Negative characters on the degree of the best approximation in Banach spaces

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NEGATIVE CHARACTERS ON THE DEGREE OF THE BEST APPROXIMATION IN BANACH SPACES

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ABSTRACT. We consider some negative characters on the degree of the best approximation associated with a total, fundamental sequence of mutually orthogonal projections in Banach spaces. Furthermore, applications are discussed under the setting of abstract Fourier expansions in Banach spaces as well as homogeneous Banach spaces which include the classical function spaces as particular cases.

1. Introduction

Let X be a Banach space with norm $\|\cdot\|_X$, and let $B[X]$ denote the Banach algebra of all bounded linear operators of X into itself with the usual operator norm $\|\cdot\|_{B[X]}$. Let $\mathbb{Z}$ denote the set of all integers, and let $\mathfrak{P} = \{P_j : j \in \mathbb{Z}\}$ be a sequence of projection operators in $B[X]$ satisfying the following conditions:

(P-1) $\mathfrak{P}$ is orthogonal, i.e., $P_j P_n = \delta_{j,n} P_n$ for all $j, n \in \mathbb{Z}$, where $\delta_{j,n}$ denotes Kronecker's symbol.

(P-2) $\mathfrak{P}$ is fundamental, i.e., the linear span of the set $\cup_{j\in\mathbb{Z}} P_j(X)$ is dense in X.

(P-3) $\mathfrak{P}$ is total, i.e., if $f \in X$ and $P_j(f) = 0$ for all $j \in \mathbb{Z}$, then $f = 0$.

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Let $\mathbb{N}$ be the set of all non-negative integers. For each $n \in \mathbb{N}$, M_n stands for the linear span of the set $\{P_j(X) : |j| \le n\}$, which is a closed linear subspace of X. For a given $f \in X$, we define

$$E_n(f) = E_n(X; f) = \inf\{\|f - g\|_X : g \in M_n\},$$

which is called the best approximation of degree n to f with respect to M_n. Then we have

$$E_0(f) \ge E_1(f) \ge \cdots \ge E_n(f) \ge E_{n+1}(f) \ge \cdots \ge 0,$$

and Condition (P-2) implies that for every $f \in X$,

$$\lim_{n\to\infty} E_n(f) = 0. \tag{1}$$

In [10], [11], [12] and [13], we studied the relation between the rapidity of convergence (1) and certain smoothness properties of f in terms of the moduli of continuity of f induced by a strongly continuous group of multiplier operators with respect to $\mathfrak{P}$. Such results are sometimes called direct (Jackson-type) theorems and inverse (Bernstein-type) theorems of the best approximation theory (cf. [1], [3], [6], [7], [16], [22]). For further general treatments and refinements of the inverse theorems, see [14] and [15].

The purpose of this paper is to consider certain negative characters about $E_n(f)$ which assert the impossibility of constructing operators with certain desirable properties. We will make the best use of our results of [17], which extensively treats the best approximation by bounded linear projections of X onto M_n. Moreover, applications are discussed under the setting of abstract Fourier expansions in Banach spaces as well as homogeneous Banach spaces (cf. [4], [8], [18], [21]) which include the Banach space $C_{2\pi}$ of all 2π-periodic, continuous functions f on the real line $\mathbb{R}$ with the norm

$$\|f\|_\infty = \max\{|f(t)| : |t| \le \pi\}$$

and the Banach space $L^p_{2\pi}$ of all 2π-periodic, p-th power Lebesgue integrable functions f on $\mathbb{R}$ with the norm

$$\|f\|_p = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^p dt\right)^{1/p} \qquad (1 \le p < \infty),$$

as special cases. Actually, further extensions are given to the classical theorems of Kharshiladze-Lozinski, Faber and Berman (cf. [3; Chap.6, Sec.5], [6; Chap.7, Sec.3]).

2. Best approximation by projections

For each $n \in \mathbb{N}$, $\mathfrak{T}_n$ denotes the set of all bounded linear operators T of X into M_n such that $T(g) = g$ for all $g \in M_n$. In other words, $\mathfrak{T}_n$ is the set of all bounded linear projections of X onto M_n and it is a closed subset of $B[X]$ such that $\alpha S + (1-\alpha)T \in \mathfrak{T}_n$ whenever $S, T \in \mathfrak{T}_n$ and α is a scalar. In particular, $\mathfrak{T}_n$ is a closed convex subset of $B[X]$. Let (Ω, μ) be a probability measure space. Let $\mathfrak{T} = \{T_t : t \in \Omega\}$ and $\mathfrak{U} = \{U_t : t \in \Omega\}$ be uniformly bounded families of operators in $B[X]$ such that for all $f \in X$ and all $T \in B[X]$, the mapping $t \mapsto T_t T U_t(f)$ is strongly μ-measurable on Ω. For any $T \in B[X]$, we define

$$\Phi_T(f) = \Phi_T(\mathfrak{T}, \mathfrak{U}; f) = \int_{\Omega} T_t T U_t(f)\, d\mu(t) \qquad (f \in X),$$

which always exists as a Bochner integral in X. Then Φ_T belongs to $B[X]$ and the uniform boundedness of $\mathfrak{T}$ and $\mathfrak{U}$ yields

$$\|\Phi_T\|_{B[X]} \leq AB\|T\|_{B[X]},$$

where

$$A = \sup\{\|T_t\|_{B[X]} : t \in \Omega\} < \infty \tag{2}$$

and

$$B = \sup\{\|U_t\|_{B[X]} : t \in \Omega\} < \infty. \tag{3}$$

From now on, we suppose that the following additional conditions

$$T_t P_j = P_j T_t \quad \text{for all } j \in \mathbb{Z},\ t \in \Omega, \tag{4}$$

$$U_t P_j = P_j U_t \quad \text{for all } j \in \mathbb{Z},\ t \in \Omega, \tag{5}$$

and

$$T_t U_t = I \quad \text{for all } t \in \Omega, \tag{6}$$

where I is the identity operator on X.

For each $n \in \mathbb{N}$, we define

$$S_n = \sum_{j=-n}^{n} P_j,$$

which belongs to $\mathfrak{T}_n$. Then (4) and (5) imply

$$S_n T_t = T_t S_n, \quad S_n U_t = U_t S_n \qquad (n \in \mathbb{N},\ t \in \Omega). \tag{7}$$

Also, for each $n \in \mathbb{N}$ we define

$$\mathfrak{T}_n^* = \{T \in \mathfrak{T}_n : \Phi_T P_j = 0 \text{ for all } j \in \mathbb{Z}, |j| > n\}.$$

By (P-1), (7) and [17; Lemma 2.1], S_n belongs to $\mathfrak{T}_n^*$. Concerning the approximation by operators in $\mathfrak{T}_n^*$, we have the following result:

Theorem 1. *([17; Theorem 2.4]) Let S be an operator in $B[X]$ such that $SU_t = U_tS$ or $ST_t = T_tS$ for all $t \in \Omega$. Then there holds*

$$\|S - S_n\|_{B[X]} \leq AB \inf\{\|S - T\|_{B[X]} : T \in \mathfrak{T}_n^*\} \qquad (n \in \mathbb{N}).$$

In particular, if $AB \leq 1$, then

$$\|S - S_n\|_{B[X]} = \min\{\|S - T\|_{B[X]} : T \in \mathfrak{T}_n^*\} \qquad (n \in \mathbb{N}),$$

which implies S_n is an operator of best approximation to S from $\mathfrak{T}_n^*$.

Let $\mathfrak{V} = \{V_t : t \in \Omega\}$ be a uniformly bounded family of operators in $B[X]$ such that for each $f \in X$, the mapping $t \mapsto V_t(f)$ is strongly μ-measurable on Ω. Let χ be a μ-integrable function on Ω and $W \in B[X]$. Then we define the convolution type operator associated with $\mathfrak{U}, \chi$ and W by

$$(\chi * W)(f) = (\chi * W)_{\mathfrak{V}}(f) = \int_\Omega \chi(t) V_t(W(f))\, d\mu(t) \qquad (f \in X), \tag{8}$$

which exists as a Bocher integral in X (cf. [8], [10]). $\chi * W$ belongs to $B[X]$ and

$$\|\chi * W\|_{B[X]} \leq C\|\chi\|_1 \|W\|_{B[X]}, \tag{9}$$

where

$$C = \sup\{\|V_t\|_{B[X]} : t \in \Omega\} < \infty \tag{10}$$

and

$$\|\chi\|_1 = \int_\Omega |\chi(t)|\, d\mu(t) < \infty.$$

Theorem 2. *Suppose that $V_uU_t = U_tV_u$ and $WU_t = U_tW$ or $V_uT_t = T_tV_u$ and $WT_t = T_tW$ for all $t, u \in \Omega$. Then we have*

$$\|\chi * W - S_n\|_{B[X]} \leq AB \inf\{\|\chi * W - T\|_{B[X]} : T \in \mathfrak{T}_n^*\} \qquad (n \in \mathbb{N}).$$

In particular, if $AB \leq 1$, then S_n is an operator of best approximation to $\chi * W$ from $\mathfrak{T}_n^*$.

Proof. Assume that $V_uU_t = U_tV_u$ and $WU_t = U_tW$ for all $t, u \in \Omega$. Then we have

$$(\chi * W)U_t = U_t(\chi * W)$$

for all $t \in \Omega$, and so the desired result follows from Theorem 1. The case of $V_uT_t = T_tV_u$ and $WT_t = T_tW$ is also similar.

Corollary 1. *There holds*

$$\|S_n\|_{B[X]} \leq AB\{\|T\|_{B[X]} : T \in \mathfrak{T}_n^*\} \qquad (n \in \mathbb{N}).$$

In particular, if $AB \leq 1$, *then*

$$\|S_n\|_{B[X]} = \min\{\|T\|_{B[X]} : T \in \mathfrak{T}_n^*\} \qquad (n \in \mathbb{N}).$$

3. Results on negative characters

In order to achieve our aim we will make the best use of the results of Section 3. For this we always here suppose that

$$\limsup_{n\to\infty} \|S_n\|_{B[X]} = +\infty. \tag{11}$$

Theorem 3. *Let* $L_n \in \mathfrak{T}_n^*$ *for each* $n \in \mathbb{N}$. *Then there exists an element* $f_0 \in X$ *for which the sequence* $\{\|L_n(f_0)\|_X\}$ *is unbounded. Also, there exists an element* $g_0 \in X$ *such that* $\{L_n(g_0)\}$ *does not converge.*

Proof. By Corollary 1, we have

$$\|S_n\|_{B[X]} \leq AB\|L_n\| \qquad (n \in \mathbb{N}),$$

and so the desired result follows from (11) and the uniform boundedness principle.

Theorem 4. *For each* $n \in \mathbb{N}$, *let* L_n *be a bounded linear operator of* X *into* M_n *such that* $\Phi_{L_n} P_j = 0$ *for every* $j \in \mathbb{Z}, |j| > n$. *Then there does not exist a nonnegative continuous function* ρ *on* $[0, \infty)$ *with* $\rho(0) = 0$ *such that*

$$\|L_n(f) - f\|_X \leq \rho(E_n(f)) \tag{12}$$

for all $f \in X$ *and all* $n \in \mathbb{N}$.

Proof. Assume that there exists a nonnegative continuous function ρ on $[0, \infty)$ with $\rho(0) = 0$ satisfying (12) for all $f \in X$ and all $n \in \mathbb{N}$. Then (1) and (12) imply $\lim_{n\to\infty} \|L_n(f) - f\|_X = 0$ for all $f \in X$. Also, again from (12) we have

$$\|L_n(g) - g\|_X \leq \rho(E_n(g)) = \rho(0) = 0$$

whenever $g \in M_n$, and so L_n belongs to $\mathfrak{T}_n^*$. This conflicts with Theorem 3.

Corollary 2. *Let L_n be as in Theorem 4. Then there does not exist a constant $K > 0$ such that*

$$\|L_n(f) - f\|_X \leq K E_n(f)$$

for all $f \in X$ and all $n \in \mathbb{N}$.

4. Applications

For any $f \in X$, we associate its (formal) Fourier series expansion

$$f \quad \sim \quad \sum_{j=-\infty}^{\infty} P_j(f). \tag{13}$$

An operator $T \in B[X]$ is called a multiplier operator on X if there exists a sequence $\{\tau_j : j \in \mathbb{Z}\}$ of scalars such that for every $f \in X$,

$$T(f) \quad \sim \quad \sum_{j=-\infty}^{\infty} \tau_j P_j(f),$$

and the following notation is used:

$$T \quad \sim \quad \sum_{j=-\infty}^{\infty} \tau_j P_j$$

(cf. [2], [8], [9], [20]). Let $M[X]$ denote the set of all multiplier operators on X, which is commutative closed subalgebra containing I and S_n, which is the n-th partial sum operator associated with the Fourier series (13).

From now on let Ω be a separable topological space and μ a Borel probability measure on Ω. Let $\mathfrak{T} = \{T_t : t \in \Omega\}$ and $\mathfrak{U} = \{U_t : t \in \Omega\}$ be families of operators in $M[X]$ satisfying (2) and (3) and having the expansions

$$T_t \quad \sim \quad \sum_{j=-\infty}^{\infty} e_j(t) P_j \qquad (t \in \Omega) \tag{14}$$

and

$$U_t \quad \sim \quad \sum_{j=-\infty}^{\infty} f_j(t) P_j \qquad (t \in \Omega), \tag{15}$$

where $\{e_j : j \in \mathbb{Z}\}$ and $\{f_j : j \in \mathbb{Z}\}$ are sequences of scalar-valued continuous functions on Ω such that

$$e_j(t) f_j(t) = 1 \quad \text{for all } j \in \mathbb{Z},\ t \in \Omega. \tag{16}$$

By (14), we have

$$\lim_{t \to u} \|T_t(g) - T_u(g)\|_X = \lim_{t \to u} |e_j(t) - e_j(u)| \|g\|_X = 0 \qquad (u \in \Omega)$$

for every $g \in P_j(X), j \in \mathbb{Z}$. Therefore, the mapping $t \mapsto T_t(f)$ is strongly continuous on Ω for each $f \in X$, since $\mathfrak{P}$ is fundamental and $\mathfrak{T}$ is uniformly bounded. Similarly, the mapping $t \mapsto U_t(f)$ is strongly continuous on Ω for each $f \in X$. Therefore, the mapping $t \mapsto T_t T U_t(f)$ is strongly continuous on Ω for each $f \in X$. Also, in view of (14), (15) and (16), there hold Conditions (4), (5) and (6). Consequently, all the results obtained in the preceding sections hold under the above setting.

Now, we assume that

$$\int_\Omega e_j(t) f_k(t)\, d\mu(t) = 0 \quad \text{for all } j,\ k,\ j \neq k. \tag{17}$$

Let $\mathfrak{V} = \{V_t : t \in \Omega\}$ be a family of operators in $M[X]$ satisfying (10) and having the expansions

$$V_t \quad \sim \quad \sum_{j=-\infty}^{\infty} v_j(t) P_j \qquad (t \in \Omega), \tag{18}$$

where $\{v_j : j \in \mathbb{Z}\}$ is a sequence of scalar-valued continuous functions on Ω and

$$W \quad \sim \quad \sum_{j=-\infty}^{\infty} \omega_j P_j.$$

Then the convolution type operator $(\chi * W)_\mathfrak{V}$ given by (8) belongs to $M[X]$ and there holds

$$(\chi * W)_\mathfrak{V} \quad \sim \quad \sum_{j=-\infty}^{\infty} c_j(\mathfrak{V}, \chi) \omega_j P_j, \tag{19}$$

where

$$c_j(\mathfrak{V}, \chi) = \int_\Omega \chi(t) v_j(t)\, d\mu(t) \qquad (j \in \mathbb{Z})$$

(cf. [10; Lemma 2], [11; Lemma 1]). Thus we have the following result (cf. [17; Corollary 3.3]):

Theorem 5. *There holds*

$$\|\chi * W - S_n\|_{B[X]} \leq AB \inf\{\|\chi * W - T\|_{B[X]} : T \in \mathfrak{T}_n\} \qquad (n \in \mathbb{N}).$$

In particular, if $AB \leq 1$, then S_n is an operator of best approximation to $\chi * W$ from $\mathfrak{T}_n$.

Proof. By [17; Lemma 3.1], we have $\mathfrak{T}_n^* = \mathfrak{T}_n$ for each $n \in \mathbb{N}$. Therefore, the desired result follows from Theorem 2.

Corollary 3. *We have*

$$\|S_n\|_{B[X]} \leq \inf\{\|T\|_{B[X]} : T \in \mathfrak{T}_n\} \qquad (n \in \mathbb{N}).$$

In particular, if $AB \leq 1$, then

$$\|S_n\|_{B[X]} = \min\{\|T\|_{B[X]} : T \in \mathfrak{T}_n\} \qquad (n \in \mathbb{N}).$$

In the following results, we always suppose that (11) holds.

Theorem 6. *Let $L_n \in \mathfrak{T}_n$ for each $n \in \mathbb{N}$. Then there exists an element $f_0 \in X$ such that the sequence $\{\|L_n(f_0)\|_X\}$ is unbounded. Also, there exists an element $g_0 \in X$ such that $\{L_n(g_0)\}$ does not converge.*

Proof. This immediately follows from [17; Lemma 3.1] and Theorem 3.

Theorem 7. *Let L_n be a bounded linear operator of X into M_n for each $n \in \mathbb{N}$. Then there does not exist a nonnegative continuous function ρ on $[0, \infty)$ with $\rho(0) = 0$, for which*

$$\|L_n(f) - f\|_X \leq \rho(E_n(f))$$

for all $f \in X$ and all $n \in \mathbb{N}$.

Proof. Use [17; Lemmas 2.3 and 3.1] and Theorem 4.

Corollary 4. *Let L_n be as in Theorem 7. Then there does not exist a constant $K > 0$ such that*

$$\|L_n(f) - f\|_X \leq K E_n(f)$$

for all $f \in X$ and all $n \in \mathbb{N}$.

If M_n is a Chebyshev subspace of X, that is, for every $f \in X$ there exists a unique element $B_n(f)$ of best approximation to f from M_n, then the mapping $B_n : X \to M_n$ is called the best approximation operator on X with respect to M_n. For general problems concerning the existence and uniqueness of elements of the best approximation in normed linear spaces, see e.g., [16], and the literatures cited there.

Remark 1. We have $B_n(g) = g$ for all $g \in M_n$ and

$$\|B_n(f)\|_X \leq 2\|f\|_X, \quad \|B_n(f) - f\|_X = E_n(f) \qquad (f \in X). \quad (20)$$

Therefore, we conclude from (20) and Corollary 4 that if (11) holds, then $\{B_n\}$ is the sequence of nonlinear operators of X onto M_n.

Next, let us improve Corollary 4 by increasing the degree of the approximating operators. For this, let $A_n \in \mathfrak{T}_n$ for each $n \in \mathbb{N}$ and we define the gliding mean operators by

$$A_{n,m} = \frac{1}{m} \sum_{j=n}^{n+m-1} A_j \qquad (n \geq 0,\ m \geq 1). \tag{21}$$

Then $A_{n,m}$ belongs to $\mathfrak{T}_n$ and

$$\begin{aligned} \|A_{n,m}(f) - f\|_X &\leq \|A_{n,m} - I\|_{B[X]} E_n(f) \\ &\leq (\|A_{n,m}\|_{B[X]} + 1) E_n(f) \qquad (f \in X) \end{aligned} \tag{22}$$

(cf. [10; Theorem 4]). Therefore, if

$$K_1 = \sup\{\|A_{n,m}\|_{B[X]} : n \geq 0,\ m \geq 1\} < \infty,$$

then we have

$$E_{n+m-1}(f) \leq \|A_{n,m}(f) - f\|_X \leq (K_1 + 1) E_n(f) \qquad (f \in X).$$

Let σ_n, $n \in \mathbb{N}$, be the Cesàro type mean operators, that is,

$$\sigma_n = A_{0,n+1} = \frac{1}{n+1} \sum_{j=0}^{n} A_j \qquad (n \in \mathbb{N}).$$

Then (21) becomes

$$A_{n,m} = \frac{1}{m}\left\{(n+m)\sigma_{n+m-1} - n\sigma_{m-1}\right\} \tag{23}$$

and we have the following de la Vallée-Poussin type estimate:

Theorem 8. *If*

$$K = \sup\{\|\sigma_n\|_{B[X]} : n \in \mathbb{N}\} < \infty, \tag{24}$$

then for all $f \in X, n \in \mathbb{N}$ and all $m \in \mathbb{N} \setminus \{0\}$,

$$E_{n+m-1}(f) \leq \|A_{n,m}(f) - f\|_X \leq \left(K\left(1 + \frac{2n}{m}\right) + 1\right) E_n(f). \tag{25}$$

Proof. By (23) and (24), we have

$$\|A_{n,m}\|_{B[X]} \leq \frac{1}{m}\left\{(n+m)\|\sigma_{n+m-1}\|_{B[X]} + n\|\sigma_{m-1}\|_{B[X]}\right\} \leq \frac{K}{m}(2n+m),$$

which together with (22) implies the desired inequality (25).

Let $A_n = S_n$ for all $n \in \mathbb{N}$. If we take $m = n \geq 1$ in (21), then Theorem 8 reduces to [10; Theorem 6]. Note that if

$$(\Omega, \mu) = \left([-\pi, \pi], \frac{1}{2\pi} dt\right)$$

and

$$v_j(t) = e^{-ijt} \qquad (j \in \mathbb{Z},\ |t| \leq \pi)$$

in (18), then (19) reduces to

$$(\chi * W)_{\mathfrak{B}} \quad \sim \quad \sum_{j=-\infty}^{\infty} \hat{\chi}(j)\omega_j P_j, \tag{26}$$

where

$$\hat{\chi}(j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \chi(t)e^{-ijt}\, dt \qquad (j \in \mathbb{Z})$$

is the j-th Fourier coefficient of χ, and (24) always holds. Indeed, let $F_n(t)$ be the Fejér kernel, i.e.,

$$F_n(t) = \frac{1}{n+1}\left\{\frac{\sin\frac{1}{2}(n+1)t}{\sin\frac{1}{2}t}\right\}^2 = 1 + 2\sum_{j=1}^{n}\left(1 - \frac{j}{n+1}\right)\cos jt$$

$$= \sum_{j=-n}^{n}\left(1 - \frac{|j|}{n+1}\right)e^{ijt}.$$

Then, in view of (26), we have

$$\sigma_n = \sum_{j=-n}^{n}\left(1 - \frac{|j|}{n+1}\right)P_j = \sum_{j=-n}^{n} \hat{F}_n(j)P_j = (F_n * I)_{\mathfrak{B}} \qquad (n \in \mathbb{N}),$$

and so (9) yields

$$\|\sigma_n\|_{B[X]} \leq C\|F_n\|_1 = C \qquad (n \in \mathbb{N}),$$

which implies (24). Also, let $D_n(t)$ be the Dirichlet kernel, i.e.,

$$D_n(t) = \frac{\sin(2n+1)\frac{1}{2}t}{\sin\frac{1}{2}t} = 1 + 2\sum_{j=1}^{n}\cos jt = \sum_{j=-n}^{n} e^{ijt}.$$

Then we have

$$S_n = \sum_{j=-n}^{n} P_j = \sum_{j=-n}^{n} \hat{D}_n(j)P_j = (D_n * I)_{\mathfrak{B}} \qquad (n \in \mathbb{N}),$$

and so

$$\|S_n\|_{B[X]} \leq C\lambda_n \qquad (n \in \mathbb{N}), \tag{27}$$

where $\lambda_n = \|D_n\|_1, n \in \mathbb{N}$, are called the Lebesgue constants. Note that

$$\frac{4}{\pi^2}\log(n+1) < \lambda_n \leq 1 + \log(2n+1)$$
$$\leq 1 + \log 3 + \log n < 3 + \log n \qquad (n \geq 1) \tag{28}$$

(cf. [1; Proposition 1.2.3], [3; Chap.6, Sec.5, Lemma], [6; Chap.1, Sec.2, Theorem 2]). Since $A_{n,1} = A_n = S_n$, (22), (27) and (28) yield the following Lebesgue type estimate:

$$\|S_n(f) - f\|_X \leq \|S_n - I\|_{B[X]} E_n(f) \leq (\|S_n\|_{B[X]} + 1)E_n(f)$$
$$\leq (1 + C(1 + \log(2n+1)))E_n(f) \leq (1 + C(3 + \log n))E_n(f)$$
$$(f \in X, \ n \geq 1)$$

(cf. [10; Theorem 5]).

Remark 2. Suppose that

$$A = \sup\{\|T_t\|_{B[X]} : t \in \mathbb{R}\} < \infty$$

and

$$T_t \quad \sim \quad \sum_{j=-\infty}^{\infty} e^{\lambda_j t} P_j \qquad (t \in \mathbb{R}), \tag{29}$$

where $\{\lambda_j : j \in \mathbb{Z}\}$ is a sequence of scalars. Then $\mathfrak{T} = \{T_t : t \in \mathbb{R}\}$ becomes a strongly continuous group of operators and there holds

$$G(f) \quad \sim \quad \sum_{j=-\infty}^{\infty} \lambda_j P_j(f) \qquad (f \in D(G)),$$

where G is the infinitesimal generator of $\mathfrak{T}$ with domain $D(G)$ ([8; Proposition 2]). Let $\Omega = [a, b] \subseteq \mathbb{R}$. Then in view of (16) and (29), (15) reduces to

$$U_t \quad \sim \quad \sum_{j=-\infty}^{\infty} e^{-\lambda_j t} P_j \qquad (t \in [a, b]).$$

Also, typical examples of the sequences $\{e_j\}$ and $\{f_j\}$ satisfying (16) and (17) are given by

$$e_j(t) = e^{-im_j\varphi(t)}, \quad f_j(t) = e^{im_j\varphi(t)} \qquad (t \in [a, b], \ j \in \mathbb{Z}),$$

where

$$\varphi(t) = \frac{2\pi}{b-a}\Big(t - \frac{1}{2}(a+b)\Big) \qquad (t \in [a, b])$$

and $\{m_j : j \in \mathbb{Z}\}$ is a sequence of integers such that $m_j \neq m_k$ whenever $j \neq k$.

Remark 3. Let $\mathfrak{G} = \{g_j, g_j^*\}_{j\in\mathbb{Z}}$ be a fundamental, total biorthogonal system on X, where $\{g_j : j \in \mathbb{Z}\}$ and $\{g_j^* : j \in \mathbb{Z}\}$ are sequences of elements in X and X^* (the dual space of X), respectively (cf. [5], [19]). That is, $\mathfrak{G}$ is a system which satisfies the following conditions:

(G-1) $\mathfrak{G}$ is fundamental, i.e., the linear span of $\{g_j : j \in \mathbb{Z}\}$ is dense in X.

(G-2) $\mathfrak{G}$ is total, i.e., if $f \in X$ and $g_j^*(f) = 0$ for all $j \in \mathbb{Z}$, then $f = 0$.

(G-3) $\mathfrak{G}$ is biorthogonal, i.e., $g_j^*(g_k) = \delta_{j,k}$ for all $j, k \in \mathbb{Z}$.

Then we define

$$P_j(f) = g_j^*(f)g_j \qquad (f \in X, j \in \mathbb{Z}),$$

which satisfies Conditions (P-1), (P-2) and (P-3). Therefore, our results obtained in this section are applied in this setting.

Finally, we restrict ourselves to the case where X is a homogeneous Banach space (cf. [4], [8], [18], [21]). That is, X a function space which satisfies the following conditions:

(H-1) X is a linear subspace of $L^1_{2\pi}$ and it is a Banach space with norm $\|\cdot\|_X$.

(H-2) X is continuously embedded in $L^1_{2\pi}$, i.e., there exists a constant $K > 0$ such that

$$\|f\|_1 \leq K\|f\|_X \quad \text{for all } f \in X.$$

(H-3) The right translation operators T_t defined by

$$T_t(f)(\cdot) = f(\cdot - t) \qquad (f \in X)$$

is isometric on X for each $t \in \mathbb{R}$.

(H-4) For each $f \in X$, the mapping $t \mapsto T_t(f)$ is strongly continuous on $\mathbb{R}$.

Typical examples of homogeneous Banach spaces are $C_{2\pi}$ and $L^p_{2\pi}, 1 \leq p < \infty$. For other examples, see [8] (cf. [4], [18], [21]).

Now take

$$(\Omega, \mu) = \left([-\pi, \pi], \frac{1}{2\pi}dt\right),$$

$$e_j(t) = e^{-ijt}, \quad f_j(t) = g_j(t) = e^{ijt} \qquad (j \in \mathbb{Z},\ |t| \leq \pi),$$

$$g_j^*(f) = \hat{f}(j) \qquad (j \in \mathbb{Z},\ f \in X)$$

(cf. Remarks 2 and 3). Then M_n is the $2n + 1$-dimensional linear subspace of X consisting of all trigonometric polynomials of degree at most n.

Consequently, all the results stated in this section hold for homogeneous Banach spaces and in particular, Corollary 3, Theorem 6 and Theorem 7 extend [3; p.212, Theorem, 6; Chap.7, Theorem 7], [3; Kharshiladze-Lozibski Theorem 1] (cf. [6; Chap.7, Theorem 8]) and [6; Chap.7, Theorem 9] to more general homogeneous Banach spaces, respectively.

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