-2\pi m)^2}{4\tau_n}\right\} \qquad (n \in \mathbb{N}_0,\ t \in \mathbb{R}),$$

where $\{\tau_n : n \in \mathbb{N}_0\}$ is a sequence of positive real numbers converging to zero. We can rewrite $k_n(t)$ as

$$k_n(t) = 1 + 2\sum_{m=1}^{\infty} e^{-\tau_n m^2} \cos mt,$$

and so $\lim_{n\to\infty} \hat{k}_n(1) = \lim_{n\to\infty} e^{-\tau_n} = 1$.

Finally, we give several examples of nonperiodic functions $h_n(t)$ satisfying (33) for which Theorem 13 can be applied, from a probabilistic point of view. These can be induced by various probability density functions as follows:

Let $\{\alpha_n : n \in \mathbb{N}_0\}$ and $\{\beta_n : n \in \mathbb{N}_0\}$ be sequences of positive real numbers, and let $q > 0$.

(9°) *Gauss type distribution*:

$$h_n(t) := \sqrt{\frac{1}{\pi\alpha_n}} \exp\Big(-\frac{t^2}{\alpha_n}\Big) \qquad (t \in \mathbb{R}).$$

Then we have

$$\mu_n(q) = \frac{1}{\sqrt{\pi}} \Gamma\Big(\frac{q+1}{2}\Big) \alpha_n^{q/2},$$

where

$$\Gamma(x) := \int_0^{\infty} t^{x-1} e^{-t}\, dt \qquad (x > 0)$$

is the gamma function. In particular, we have

$$\mu_n(1) = \sqrt{\frac{\alpha_n}{\pi}}, \qquad \mu_n(2) = \frac{\alpha_n}{2}$$

and

$$\mu_n(2m) = \Big(m - \frac{1}{2}\Big)\Big(m - \frac{3}{2}\Big)\Big(m - \frac{5}{2}\Big) \cdots \frac{3}{2}\frac{1}{2}\alpha_n^m \qquad (m \in \mathbb{N}).$$

(10°) *Laplace type distribution*:

$$h_n(t) := \frac{1}{2\alpha_n} \exp\Big(-\frac{|t|}{\alpha_n}\Big) \qquad (t \in \mathbb{R}).$$

Then we have

$$\mu_n(q) = q\Gamma(q)\alpha_n^q.$$

In particular, we have

$$\mu_n(m) = m!\alpha_n^m \qquad (m \in \mathbb{N}),$$

and so

$$\mu_n(1) = \alpha_n, \qquad \mu_n(2) = 2\alpha_n^2.$$

(11°) *Student (t) type distribution*:

$$h_n(t) := \sqrt{\frac{\alpha_n}{\pi}} \frac{\Gamma(\beta_n)}{\Gamma(\beta_n - 1/2)} (1 + \alpha_n t^2)^{-\beta_n} \qquad (t \in \mathbb{R}).$$

Then we have

$$\mu_n(q) = \frac{\Gamma\left(\frac{q+1}{2}\right)}{\sqrt{\pi}} \left(\frac{1}{\sqrt{\alpha_n}}\right)^q \frac{\Gamma\left(\beta_n - \frac{q+1}{2}\right)}{\Gamma\left(\beta_n - \frac{1}{2}\right)},$$

and so

$$\mu_n(1) = \frac{1}{\sqrt{\pi \alpha_n}} \frac{\Gamma(\beta_n - 1)}{\Gamma\left(\beta_n - \frac{1}{2}\right)}, \qquad \mu_n(2) = \frac{1}{\alpha_n(2\beta_n - 3)}.$$

(12°) *Gamma type ditribution*:

$$h_n(t) := \begin{cases} \frac{\beta_n^{\alpha_n}}{\Gamma(\alpha_n)} t^{\alpha_n - 1} e^{-\beta_n t} & (t > 0) \\ 0 & (t \leq 0). \end{cases}$$

Then we have

$$\mu_n(q) = \frac{1}{\beta_n^q} \frac{\Gamma(q + \alpha_n)}{\Gamma(\alpha_n)}.$$

In particular, we have

$$\mu_n(m) = \frac{1}{\beta_n^m} \prod_{i=0}^{m-1} (\alpha_n + i) \qquad (m \in \mathbb{N}),$$

and so

$$\mu_n(1) = \frac{\alpha_n}{\beta_n}, \qquad \mu_n(2) = \frac{\alpha_n(\alpha_n + 1)}{\beta_n^2}.$$

(13°) *Beta type distribution*:

$$h_n(t) := \begin{cases} \frac{1}{B(\alpha_n, \beta_n)} t^{\alpha_n - 1}(1 - t)^{\beta_n - 1} & (0 < t < 1) \\ 0 & (t \leq 0 \text{ or } 1 \leq t), \end{cases}$$

where

$$B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1}\, dt \qquad (x, y > 0)$$

is the beta function. Then we have

$$\mu_n(q) = \frac{B(\alpha_n + q, \beta_n)}{B(\alpha_n, \beta_n)} = \frac{\Gamma(\alpha_n + \beta_n)}{\Gamma(\alpha_n)} \frac{\Gamma(\alpha_n + q)}{\Gamma(\alpha_n + \beta_n + q)}.$$

In particular, we have

$$\mu_n(m) = \prod_{i=0}^{m-1} \frac{\alpha_n + i}{\alpha_n + \beta_n + i} \qquad (m \in \mathbb{N}),$$

and so

$$\mu_n(1) = \frac{\alpha_n}{\alpha_n + \beta_n}, \qquad \mu_n(2) = \frac{\alpha_n(\alpha_n + 1)}{(\alpha_n + \beta_n)(\alpha_n + \beta_n + 1)}.$$

(14°) *Landau type distribution*:

$$h_n(t) := \begin{cases} \frac{\alpha_n}{2B\left(\frac{1}{\alpha_n}, \beta_n\right)}(1 - |t|^{\alpha_n})^{\beta_n - 1} & (|t| \leq 1) \\ 0 & (|t| > 1). \end{cases}$$

Then we have

$$\mu_n(q) = \frac{\Gamma\left(\frac{q+1}{\alpha_n}\right)}{\Gamma\left(\frac{1}{\alpha_n}\right)} \frac{\Gamma\left(\beta_n + \frac{1}{\alpha_n}\right)}{\Gamma\left(\beta_n + \frac{q+1}{\alpha_n}\right)},$$

and so

$$\mu_n(1) = \frac{\Gamma\left(\frac{2}{\alpha_n}\right)}{\Gamma\left(\frac{1}{\alpha_n}\right)} \frac{\Gamma\left(\beta_n + \frac{1}{\alpha_n}\right)}{\Gamma\left(\beta_n + \frac{2}{\alpha_n}\right)}, \qquad \mu_n(2) = \frac{\Gamma\left(\frac{3}{\alpha_n}\right)}{\Gamma\left(\frac{1}{\alpha_n}\right)} \frac{\Gamma\left(\beta_n + \frac{1}{\alpha_n}\right)}{\Gamma\left(\beta_n + \frac{3}{\alpha_n}\right)}.$$

In particular, if $\alpha_n = 2$, then

$$\mu_n(q) = \frac{\Gamma\left(\frac{q+1}{2}\right)}{\sqrt{\pi}} \frac{\Gamma\left(\beta_n + \frac{1}{2}\right)}{\Gamma\left(\beta_n + \frac{q+1}{2}\right)},$$

and so

$$\mu_n(1) = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\beta_n + \frac{1}{2}\right)}{\beta_n \Gamma(\beta_n)}, \qquad \mu_n(2) = \frac{1}{2\beta_n + 1},$$

and furthermore, if $\beta_n = n + 1$, then

$$\mu_n(q) = \Gamma\left(\frac{q+1}{2}\right) \frac{\left(n + \frac{1}{2}\right)\left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right)\left(n - \frac{5}{2}\right) \cdots \frac{3}{2}\frac{1}{2}}{\Gamma\left(n + \frac{q+3}{2}\right)},$$

and so

$$\mu_n(1) = \frac{\left(n + \frac{1}{2}\right)\left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right)\left(n - \frac{5}{2}\right) \cdots \frac{3}{2}\frac{1}{2}}{(n+1)!}, \quad \mu_n(2) = \frac{1}{2n + 3}.$$

Also, if $\ell_n := 1/\alpha_n \in \mathbb{N}$, then

$$\mu_n(q) = \frac{\Gamma(\ell_n(q+1))}{\Gamma(\ell_n)} \frac{\Gamma(\beta_n + \ell_n)}{\Gamma(\beta_n + \ell_n(q+1))},$$

and so

$$\mu_n(1) = \prod_{i=\ell_n}^{2\ell_n - 1} \frac{i}{\beta_n + i}, \qquad \mu_n(2) = \prod_{i=\ell_n}^{2\ell_n - 1} \frac{1}{\beta_n + i} \prod_{i=2\ell_n}^{3\ell_n - 1} \frac{i}{\beta_n + i}.$$

(15°) *Weibull type distribution*:

$$h_n(t) := \begin{cases} \frac{\beta_n}{\alpha_n} t^{\beta_n - 1} \exp\left(-\frac{t^{\beta_n}}{\alpha_n}\right) & (t > 0) \\ 0 & (t \leq 0). \end{cases}$$

Then we have

$$\mu_n(q) = \frac{q\alpha_n^{q/\beta_n}}{\beta_n} \Gamma\left(\frac{q}{\beta_n}\right),$$

and so

$$\mu_n(1) = \frac{\alpha_n^{1/\beta_n}}{\beta_n} \Gamma\left(\frac{1}{\beta_n}\right), \qquad \mu_n(2) = \frac{2\alpha_n^{2/\beta_n}}{\beta_n} \Gamma\left(\frac{2}{\beta_n}\right).$$

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